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A CLASSIFICATION OF THE CUBIC S-REGULAR GRAPHS OF ORDERS $\mathbf{12p}$ and $\mathbf{12p^2}$

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Abstract. A graph is called *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, we classify all connected cubic s-regular graphs of order $12p$ and $12p^2$ for each $s \geq 1$ and each prime p.

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1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph X, we denote by $V(X),\ E(X),\ A(X)$ and $\mathrm{Aut}(X)$ the *vertex set*, the *edge set*, the *arc set* and *the full automorphism group* of X, respectively.

An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. A graph X is said to be *s-regular* if $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X. Tutte [15] showed that every finite connected cubic symmetric graph is s-regular for some s, $1 \leq s \leq 5$. A subgroup of $\mathrm{Aut}(X)$ is said to be s-regular if it acts regularly on the set of s-arcs in X.

The classification of cubic symmetric graphs of different orders is given in many papers. Conder and Dobcsanyi [2, 3] classified the cubic s-regular graphs up to order 2048. Cheng and Oxley [1] classified symmetric graphs of order $2p$. The cubic s-regular graphs of order $2p^2, 2p^3, 4p^2, 6p^2, 8p^2, 10p, 10p^2, 14p$ and $16p$ were classified in [4-9, 12 ,13]. In this paper we will classify all connected s-regular graphs of order $12p$ and $12p^2$ where p is a prime.

The following is the main result of this paper.

Theorem 1.1. *Let p be a prime. Let X be a connected cubic symmetric graph.*

(1) *If X has order $12p$, then X is isomorphic to one of the 2-regular graphs F_{24}, F_{60}, F_{84} or the 4-regular graph F_{204}.*

(2) *If X has order $12p^2$, then X is isomorphic to one of the 2-regular graphs F_{48} and F_{108}.*

2. Primary Analysis

Let X be a graph and N a subgroup of $\mathrm{Aut}(X)$. Denote by X_N *the quotient graph* corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in X_N whenever there is an edge between those orbits in X.

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}_K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}_K$ is the composition ph of p and h.

Proposition 2.1. [11, Theorem 9] *Let X be a connected symmetric graph of prime valency and G an s-regular subgroup of $Aut(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of $Aut(X_N)$, where X_N is the quotient graph of X corresponding to the orbits of N. Furthermore, X is a N-regular covering of X_N.*

By [14, Theorems 10.1.5 and 10.1.6], we have the following lemma.

Proposition 2.3. *Let G be a finite group, if G has an abelian sylow p-subgroup then p does not divide $|G' \bigcap Z(G)|$.*

3. Proof of Theorem 1.1

By [2, 3], we have the following Lemma.

Lemma 3.1. *Let p be a prime. Let X be a connected cubic symmetric graph.*

(1) *If X has order $12p$ and $p < 67$, then X is isomorphic to one of the 2-regular graphs F_{24}, F_{60}, F_{84} with orders $24, 60, 84$ respectively or the 4-regular graph F_{204} with order 204.*

(2) *If X has order $12p^2$ and $p < 17$, then X is isomorphic to one of the 2-regular graphs F_{48} and F_{108} with orders 48 and 108 respectively.*

Lemma 3.2. *Let p be a prime. Then there is no cubic symmetric graph of order $12p$ for $p \geq 67$.*

Proof. Suppose that there exist a cubic symmetric graph X of order $12p$ with $p \geq 67$. Set $A := Aut(X)$. Since X is symmetric, by Tutte [15], X is at most 5-regular. Thus $|A| = 2^{s+1}.3^2.p$ for some integer $1 \leq s \leq 5$. Let q be a prime. By [10, pp.12-14], if there exist a simple $\{2, 3, q\}$-group then $q = 5, 7, 13$ or 17. Thus A is solvable. Let N be a minimal normal subgroup of A. Then N is an elementary abelian r-group, where $r = 2, 3, p$. Hence N has more than two orbits on $V(X)$ and by Proposition 2.1, it is semiregular. Therefore $|N| = 2, 4, 3$ or p. In each case, with use of Proposition 2.1, we get a contradiction.

If $|N| = p$, then by Proposition 2.1, the quotient graph X_N of X corresponding to the orbits of N is a connected cubic symmetric graph of order 12, which is impossible by [2]. Thus $O_p(A) = 1$.

If $|N| = 4$, then Proposition 2.1, implies that the quotient graph X_N corresponding to orbits of N has odd number of vertices and valency 3, which is impossible.

If $|N| = 3$, then by Proposition 2.1, the quotient graph X_N is a connected cubic symmetric graph of order $4p$. But by [4, Theorem 6.2], there is no cubic symmetric graph of this order for $p \geq 11$, which is a contradiction. Thus $O_3(A) = 1$.

If $|N| = 2$. Then by Proposition 2.1, A/N is an s-regular subgroup of $Aut(X_N)$. Let T/N be a minimal normal subgroup of A/N. By the same argument as above one may prove that T/N is elementary abelian and $|T/N| =$ 3 or p. Consequently $|T| = 6$ or $2p$. It follows that T has a normal subgroup of order 3 or p which is characteristic in T and hence is normal in A, contradicting $O_3(A) = O_p(A) = 1$.

Lemma 3.3. *Let p be a prime. Then there is no cubic symmetric graph* *of order* $12p^2$ *for* $p \geq 17$.

Proof. Suppose that there exist a cubic symmetric graph X of order $12p^2$ with $p \geq 17$. Set $A := Aut(X)$. Hence $|A| = 2^{s+1}.3^2.p^2$ for some $1 \leq s \leq 5$. Let P be a Sylow p-subgroup of A and $N_A(P)$ the normalizer of P in A. By Sylow's theorem the number of Sylow p- subgroup of A is $np + 1$ and also $np + 1 = |A : N_A(P)|$ where n is integer.

If $np + 1 = 1$, then $P \lhd A$ and by proposition 2.1, the quotient graph X_P of X corresponding to the orbits of P is a connected cubic symmetric graph of order 12, which is impossible by [2]. Thus we may assume that $np+1 > 1$ and so P is not normal in A. Since $|A|$ is divisor of $48.12p^2$, one has $np+1 \mid 2^6.3^2$. It follows that np is one of the following: $287 = 7 \times 41, 191, 143 = 11 \times 13, 95 = 5 \times 19, 71, 47, 35 = 5 \times 7, 31, 23, 17$ or $15 = 3 \times 5$. Since $p \geq 17$, there are three possible cases:
I) $p = 17, 23, 31, 47, 71$ or 191 and $n = 1$,
II) $p = 19$ and $n = 5$ or
III) $p = 41$ and $n = 7$.

Case I: $p = 17, 23, 31, 47, 71$ or 191 and $n = 1$.

Let $H = N_A(P)$. By Considering the right multiplication action of A on the set of right cosets of H in A, we have $|A/H_A| \mid (p + 1)!$, where H_A is the largest normal subgroup of A in H. Thus $p \mid |H_A|$, because A is divisible by p^2. Since P is not normal in A, one has $p^2 \nmid |H_A|$, and by Proposition 2.1, H_A is semiregular. It follows that $|H_A| \mid 12p$. Let L be a Sylow p- subgroup of H_A, Clearly, L is characteristic in H_A and so $L \lhd A$. Set $C := C_A(L)$, where $C_A(L)$ is the centralizer of L in A. Since Sylow p-subgroups of A are abelian, $p^2 \mid |C|$. By Proposition 2.2, $C' \bigcap L = 1$, where C' is the derived subgroup of C. This force $p^2 \nmid |C'|$ and so C' has more than two orbits on $V(X)$. Clearly C' is characteristic in C and so $C' \lhd A$. Then by Proposition 2.1, it is semiregular and hence $|C'| \mid 12p$. Let K/C' be a Sylow p-subgroup of C/C'. Since C/C' is abelian, we have $K/C' \lhd C/C'$. Note that $p^2 \mid |K|$ and $|K| \mid 12p^2$. Then by Sylow's theorem K has a normal subgroup of order p^2, which is characteristic in K, because $p \geq 17$. Therefore the normal Sylow p-subgroup of K is normal in C and also in A, because $K \lhd C$ and $C \lhd A$, contradicting $P \ntriangleleft A$.

Case II: $p = 19$ and $n = 5$.

In this case $|A : N_A(P)| = 2^5.3$. Thus 2^5 is a divisor of $|A|$. It follows that X is at least 4-regular. Let q be a prime. By [10, pp.12-14], if there exist a simple $\{2, 3, q\}$-group then $q = 5, 7, 13$ or 17. Thus A is solvable. Let N be a minimal normal subgroup of A and X_N the quotient graph of X corresponding to the orbits of N. Thus N is an elementary abelian r-group, where $r = 2, 3$ or 19. Hence $|N| = 2, 3$ or 19. If $|N| = 19$, then Proposition 2.1, implies that the quotient graph X_N of X corresponding to the orbits of N is a connected cubic symmetric graph of order 12×19 and if $|N| = 3$, then X_N is a connected cubic symmetric graph of order 4×19^2. By [2] both are impossible. If $|N| = 2$, then by Proposition 2.1, X_N is at least 4-regular graph of order 6×19^2, which is impossible by [5, Theorem 5.3].

Case III: $p = 41$ and $n = 7$.

In this case $|A : N_A(P)| = 2^5.3^2$. Thus 2^5 is a divisor of $|A|$. It follows that X is at least 4-regular. In this case by the argument as in the case II a similar contradiction is obtained.

Proof of Theorem 1.1. It follows by Lemmas 3.1, 3.2 and 3.3.

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