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PSEUDO PROJECTIVELY FLAT MANIFOLDS SATISFYING CERTAIN CONDITION ON THE RICCI TENSOR

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Abstract. In this paper we consider pseudo projectively flat Riemannian manifold whose Ricci tensor S satisfies the condition $S(X,Y) = rT(X)T(Y)$, where r is the scalar curvature, T is a non-zero 1-form defined by $g(X,\xi) = T(X)$, ξ is a unit vector field and prove that the manifold is of pseudo quasi constant curvature, integral curves of the vector field ξ are geodesic and ξ is a proper concircular vector field, manifold is locally product type and it can be expressed as a warped product $IXe^qM^\star$ where $M^\star$ is an Einstein manifold.

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1. Introduction

In 2006, De and Matsuyama studied quasi conformally flat manifolds [2] satisfying the condition

$$S(X,Y) = rT(X)T(Y) \tag{1}$$

where r is the scalar curvature and T is a 1-form defined by $T(X) = g(X,\xi)$, and ξ is a unit vector field. The present paper deals with the pseudo projectively flat manifold $(M^n, g)(n > 3)$ whose Ricci tensor S satisfies the condition (1.1). For the scalar curvature r we suppose that $r \neq 0$ for each point of M and we have proved that the manifold is of pseudo quasi constant curvature, the integral curves of the vector field ξ are geodesic and ξ is a proper concircular vector field. The manifold is a locally product manifold and can be expressed as a locally warped product $IXe^qM^\star$ where $M^\star$ is an Einstein manifold.

From [5] we know that a pseudo-projective curvature tensor $\bar{P}$ is defined by

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]$$

$$-\frac{r}{n}[\frac{a}{n-1}+b][g(Y,Z)X-g(X,Z)Y] \tag{2}$$

where a, b are constants such that $a, b \neq 0$; R, S and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor and the scalar curvature respectively. We have defined pseudo quasi constant curvature as follows

Definition 1. A Riemannian manifold $(M^n, g)(n > 3)$ is said to be of pseudo quasi-constant curvature if it is pseudo projectively flat and its curvature tensor R of type (0,4) satisfies the condition

$$\acute{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+P(Y,Z)g(X,W) - P(X,Z)g(Y,W) \tag{3}$$

where a is constant and $g(R(X,Y)Z, W) = \acute{R}(X,Y,Z,W)$ and P is a non-zero (0,2) tensor.

From (1.2) we obtain

$$(\nabla_W \bar{P})(X,Y)Z = a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y]$$
$$-\frac{dr(W)}{n}[\frac{a}{n-1}+b][g(Y,Z)X-g(X,Z)Y] \tag{4}$$

where ∇ is the covariant differentiation with respect to the Riemannian metric g. We know that

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z).$$

Hence contracting (1.4) we obtain

$$(div\bar{P})(X,Y)Z = (a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)]$$
$$-\frac{1}{n}[\frac{a}{n-1}+b][g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{5}$$

Here we consider pseudo projectively flat manifold i.e., $\bar{P}(X,Y)Z = 0$. Hence $(div\bar{P})(X,Y)Z = 0$ where 'div' denotes the divergence. If $a + b \neq 0$, then from (1.5) it follows that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \alpha[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{6}$$

where $\alpha = \frac{1}{n(a+b)}[\frac{a}{n-1}+b]$.

2.Pseudo projectively flat manifold satisfying the condition (1.1)

Proposition 2.1. *A pseudo projectively flat manifold satisfying* $S(X,Y) = rT(X)T(Y)$ *under the assumption of* $r \neq 0$ *is a manifold of pseudo quasi-constant curvature.*

Proof. From (1.2) we get

$$\bar{P}(X,Y,Z,W) = a\acute{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]$$
$$- \frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \quad (7)$$

If the manifold is pseudo projectively flat, then we obtain

$$\acute{R}(X,Y,Z,W) = \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)]$$
$$+ \frac{r}{an}[\frac{a}{n-1} + b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \quad (8)$$

which implies that it is a manifold of pseudo quasi-constant curvature.

Theorem No 2.1.*In a pseudo projectively flat Riemannian manifold satisfying* $S(X,Y) = rT(X)T(Y)$ *under the assumption of* $r \neq 0$*, the integral curves of the vector field* ξ *are geodesic.*

Proof. Differentiating covariantly to (1.1) along Z we have

$$(\nabla_Z S)(X,Y) = dr(Z)T(X)T(Y) + r[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)] \quad (9)$$

Substituting (2.3) in (1.6), we obtain

$$dr(Z)T(X)T(Y) + r[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)]$$
$$-dr(X)T(Z)T(Y) - r[(\nabla_X T)(Z)T(Y) + T(Z)(\nabla_X T)(Y)]$$
$$= \alpha[g(X,Y)dr(Z) - g(Y,Z)dr(X)] \quad (10)$$

Now putting $Y = Z = e_i$ in the above expression where $\{e_i\}$ define an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$\alpha(1-n)dr(X) = dr(\xi)T(X) + r(\nabla_\xi T)(X) + rT(X)(\delta\ T) - dr(X) \quad (11)$$

where $\delta\ T = (\nabla_{e_i} T)(e_i)$.

Again $Y = Z = \xi$ in (2.4), we have

$$r(\nabla_\xi T)(X) = (\alpha - 1)[dr(\xi)T(X) - dr(X)] \tag{12}$$

Substituting (2.6) in (2.5), we get

$$\alpha(n-2)dr(X) + \alpha\ dr(\xi)T(X) + rT(X)(\delta\ T) = 0 \tag{13}$$

Now putting $X = \xi$ in (2.7), it yields

$$\alpha(n-1)dr(\xi) + r(\delta\ T) = 0 \tag{14}$$

From (2.7) and (2.8) it follows that

$$\alpha\ dr(X) = \alpha\ dr(\xi)T(X)$$

since $\alpha \neq 0$, we have

$$dr(X) = dr(\xi)T(X) \tag{15}$$

Putting $Y = \xi$ in (2.4) and using (2.9) we get

$$(\nabla_X T)(Z) - (\nabla_Z T)(X) = 0 \tag{16}$$

since $r \neq 0$.
This means that the 1-form T defined by $g(X, \xi) = T(X)$ is closed, i.e., $dT(X, Y) = 0$.
Hence it follows that

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X) \tag{17}$$

for all X.

Now putting $Y = \xi$ in (2.11), we obtain

$$g(\nabla_X \xi, \xi) = g(\nabla_\xi \xi, X) \tag{18}$$

Since $g(\nabla_X \xi, \xi) = 0$, from (2.12) it follows that $g(\nabla_\xi \xi, X) = 0$ for all X.

Hence $\nabla_\xi \xi = 0$. This means that the integral curves of the vector field ξ are geodesic.

Theorem No 2.2. *In a pseudo projectively flat manifold satisfying*

$$S(X, Y) = rT(X)T(Y)$$

under the assumption of $r \neq 0$, the vector field ξ is a proper concircular vector field.

Proof. From (2.6), by virtue of (2.9) we get

$$(\nabla_\xi T)(X) = 0 \tag{19}$$

From (2.9) and (2.10) in (2.4), we get

$$r[T(Z)(\nabla_X T)(Y) - (\nabla_Z T)(Y)T(X)] = \alpha\ dr(\xi)[g(Y,Z)T(X) - g(X,Y)T(Z)]$$

Now putting $Z = \xi$ in the above expression, we have

$$(\nabla_X T)(Y) = \frac{\alpha}{r} dr(\xi)[T(X)T(Y) - g(X,Y)] \tag{20}$$

If we consider the scalar function $f = \frac{\alpha}{r} dr(\xi)$, differentiating covariantly along X We get

$$(\nabla_X f) = \frac{\alpha}{r^2}[dr(\xi)T(\nabla_X \xi)r - dr(\xi)dr(X)] + \frac{\alpha}{r} d^2 r(\xi, X) \tag{21}$$

From (3.9) it follows that

$$d^2 r(Y,X) = d^2 r(\xi, Y)T(X) + dr(\xi)T(\nabla_Y \xi)T(X) + dr(\xi)(\nabla_Y T)(X)$$

from which we get

$$d^2 r(\xi, Y)T(X) = d^2 r(\xi, X)T(Y) \tag{22}$$

Now putting $X = \xi$ in (2.16) we obtain $d^2 r(\xi, Y) = d^2 r(\xi, \xi)T(Y)$ since $T(\xi) = 1$.

Thus from (2.15) by using(2.9) it follows that

$$(\nabla_X f) = \mu\ T(X) \tag{23}$$

where $\mu = \frac{\alpha}{r}[d^2 r(\xi, \xi) - \frac{dr(\xi)}{r} dr(\xi)]$

By considering $\omega(X) = fT(X)$, (2.14) it can be written as

$$(\nabla_X T)(Y) = -fg(X,Y) + \omega(X)T(Y) \tag{24}$$

since T is closed, ω is obviously closed.

This means that the vector field ξ defined by $g(X, \xi) = T(X)$ is a proper concircular vector field ([4], [6]).

Theorem No 2.3. *If a pseudo projectively flat manifold satisfies* $S(X,Y) = rT(X)T(Y)$ *under the assumption of* $r \neq 0$*, the manifold is a locally product manifold.*

Proof. From (2.18) it follows that

$$\nabla_X \xi = -fX + \omega(X)\xi \tag{25}$$

Let $\xi^{\perp}$ denote the (n-1) dimensional distribution in a pseudo projectively flat manifold orthogonal to ξ.
If X and Y belong to $\xi^{\perp}$, then

$$g(X, \xi) = 0 \tag{26}$$

and

$$g(Y, \xi) = 0 \tag{27}$$

Since $(\nabla_X g)(Y, \xi) = 0$, it follows from (2.19) and (2.21) that

$$-g(\nabla_X Y, \xi) = +g(\nabla_X \xi, Y) = -fg(X, Y)$$

Similarly, we getm

$$-g(\nabla_Y X, \xi) = +g(\nabla_Y \xi, X) = -fg(X, Y).$$

Hence we have

$$g(\nabla_X Y, \xi) = g(\nabla_Y X, \xi) \tag{28}$$

Now $[X, Y] = \nabla_X Y - \nabla_Y X$ and therefore by (2.22) we obtain $g([X, Y], \xi) = 0$.

Hence $[X, Y]$ is orthogonal to ξ ; i.e., $[X, Y]$ belongs to $\xi^{\perp}$.
Thus the distribution is involutively by [1]. Hence from Frobenius' theorem on [1] it follows that $\xi^{\perp}$ is integrable.

This implies the pseudo projectively flat manifold is a locally product manifold.

Theorem No 2.4. *A pseudo projectively flat manifold satisfying $S(X, Y) = rT(X)T(Y)$ under the assumption of $r \neq 0$ can be expressed as a locally warped product $IXe^q M^\star$, where $M^\star$ is an Einstein manifold.*

Proof. If a pseudo projectively flat manifold satisfies $S(X, Y) = rT(X)T(Y)$ under the assumption of $r \neq 0$, then in view of proposition 2.1, theorem 2.2 and theorem 2.3 we obtain

$$g(\nabla_X Y, \xi) = -(\nabla_X T)(Y)$$

for the local vector fields X, Y belonging to $\xi^{\perp}$.
Then from (2.17) the second fundamental form k for each leaf satisfies

$$k(X, Y) = fg(X, Y) = \frac{\alpha}{r} dr(\xi) g(X, Y).$$

Hence we know that each leaf is totally umbilic. Therefore each leaf is a manifold of constant curvature. Hence it must be a locally warped product $IXe^{q}M^{\star}$, where $M^{\star}$ is a Einstein manifold (by [6], [3]).

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