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HAMILTONIAN STRUCTURE AND NEW EXACT SOLITON SOLUTIONS OF SOME KORTEWEG – DE VRIES EQUATIONS

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Abstract. In this paper, we discuss the Hamiltonian structure of Korteweg–de Vries equation, modified Korteweg–de Vries equation, and generalized Korteweg–de Vries equation. We proposed the Sine-function algorithm to obtain the exact solution for non-linear partial differential equations. This method is used to obtain the exact solutions for KdV, mKdV and GKdV equations. Also, we have applied the method to Burgers equation which does not admit Hamiltonian structure.

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1. Introduction

Recently, in a review article Praught and Smirnov [1], discussed the multi-Hamiltonian structure of the Korteweg–de Vries equation and the history of Lenard recursion formula for the construction of higher order Korteweg–de Vries equations, so that the higher order Korteweg–de Vries equations also has the same conserved quantities as the basic Korteweg–de Vries equation have. This led to the KdV hierarchy.

The Korteweg–de Vries equation, an evolution equation in one space dimension is named after the Dutch mathematicians Korteweg and de Vries [2], however it was even earlier discovered by Boussinesq [3]. Initially the KdV equation was proposed as a model equation for long surface waves of water in a narrow and shallow channel. The objective of study the KdV equation was to obtain Solitary wave solutions of the type discovered in nature Russell [4]. Later it was found that this equation also models waves in other homogeneous, weakly nonlinear and weakly dispersive media. Since, the mid sixties the KdV equation received a lot of attention in the aftermath of computational experiments of Kruskel and Zabusky [5], which led to the discovery of the interaction properties of the Solitary wave solutions and in turn to the understanding of KdV equation as an infinite dimensional integrable Hamiltonian system.

In recent years, a lot of attention is made by many mathematicians and scientists to develop the methods for exact Solitary wave solutions of partial differential equations, such as, tanh method [6,7], the extended tanh-function method [8,9], the modified extended tanh-function method [10-14], variational iteration method [15-20], First Integral Method [21,22].

The purpose of this paper is to review the Hamiltonian structure of the GKdV equation and to get mKdV and KdV equations as its spectial cases. Then we construct the new exact Solitary wave solutions of GKdV, mKdV, and KdV equatios using Sine-function method.

2. Hamiltonian Structure

Initially it was observed by Gardner [23], Faddeev and Zakharov [24] that the KdV equation can be put in Hamiltonian form. The following Lemma is used for the Hamiltonian structure of KdV equation.

Lemma [25]. *If* $H(u) = \int_{-\infty}^{\infty} F(u, u_x, u_{xx}...)dx$ *then the corresponding vector field is given by*

$$X_H(u) = \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u}\right),$$

where $\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\delta F}{\delta u_{xx}}\right) - ...$ *denotes the* L^2*-gradient of* H.

We take the underlying phase space as the Sobolev space, endowed with the Poisson bracket (in case of KdV equation, called KdV bracket) proposed by Gardner:

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right)dx,$$

where F and G are the differentiable functions of the corresponding phase space with L^2-gradients. The corresponding Hamiltonian equations become $u_t = \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u}\right)$.

Example 1. The general Korteweg–de Vries (GKdV) equation. The general Korteweg–de Vries (GKdV) equation [26] for long waves in shallow water has the form:

$$u_t + \epsilon u^p u_x + \nu u_{xxx} = 0.$$

Using the above Lemma we find that the corresponding Hamiltonian for this equation is given by

$$H(u) = \int_{-\infty}^{\infty}\left(\frac{-\epsilon}{p^2 + 3p + 2}u^{p+2} + \frac{1}{2}\nu u_x^2\right)dx.$$

Case I. When $\epsilon = -6$, $p = 1$, $\nu = 1$, the Hamiltonian $H(u)$, reduces to the Hamiltonian for KdV equation as:

$$H_1(u) = \int_{-\infty}^{\infty} \left(u^3 + \frac{1}{2} u_x^2 \right) dx$$

with the corresponding Korteweg–de Vries (KdV) equation: $u_t - 6uu_x + u_{xxx} = 0$, which is the same Hamiltonian and Hamilton equation as given by Kappeller and Poschel [27].

Case II. When $\epsilon = -6$, $p = 2$, $\nu = 1$, the Hamiltonian $H(u)$, reduces to the Hamiltonian for mKdV equation as:

$$H_2(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2} u^4 + \frac{1}{2} u_x^2 \right) dx$$

with the corresponding modified Korteweg–de Vries (mKdV) equation:

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

3. The Sine function Method

Consider the nonlinear partial differential equation of the form:

$$F(u, u_t, u_x, u_{xx}, u_{xxt}, ...) = 0, \tag{1}$$

where $u(x,t)$ is the solution of nonlinear partial differential equation (1). We use the transformations

$$u(x,t) = f(\xi), \xi = x - ct. \tag{2}$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \ ... \tag{3}$$

Eq. (3) changes Eq. (1) in the form

$$G(f, f', f'', f''', ...) = 0 \tag{4}$$

The solution of Eq. (4) can be expressed in the form:

$$f(\xi) = \lambda \sin^{\alpha}(\mu\xi), \mid \xi \mid \leq \frac{\pi}{\mu} \tag{5}$$

where λ, α and μ are unknown parameters which are to be determined. Thus we have:

$$f' = \frac{df(\xi)}{d\xi} = \lambda\alpha\mu \sin^{\alpha-1}(\mu\xi)\cos(\mu\xi), \tag{6}$$

$$f'' = \frac{d^2 f(\xi)}{d\xi^2} = -\lambda\mu^2\alpha \sin^{\alpha}(\mu\xi) + \lambda\mu^2\alpha(\alpha-1)\sin^{\alpha-2}(\mu\xi) - \lambda\mu^2\alpha(\alpha-1)\sin^{\alpha}(\mu\xi). \tag{7}$$

Using Eq. (5) in Eq. (4) we obtain a trigonometric equation in terms of $\sin^{\alpha}(\mu\xi)$. To determine the parameters first we determine α by balancing the exponents of each pair of sine. Then collecting all terms of the same power in the form $\sin^{\alpha}(\mu\xi)$ and then equating their coefficients equal to zero we get system of algebraic equations among the unknowns λ, α and μ. Finally, the problem is reduced to a system of algebraic equations that can be solved to obtain the unknown parameters λ, α and μ. Hence, the solution considered in Eq. (5) is obtained. The above analysis yields the following theorem.

Theorem 1. *The exact analytical solution of the nonlinear partial differential equations (1) can be determined in the form given by Eq. (5) where all constants are found from the algebraic equations.*

4. Applications

In order to illustrate the effectiveness of the proposed method two examples are illustrated as follows.

Example 2. The general Korteweg–de Vries (GKdV) equation. The general Korteweg–de Vries (GKdV) equation [26] for long waves in shallow water has the form:

$$u_t + \epsilon u^p u_x + \nu u_{xxx} = 0. \tag{8}$$

Using the transformation $u(x,t) = f(\xi)$ and $\xi = x - ct$, Eq.(8)reduces to:

$$-c\frac{df(\xi)}{d\xi} + \epsilon f^p(\xi)\frac{df(\xi)}{d\xi} + \nu\frac{d^3 f(\xi)}{d\xi^3} = 0. \tag{9}$$

Integrating Eq. (9), gives

$$-cf(\xi) + \frac{\epsilon}{p+1}(f(\xi))^{p+1} + \nu\frac{d^2 f(\xi)}{d\xi^2} = 0. \tag{10}$$

Substituting Eq. (5) and (7) into Eq. (10) gives:

$$-c\lambda\sin^{\alpha}(\mu\xi) + \frac{\epsilon\lambda^{p+1}}{p+1}\sin^{(p+1)\alpha}(\mu\xi) - \nu\lambda\alpha\mu^2\sin^{\alpha}(\mu\xi) +$$

$$+\nu\lambda\mu^2\alpha(\alpha-1)\sin^{\alpha-2}(\mu\xi)-\nu\lambda\mu^2\alpha(\alpha-1)\sin^{\alpha}(\mu\xi)=0. \tag{11}$$

Eq. (11) is satisfied only if the following system of algebraic equations holds:

$$\begin{gathered}(p+1)\alpha=\alpha-2,\\ -c\lambda-\nu\lambda\alpha\mu^2-\nu\lambda\mu^2\alpha(\alpha-1)=0,\\ \frac{\epsilon\lambda^{p+1}}{p+1}+\nu\lambda\mu^2\alpha(\alpha-1)=0.\end{gathered} \tag{12}$$

Solving the system of equations (12), we obtain:

$$\alpha=-\frac{2}{p},\mu=\pm\frac{\iota p\sqrt{c}}{2\sqrt{\nu}},\lambda=2^{-1/p}\left(\frac{c(2+3p+p^2)}{\epsilon}\right)^{1/p}. \tag{13}$$

Substituting Eq. (13) into Eq. (5) we obtain the exact soliton solution of the GKdV equation in the form:

$$u(x,t)=2^{-1/p}\left(\frac{c(2+3p+p^2)}{\epsilon}\right)^{1/p}\sin^{-2/p}\left(\pm\frac{\iota p\sqrt{c}}{2\sqrt{\nu}}(x-ct)\right). \tag{14}$$

Case I. Putting $\epsilon=-6$, $p=1$, $\nu=1$; in equation (14) we obtain the exact soliton solution of the KdV equation in the form

$$u(x,t)=-\frac{c}{2}\text{cosec}^2\left(\pm\frac{\iota\sqrt{c}}{2}(x-ct)\right). \tag{15}$$

Case II. Putting $\epsilon=-6$, $p=2$, $\nu=1$; in equation (14) we obtain the exact soliton solution of the mKdV equation in the form

$$u(x,t)=\iota\sqrt{c}\sin^{-1}(\pm\iota\sqrt{c}(x-ct)). \tag{16}$$

Example 3. The Burgers equation. Finally, consider the well-known Burgers equation in the form

$$u_t=uu_x-ku_{xx}=0. \tag{17}$$

Using the transformation $u(x,t)=f(\xi)$ and $\xi=x-ct$, Eq.(17) reduces to:

$$-c\frac{df(\xi)}{d\xi}+f(\xi)\frac{df(\xi)}{d\xi}-k\frac{d^2f(\xi)}{d\xi^2}=0. \tag{18}$$

Integrating Eq. (18), gives

$$-cf(\xi)+\frac{f^2(\xi)}{2}-k\frac{df(\xi)}{d\xi}=0. \tag{19}$$

In a similar manner to solve Eq. (19) by Sine-function method, we obtain the system of equations as follows.

Case I. $4\alpha=2\alpha-2,\ c^2\lambda^2-k^2\lambda^2\alpha^2\mu^2=0,\ \frac{\lambda^4}{4}-k^2\lambda^2\alpha^2\mu^2=0.$

Case II. $3\alpha=2\alpha-2,\ -c\lambda^3-k^2\lambda^2\alpha^2\mu^2=0,\ c^2\lambda^2-k^2\lambda^2\alpha^2\mu^2=0.$

Thus we obtain the exact soliton solution of the Burgers equation in the form:

$$u(x,t)=\pm 2c\sin^{-1}\left(\pm\frac{c}{k}(x-ct)\right),$$

$$u(x,t)=-c\mathrm{cosec}^2\left(\frac{c}{2k}(x-ct)\right). \tag{20}$$

5. Conslusions

In this paper, we have discussed the Hamiltonian structure of GKdV equation and then get the same Hamiltonian structure of KdV equation, and mKdV equation given in [27], as its particular cases. The Sine-function method has been successfully applied to find the solution for four nonlinear partial differential equations such as GKdV, KdV, mKdV, and Burgers equations. The Sine-function method is used to find new exact solution. Thus, it is possible that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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