FINDING A MINIMUM DOMINATING SET BY TRANSFORMING DOMINATION OF VERTICES

Mahmoud Saoud, Jebran Jebran

Abstract. The dominating set decision problem is NP-hard, there is no formula for the domination number of a graph. In this paper, we will introduce the concept of transforming the domination from a vertex in a dominating set D of a graph G to a vertex in $V-D$, where G is a simple connected graph. And we'll give an algorithm using this transformation to obtain a minimum dominating set of a graph G, in particular, we'll illustrate the algorithm for the Cartesian product of two paths.

Keywords: Dominating set, domination number, transformation of domination of a vertex, redundant vertex of a dominating set, Cartesian product of two paths.

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1. Introduction

A graph $G=(V,E)$ is a mathematical structure which consists of two sets V and E, where V is finite nonempty, and every element of E is an unordered pair $\{u,v\}$ of distinct elements of V, we simply write uv instead of $\{u,v\}$. The elements of V are called vertices, while the elements of E are called edges. The order of G is the cardinality $|V|$ of its vertex-set, the size of G is the cardinality $|E|$ of its edge-set. Two vertices u and v of a graph G are said to be adjacent if $uv\in E$. For a vertex v of G, the neighborhood of v is the set of all vertices of G which are adjacent to v, the neighborhood of v is denoted by $N(v)$. The closed neighborhood of v is $\overline{N}(v)=N(v)\cup\{v\}$. If D is a set of vertices of G, then the neighborhood of D is $N(D)=\underset{v\in D}{\cup}N(v)$, and $\overline{N}(D)=\underset{v\in D}{\cup}\overline{N}(v)$, The degree of a vertex v is $d(v)=|N(v)|$. Let $G=(V,E)$ be a graph, a set $D\subseteq V$ is called a dominating set of G if every vertex in $V-D$ is adjacent to at least one vertex of D, *i.e.* if $\overline{N}(D)=V$. A dominating set D of G is said to be a minimum dominating set of G if $|D|\leq|D_1|$ for any dominating set D_1 of G. A minimal dominating set in a graph G is a dominating set that contains no dominating set as a proper subset. The cardinality of a minimum dominating set of G is known as the domination number of G, and is denoted by $\delta(G)$.

2.transformation of domination of vertices

Definition 1. *Let D be a dominating set of a graph $G=(V,E)$. We define the function C_D, which we call the weight function, as follows: $C_D : V \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, $C_D(v) = \left|\widetilde{N}(v)\right|$, where $\widetilde{N}(v) = \{w \in D : vw \in E \text{ or } w = v\}$, i.e. the weight of v is the number of vertices in D which dominates v.*

Definition 2. *Let D be a dominating set of a graph $G=(V,E)$, and let $v \in D$. Then we say that v has a moving domination if there exists a vertex $w \in N(v) - D$ such that $wu \in E$ for every vertex $u \in \{x \in N(v) : C_D(x) = 1\}$. Note that a vertex $v \in D$ does not have a moving domination if for any $w \in N(v) - D$, there exists at least one vertex in $N(v)$ whose weight is 1, but is not adjacent to w. If the vertex v has a moving domination and hence there exists a vertex $w \in N(v) - D$ with $wu \in E$ for any vertex $u \in \{x \in N(v) : C_D(x) = 1\}$, note that the domination can be transformed from v to the vertex w in the sence that $(D - \{v\}) \cup \{w\}$ is also a dominating set of G.*

Definition 3. *Let D be a dominating set of a graph G. Then for any vertex $v \in D$, we define the region of movement of v to be the set $N_1(v) = \{w \in N(v) - D : wt \in E$ for every $t \in N(v)$ with $C_D(t) = 1\}$, i.e. a vertex w is in the region of movement of v if and only if the domination of v can be transformed to w.*

Definition 4. *Let D be a dominating set of a graph G. We say that a vertex $v \in D$ is a redundant vertex of D if $C_D(w) \geq 2$ for every vertex $w \in \overline{N}(v)$.*

Definition 5. *Let D be a dominating set of a graph G, and let v be a vertex in D which has a moving domination. We say that v is inefficient if transforming the domination from v to any vertex in the region of movement of v would not produce any redundant vertex.*

3. An algorithm for finding a minimum dominating set of a graph G using transformation of domination of vertices

1. Let $G=(V,E)$ be a graph of order greater than 1, $|V| = l$.
2. Let $D = V$ be a dominating set of G. Then for any vertex $v \in D$ we have $C_D(v) = d(v) + 1 \geq 2$.
3. Pick a vertex v_1 of D, and delete from D all vertices w , $w \in N(v_1)$. Then, for $n < l$, pick a vertex $v_n \in D - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$ and delete from D all vertices w , $w \in N(v_n) - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$.
4. If D contains a redundant vertex, then delete it. Repeat this process until D has no redundant vertex .

5. Transform domination from vertices of D which have moving domination to vertices in $V-D$ to obtain redundant vertices and go to step **4.** If no redundant vertex can be obtained by transformation of domination of vertices of D, then we stop, and the obtained dominating set D satisfies:

For every $v \in D, \exists w \in \overline{N}(v)$ such that $C_D(w) = 1$, this implies that D is a minimal dominating set.

Note. A dominating set D obtained by applying the previous algorithm satisfies :
(a) There is no vertex in D which has a moving domination,or

(b) D has vertices with moving domination but they are inefficient. We conjecture that a dominating set obtained by the previous algorithm is not only a minimal dominating set but also a minimum one.

Theorem. Let D be a dominating set of a graph $G = (V, E)$, which is obtained by the previous algorithm. Then D is a minimum dominating set of G if and only if there exists status of vertices of D satisfies: $\forall v \in D, \exists w \in V - D$ such that $N(w) \cap D = \{v\}$.

Proof. Suppose that D is a minimum dominating set of G. Then for any $v \in D$, the set $D-$ $\{v\}$ is not a dominating set of G. Thus there exists $w \in V - D$ which is dominated only by v, i.e. $N(w) \cap D = \{v\}$.

Conversely, suppose that for any $v \in D$, there exists $w \in V - D$ such that $N(w) \cap D = \{v\}$. Then for any $v \in D$, the set $D-$ $\{v\}$ is not a dominating set of G. Hence D is a minimum dominating set of G.

Example 1.
1. Let $G = (V, E)$ be the graph depicted in Figure 1, $|V| = 18$

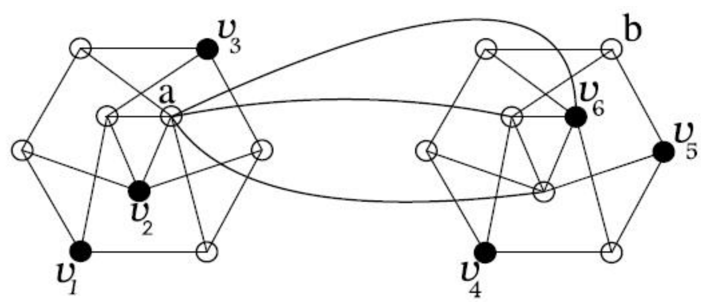

Fig.1

2. Let $D = V$.
3. pick a vertex $v_1 \in D$, and Delete from D all vertices w , $w \in N(v_1)$, then, for $2 < n < 18$,pick a vertex $v_n \in D - \overset{n-1}{\underset{i=1}{\cup}} \overline{N}(v_i)$ and delete from D all vertices w,

$w \in N(v_n) - \overline{\bigcup_{i=1}^{n-1} N}(v_i)$. We obtain the dominating set $D = \{v_1, v_2, ..., v_6\}$, see Figure 1.

4. Transform the domination from the vertices: from v_2 to the vertex a, and from v_5 to the vertex b, and delete redundant vertex v_6 . We obtain the dominating set $D = \{v_1, v_2, ..., v_5\}$ which has no redundant vertices, see Figure 2.

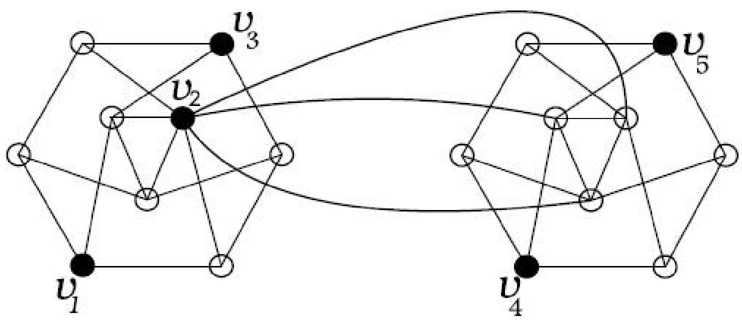

Fig. 2

5. The vertices v_1 and v_3 are the only vertices of D with moving domination , but they are inefficient. Therefore we stop , and we obtain the dominating set $D = \{v_1, v_2, ..., v_5\}$. D is indeed a minimum dominating set and hence $\delta(G) = 5$.

4.Applying the algorithm on the Cartesian product of two paths

For two vertices v_0 and v_n of a graph $G = (V, E)$, a $v_0 - v_n$ walk is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, ..., e_n, v_n$ such that consecutive vertices and edges are incident . A path is a walk in which no vertex is repeated . A path with n vertices is denoted by P_n , it has $n - 1$ edges . The length of P_n is $n - 1$.

The Cartesian product $P_n \times P_m$ of two paths is the graph with vertex set $V = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ where $(u_1, u_2)(v_1, v_2)$ is an edge of $P_n \times P_m$ if $|u_1 - v_1| + |u_2 - v_2| = 1$.

4.1.Characterization of the vertex which has moving domination in the Cartesian product of two paths

Let $v \in D$, where D is a dominating set of the Cartesian product $P_n \times P_m$ of two paths, which has no redundant vertex.

Then a vertex $v \in D$ has a moving domination if and only if one of the following two cases occurs :

Case (1): For every vertex $w \in N(v)$, we have $C_D(w) \geq 2$.

In this case, since v is not a redundant vertex of D , we must have $N(v) - D \neq \phi$ and hence the domination of v can be transformed to any vertex in $N(v) - D$.

Case (2): There exists exactly one vertex $u \in N(v)$ such that $C_D(u) = 1$. In this case , the domination of v can be transformed only to u .

This characterization of vertices of $P_n \times P_m$ which have moving domination will simplify applying step 5 of the algorithm .

Example 2.

Let (n, m) be the vertex in the $n-th$ row and in the $m-th$ column of the graph $G = P_7 \times P_{10}$.

1. Let $G = P_7 \times P_{10}$

2. Let $D = V$,$|V| = 70$.

3. pick a vertex $v_1 = (1,1)$ of D ,and delete from D all vertices w , $w \in N(v_1)$, then, for $2 \leq n < 70$, pick a vertex $v_n \in D - \overset{n-1}{\underset{i=1}{\cup}} \overline{N}(v_i)$, and delete from D all vertices w , $w \in N(v_n) - \overset{n-1}{\underset{i=1}{\cup}} \overline{N}(v_i)$. We obtain the dominating set $D = \{v_1, v_2, ..., v_{22}\}$, see Figure 3.

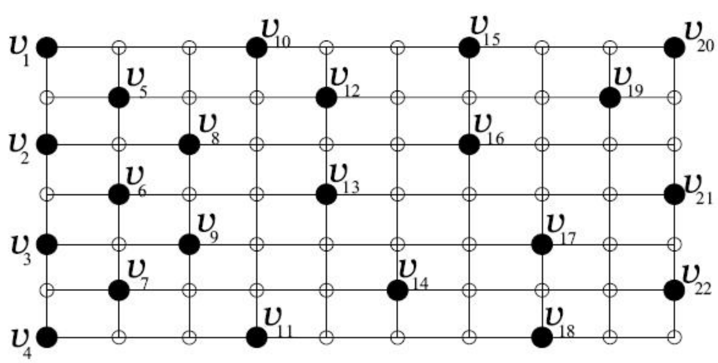

Fig. 3

4. Since every vertex in D has weight 1 , D has no redundant vertices.

5. Transform the domination from the vertex v_5 to the vertex $(3,2)$, and delete from D the resulting redundant vertices v_2 and v_6. then, transform the domination from v_7 to the vertex $(7,2)$, and delete from D the resulting redundant vertex v_4. Then, transform the domination from the vertices: from v_{12} to the vertex $(2,6)$, and from v_{15} to the vertex $(1,8)$, and from v_{20} to the vertex $(2,10)$, and from v_{21} to the vertex $(4,9)$, and delete from D the resulting redundant vertex v_{19} . This produces the new dominating set D illustrated in Figure 4.

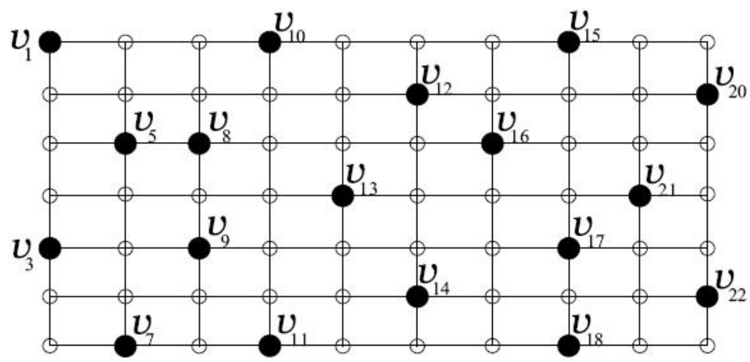

Fig. 4

Note that the vertex v_{17} is the only vertex of D with moving domination , but it is inefficient . Therefore the set D indicated in Figure 4 (black circles) is a dominating set of $G = P_7 \times P_{10}$. Note that D is a minimum dominating set and hence $\delta(P_7 \times P_{10}) = 18$.

Note: The domination number of the graph $P_n \times P_m$ when min $\{n, m\} \leq 3$ can be computed by the simple formulas :

$\delta\left(P_1 \times P_n\right) = \left\lceil \frac{n}{3} \right\rceil$, for $n \geq 1$,

$\delta\left(P_2 \times P_n\right) = \left\lceil \frac{n+1}{2} \right\rceil$, for $n \geq 2$,

$\delta\left(P_3 \times P_n\right) = \left\lfloor \frac{3n}{4} \right\rfloor + 1$, for $n \geq 3$,

where $\lceil x \rceil$ is the smallest integer greater than or equal to x , and $\lfloor x \rfloor$ is the largest integer less than or equal to x.

For the graph $P_n \times P_m$ where both $n, m \geq 4$, we apply the previous algorithm .

We have the impression that this algorithm can be applied for any graph of order greater than 1 to obtain a minimum domination set .

5.Conclusion

We believe that the concept of transformation of domination of vertices, which we introduce, plays an important role in finding a minimum dominating set from a given one .

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Mahmoud Saoud
Department of Mathematics
Ecole Normale Supérieure
BP 92 Kouba; 16050 Alger, Algéria
email:*saoud_m@yahoo.fr*

Jebran Jebran
Department of Mathematics
University of Damascus, Faculty of Science
Damascus-Syria