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# THE ORLICZ SPACE OF $\chi^{\pi}$

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Abstract. In this paper we introduced the Orlicz space of $\chi^{\pi}$. We establish some inclusion relations, topological results and we characterize the duals of the Orlicz of $\chi^{\pi}$ sequence spaces.

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## 1. Introduction

A complex sequence, whose $k^{th}$ terms is $x_k$ is denoted by $\{x_k\}$ or simply $x$. Let w be the set of all sequences $x = (x_k)$ and $\phi$ be the set of all finite sequences. Let $\ell_{\infty}, c, c_0$ be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of $\ell_{\infty}, c, c_0$ we have
$\|x\| = \overset{sup}{k} \; |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_{\infty}$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x$ is called entire sequence if $lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by $\Gamma$.$\chi$ was discussed in Kamthan [19]. Matrix transformation involving $\chi$ were characterized by Sridhar [20] and Sirajiudeen [21]. Let $\chi$ be the set of all those sequences $x = (x_k)$ such that $(k! |x_k|)^{1/k} \to 0$ as $k \to \infty$. Then $\chi$ is a metric space with the metric

$$d(x, y) = sup_k \left\{ (k! |x_k - y_k|)^{1/k} : k = 1, 2, 3, \cdots \right\}$$

Orlicz [4] used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary[6], Mursaleen et al.[7], Bektas and Altin[8], Tripathy et al.[9], Rao and subramanian[10] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[11].

Recall([4],[11]) an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called modulus function, introduced by Nakano[18] and further discussed by Ruckle[12] and Maddox[13] and many others.

An Orlicz function $M$ is said to satisfy $\Delta_2-$ condition for all values of $u$, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$. The $\Delta_2-$ condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of $u$ and for $\ell > 1$. Lindenstrauss and Tzafriri[5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for some \, \rho > 0 \right\}. \tag{1}$$

The space $\ell_M$ with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \leq p < \infty$, the space $\ell_M$ coincide with the classical sequence space $\ell_p$. Given a sequence $x = \{x_k\}$ its $n^{th}$ section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ...)$, 1 in the $n^{th}$ place and zero's else where; and $s^{(k)} = (0, 0, ..., 1, -1, 0, ...)$, 1 in the $n^{th}$ place,-1 in the $(n+1)^{th}$place and zero's else where. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k \, (k = 1, 2, 3, \ldots)$ are continuous. We recall the following definitions [see [15]].

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric-space $(X, d)$ is said to have AK (or sectional convergence) if and only if $d\left(x^{(n)}, x\right) \to 0$ as $n \to \infty$.[see[15]] The space is said to have AD (or) be an AD space if $\phi$ is dense in $X$. We note that AK implies AD by [14].

If $X$ is a sequence space, we define

(i)$X'$ = the continuous dual of $X$.

(ii)$X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \, for each \, x \in X\}$;

(iii)$X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \, is \, convergent, \, for each \, x \in X\}$;

(iv)$X^{\gamma} = \left\{a = (a_k) : \sup_n \left|\sum_{k=1}^{n} a_k x_k\right| < \infty, \, for each \, x \in X\right\}$;

(v)Let $X$ be an FK-space$\supset \phi$. Then $X^f = \left\{f(\delta^{(n)}) : f \in X'\right\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the $\alpha-$(or Kö the-T öeplitz)dual of $X$, $\beta-$ (or generalized Kö the-T öeplitz)dual of $X$, $\gamma-$dual of $X$. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta, or \, \gamma$.

**Lemma 1.1.** $(See\,(15, Theorem 7.27))$. *Let $X$ be an FK-space $\supset \phi$. Then*
(i) $X^{\gamma} \subset X^{f}$.
(ii) If $X$ has AK, $X^{\beta} = X^{f}$.
(iii) If $X$ has AD, $X^{\beta} = X^{\gamma}$.

## 2.Definitions and Prelimiaries

Let $w$ denote the set of all complex double sequences $x = (x_k)_{k=1}^{\infty}$ and $M : [0,\infty) \to [0,\infty)$ be an Orlicz function, or a modulus function. Let

$$\chi_M^{\pi} = \left\{ x \in w : lim_{k\to\infty} \left( M \left( \frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right) = 0\, for some\, \rho > 0 \right),$$

$$\Gamma_M^{\pi} = \left\{ x \in w : lim_{k\to\infty} \left( M \left( \frac{|x_k|^{1/k}}{\pi_k^{1/k}\rho} \right) \right) = 0\, for some\, \rho > 0 \right)$$

and $\Lambda_M^{\pi} = \left\{ x \in w : sup_k \left( M \left( \frac{|x_k|^{1/k}}{\pi_k^{1/k}\rho} \right) \right) < \infty\, for some\, \rho > 0 \right)$

The space $\chi_M^{\pi}$ is a metric space with the metric

$$d\,(x,\,y) = inf \left\{ \rho > 0 : sup_k \left( M \left( \frac{(k!\,|x_k - y_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right) \leq 1 \right\} \tag{3}$$

The space $\Gamma_M^{\pi}$ and $\Lambda_M^{\pi}$ is a metric space with the metric

$$d\,(x,\,y) = inf \left\{ \rho > 0 : sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\pi_k^{1/k}\rho} \right) \right) \leq 1 \right\} \tag{4}$$

## 3.Main Results

**Proposition 3.1.** $\chi_M^{\pi} \subset \Gamma_M^{\pi}$, *with the hypothesis that* $M\left(\frac{|x_k|}{\pi_k^{1/k}\rho}\right) \leq M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)$

*Proof.* Let $x \in \chi_M^{\pi}$. Then we have the following implications

$$M\left(\frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \to 0\, as\, k \to \infty \tag{5}$$

But $M\left(\frac{|x_k|}{\pi_k^{1/k}\rho}\right) \leq M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)$, by our assumption, implies that
$\Rightarrow M\left(\frac{|x_k|^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, k \rightarrow \infty$, by (5).
$\Rightarrow x \in \Gamma_M^{\pi}$
$\Rightarrow \chi_M^{\pi} \subset \Gamma_M^{\pi}$. This completes the proof.

**Proposition 3.2.** *$\chi_M^{\pi}$ has AK where $M$ is a modulus function.*

*Proof.* Let $x = \{x_k\} \in \chi_M^{\pi}$, but then $\left\{M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} \in \chi$, and hence

$$sup_{k\geq n+1} M\left(\frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, n \rightarrow \infty. \tag{6}$$

$d\left(x, x^{[n]}\right) = sup_{k\geq n+1}\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, n \rightarrow \infty$, by using (6)
$\Rightarrow x^{[n]} \rightarrow x\, as\, n \rightarrow \infty$, implying that $\chi_M^{\pi}$ has AK. This completes the proof.

**Proposition 3.3.** *$\chi_M^{\pi}$ is solid.*

*Proof.* Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \chi_M^{\pi}$.
$M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \leq M\left(\frac{(k!|y_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)$, because $M$ is non-decreasing.
But $M\left(\frac{(k!|y_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \in \chi$, because $y \in \chi_M^{\pi}$.
That is $M\left(\frac{(k!|y_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, k \rightarrow \infty$ and $M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, k \rightarrow \infty$. Therefore $x = \{x_k\} \in \chi_M^{\pi}$. This completes the proof.

**Proposition 3.4.** *Let $M$ be an Orlicz function which satisfies $\Delta_2-$ condition. Then $\chi \subset \chi_M^{\pi}$.*

*Proof.*Let

$$x \in \chi \tag{7}$$

Then $(k!\,|x_k|)^{1/k} \leq \epsilon$ sufficiently large $k$ and every $\epsilon > 0$. But then by taking $\rho \geq \frac{1}{2}$
$M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \leq M\left(\frac{\epsilon}{\rho}\right) \leq M\,(2\epsilon)$(because $M$ is non-decreasing)

$$M\left(\frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \leq KM\,(\epsilon)\; by \Delta_2 - condition,\; for\, some\, K > 0 \leq \epsilon \tag{8}$$

$\Rightarrow M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \rightarrow 0\, as\, k \rightarrow \infty$ (by defining $M\,(\epsilon) < \frac{\epsilon}{k}$). Hence $x \in \chi_M^{\pi}$. From (7) and since

$$x \in \chi_M^{\pi} \tag{9}$$

we get $x \subset \chi_M^{\pi}$. This completes the proof.

**Proposition 3.5.** *If $M$ is a modulus function, then $\chi_M^{\pi}$ is linear set over the set of complex numbers $\mathbb{C}$*

*Proof.* Let $x, y \in \chi_M^{\pi}$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some $\rho_3$ such that

$$M\left(\frac{(k!\,|\alpha x_k + \beta y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right) \to 0\, as\, k \to \infty. \tag{10}$$

Since $x, y \in \chi_M^{\pi}$, there exists some positive $\rho_1$ and $\rho_2$ such that

$$M\left(\frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \to 0\, as\, k \to \infty\, and\, M\left(\frac{(k!\,|y_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) \to 0\, as\, k \to \infty. \tag{11}$$

Since $M$ is a non decreasing modulus function, we have
$M\left(\frac{(k!|\alpha x_k+\beta y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right) \leq M\left(\frac{(k!|\alpha x_k|)^{1/k}}{\pi_k^{1/k}\rho_3} + \frac{(k!|\beta y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right) \leq M\left(\frac{|\alpha|(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho_3} + \frac{|\beta|(k!|y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right)$
Take $\rho_3$ such that $\frac{1}{\rho_3} = min\left\{\frac{1}{|\alpha|}\frac{1}{\rho_1}, \frac{1}{|\beta|}\frac{1}{\rho_2}\right\}$. Then

$M\left(\frac{(k!|\alpha x_k+\beta y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right) \leq M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho_1} + \frac{(k!|y_k|)^{1/k}}{\pi_k^{1/k}\rho_2}\right) \to 0$ (by(11)).
Hence $M\left(\frac{(k!|\alpha x_k+\beta y_k|)^{1/k}}{\pi_k^{1/k}\rho_3}\right) \to 0\, as\, k \to \infty$. So $(\alpha x + \beta y) \in \chi_M^{\pi}$. Therefore $\chi_M^{\pi}$ is linear. This completes the proof.

**Definition 3.6.** *Let $p = (p_k)$ be any sequence of positive real numbers. Then we define $\chi_M^{\pi}(p) = \left\{x = (x_k) : \left(M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right) \to 0\, as\, k \to \infty\right\}$. Suppose that $p_k$ is a constant for all $k$, then $\chi_M^{\pi}(p) = \chi_M^{\pi}$.*

**Proposition 3.7.** *Let $0 \leq p_k \leq q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded. Then $\chi_M^{\pi}(q) \subset \chi_M^{\pi}(p)$.*

*Proof.* Let

$$x \in \chi_M^{\pi}(q) \tag{12}$$

$$\left(M\left(\frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right)^{q_k} \to 0\, as\, k \to \infty. \tag{13}$$

Let $t_k = \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$. Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k, (t_k \geq 1) \\ 0, (t_k < 1) \end{cases} \quad and\, v_k = \begin{cases} 0\,(t_k \geq 1) \\ t_k, (t_k < 1) \end{cases} \tag{14}$$

$t_k = u_k + v_k; t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k^{\lambda_k} \leq u_k \leq t_k\, and\, v_k^{\lambda_k} \leq v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k},\, then\, t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$

$\left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right)^{q_k} \right)^{\lambda_k} \leq \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{q_k}$

$\Rightarrow \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right)^{q_k} \right)^{p_k/q_k} \leq \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{q_k}$

$\Rightarrow \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{p_k} \leq \left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{q_k}$

But $\left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{q_k} \to 0\, as\, k \to \infty$. (by (13))

Therefore $\left( M\left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{p_k} \to 0\, as\, k \to \infty$. Hence

$$x \in \chi_M^{\pi}(p) \tag{15}$$

From (12) and (15) we get $\chi_M^{\pi}(q) \subset \chi_M^{\pi}(p)$. This completes the proof.

**Proposition 3.8.** (a) *Let* $0 < inf p_k \leq p_k \leq 1$. *Then* $\chi_M^{\pi}(p) \subset \chi_M^{\pi}$
(b) *Let* $1 \leq p_k \leq sup p_k < \infty$. *Then* $\chi_M^{\pi} \subset \chi_M^{\pi}(p)$

*Proof.* (a)Let $x \in \chi_M^{\pi}(p)$

$$\left( M\left( \frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{p_k} \to 0\, as\, k \to \infty. \tag{16}$$

Since $0 < inf p_k \leq p_k \leq 1$

$$\left( M\left( \frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right) \leq \left( M\left( \frac{(k!\,|x_k|)^{1/k}}{\pi_k^{1/k}\rho} \right) \right)^{p_k} \tag{17}$$

From (16) and (17) it follows that $x \in \chi_M^{\pi}$.
Thus $\chi_M^{\pi}(p) \subset \chi_M^{\pi}$. We have thus proven (a). (b) Let $p_k \geq 1$ for each $k$ and

$supp_k < \infty$
Let $x \in \chi_M^\pi$

$$\left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \to 0 \, as \, k \to \infty. \tag{18}$$

Since $1 \leq p_k \leq supp_k < \infty$ we have

$$\left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k} \leq \left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \tag{19}$$

$\left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k} \to 0 \, as \, k \to \infty$. by using (18).
Therefore $x \in \chi_M^\pi (p)$. This completes the proof.

**Proposition 3.9.** *Let $0 < p_k \leq q_k < \infty$ for each $k$. Then $\chi_M^\pi (p) \subseteq \chi_M^\pi (q)$.*

*Proof.* Let $x \in \chi_M^\pi (p)$

$$\left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k} \to 0 \, as \, k \to \infty. \tag{20}$$

This implies that $\left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) \leq 1$ for sufficiently large $k$.
Since $M$ is non-decreasing, we get

$$\left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{q_k} \leq \left( M \left( \frac{(k! \left| x_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{p_k} \tag{21}$$

$\Rightarrow \left( M \left( \frac{(k! |x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right)^{q_k} \to 0 \, as \, k \to \infty$. (by using (20)).
$x \in \chi_M^\pi (q)$
Hence $\chi_M^\pi (p) \subseteq \chi_M^\pi (q)$. This completes the proof.

**Proposition 3.10.** *$\chi_M^\pi (p)$ is a $r-$ convex for all $r$ where $0 \leq r \leq inf p_k$. Moreover if $p_k = p \leq 1 \forall k$, then they are $p-$ convex.*

*Proof.* We shall prove the Theorem for $\chi_M^\pi (p)$.
Let $x \in \chi_M^\pi (p)$ and $r \in (0, lim_{n \to \infty} p_n)$
Then, there exists $k_0$ such that $r \leq p_k \forall k > k_0$.
Now, define

$$g^* (x) = inf \left\{ \rho : M \left( \frac{(k! \left| x_k - y_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right)^r + M \left( \frac{(k! \left| x_k - y_k \right|)^{1/k}}{\pi_k^{1/k} \rho} \right)^{p_n} \right\} \tag{22}$$

Since $r \leq p_k \leq 1 \forall k > k_0$
$g^*$ is subadditive: Further, for $0 \leq |\lambda| \leq 1; |\lambda|^{p_k} \leq |\lambda|^r \, \forall k > k_0$.

$$g^* (\lambda x) \leq |\lambda|^r \cdot g^* (x) \tag{23}$$

Now, for $0 < \delta < 1$,

$$U = \{x : g^* (x) \leq \delta\}, \, which\, is\, an\, absolutely\, r - convex\, set,\, for \tag{24}$$

$$|\lambda|^r + |\mu|^r \leq 1 \, x, y \in U \tag{25}$$

Now
$g^* (\lambda x + \mu y) \leq g^* (\lambda x) + g^* (\mu y)$
$\leq |\lambda|^r g^* (x) + |\mu|^r g^* (y)$
$\leq |\lambda|^r \delta + |\mu|^r \delta$ using (23) and (24)
$\leq (|\lambda|^r + |\mu|^r) \delta$
$\leq 1 \cdot \delta$, by using (25)
$\leq \delta$. If $p_k = p \leq 1 \forall k$ then for $0 < r < 1$,
$U = \{x : g^* (x) \leq \delta\}$ is an absolutely $p-$ convex set.
This can be obtained by a similar analysis and there fore we omit the details. This completes the proof.

**Proposition 3.11.** $(\chi_M^{\pi})^{\beta} = \Lambda$

*Proof: Step 1:* $\chi_M^{\pi} \subset \Gamma_M^{\pi}$ by Proposition 3.1;
$\Rightarrow (\Gamma_M^{\pi})^{\beta} \subset (\chi_M^{\pi})^{\beta}$. But $(\Gamma_M^{\pi})^{\beta} = \Lambda$

$$\Lambda \subset (\chi_M^{\pi})^{\beta} \tag{26}$$

**Step 2:** Let $y \in (\chi_M^{\pi})^{\beta}$ we have $f (x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \chi_M^{\pi}$.
We recall that $s^{(k)}$ has $\frac{\pi_k^{1/k}}{k!}$ in the $k^{th}$ place and zero's elsewhere, with
$x = s^{(k)}, \left( M \left( \frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k} \rho} \right) \right) = \left\{ 0, 0, \cdots M \left( \frac{(1)^{1/k}}{\rho} \right), 0, \cdots \right\}$
which converges to zero. Hence $s^{(k)} \in \chi_M^{\pi}$. Hence $d \left( s^{(k)}, 0 \right) = 1$.
But $|y_k| \leq \|f\| \, d \left( s^{(k)}, 0 \right) < \infty \forall k$. Thus $(y_k)$ is a bounded sequence and hence an analytic sequence. In other words $y \in \Lambda$.

$$(\chi_M^{\pi})^{\beta} \subset \Lambda \tag{27}$$

**Step 3** From (26) and (27) we obtain $(\chi_M^{\pi})^{\beta} = \Lambda$.
This completes the proof.

**Proposition 3.12.** $(\chi_M^{\pi})^{\mu} = \Lambda$ *for* $\mu = \alpha, \beta, \gamma, f$.

*Proof.* Step 1: $\chi_M^{\pi}$ has AK by Proposition 3.2. Hence by Lemma 1.1 (i) we get $(\chi_M^{\pi})^{\beta} = (\chi_M^{\pi})^{f}$ . But $(\chi_M^{\pi})^{\beta} = \Lambda$. Hence

$$(\chi_M^{\pi})^{f} = \Lambda \tag{28}$$

**Step 2:** Since AK $\Rightarrow$ AD. Hence by Lemma 1.1 (iii) we get $(\chi_M^{\pi})^{\beta} = (\chi_M^{\pi})^{\gamma}$. Therefore

$$(\chi_M^{\pi})^{\gamma} = \Lambda \tag{29}$$

**Step 3:** $\chi_M^{\pi}$ is normal by Proposition 3.3 Hence by Proposition 2.7 [16]. we get

$$(\chi_M^{\pi})^{\alpha} = (\chi_M^{\pi})^{\gamma} = \Lambda. \tag{30}$$

From (28), (29) and 930) we have $(\chi_M^{\pi})^{\alpha} = (\chi_M^{\pi})^{\beta} = (\chi_M^{\pi})^{\gamma} = (\chi_M^{\pi})^{f} = \Lambda$.

**Proposition 3.13.** The dual space of $\chi_M^{\pi}$ is $\Lambda$. In other words $(\chi_M^{\pi})^{*} = \Lambda$.

*Proof.* We recall that $s^{(k)}$ has $\frac{\pi_k^{1/k}}{k!}$ has the $k^{th}$ place zero's else where, with $x = s^{(k)}, \left(M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right) = \left\{0, 0, \cdots M\left(\frac{(1)^{1/k}}{\rho}\right), 0, \cdots\right\}$
Hence $s^{(k)} \in \chi_M^{\pi}$. We have $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \chi_M^{\pi}$ and $f \in (\chi_M^{\pi})^{*}$, where $(\chi_M^{\pi})^{*}$ is the dual space of $\chi_M^{\pi}$. Take $x = s^{(k)} \in \chi_M^{\pi}$. Then

$$|y_k| \le \|f\| \, d\left(s^{(k)}, 0\right) < \infty \, for \, all \, k. \tag{31}$$

Thus $(y_k)$ is a bounded sequence and hence an analytic sequence. In other words, $y \in \Lambda$. Therefore $(\chi_M^{\pi})^{*} = \Lambda$. This completes the proof.

**Lemma 3.14.** [15, Theorem 8.6.1] $Y \supset X \Leftrightarrow Y^f \subset X^f$ *where* $X$ *is an AD-space and* $Y$ *an FK-space.*

**Proposition 3.15.** *Let* $Y$ *be any FK-space* $\supset \phi$. *Then* $Y \supset \chi_M^{\pi}$ *if and only if the sequence* $s^{(k)}$ *is weakly analytic.*

*Proof.* The following implications establish the result.
$Y \supset X \Leftrightarrow Y^f \subset (\chi_M^{\pi})^{f}$, since $\chi_M^{\pi}$ has AD and by Lemma 3.14.
$\Leftrightarrow Y^f \subset \Lambda$, since $(\chi_M^{\pi})^{f} = \Lambda$.
$\Leftrightarrow$ $for \, each \, f \in Y'$, the topological dual of $Y$. Therefore $f\left(s^{(k)}\right) \in \Lambda$.
$\Leftrightarrow f\left(s^{(k)}\right)$ is analytic
$\Leftrightarrow s^{(k)}$ is weakly analytic.
This completes the proof.

**Proposition 3.16.** *$\chi^{\pi}_M$ is a complete metric space under the metric* $d(x,y) = sup_k \left\{ M\left(\frac{(k!|x_k-y_k|)^{1/k}}{\pi_k^{1/k}\rho}\right) : k = 1,2,3,\cdots \right\}$ *where* $x = (x_k) \in \chi^{\pi}_M$ *and* $y = (y_k)\, \chi^{\pi}_M$.

*Proof.* Let $\{x^{(n)}\}$ be a Cauchy sequence in $\chi^{\pi}_M$.
Then given any $\epsilon > 0$ there exists a positive integer $N$ depending on $\epsilon$ such that $d\left(x^{(n)}, x^{(m)}\right) < \epsilon$ for all $n \geq N$ and for all $m \geq N$. Hence
$sup_k \left\{ M\left(\frac{\left(k!\left|x_k^{(n)} - x_k^{(m)}\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} < \epsilon$ for all $n \geq N$ and for all $m \geq N$.
Consequently $\left(M\left(\frac{\left(k!\left|x_k^{(n)}\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right)$ is a Cauchy sequence in the metric space $\mathbb{C}$ of complex numbers. But $\mathbb{C}$ is complete. So,

$$\left(M\left(\frac{\left(k!\left|x_k^{(n)}\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right) \to \left(M\left(\frac{(k!\left|x_k\right|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right) \text{ as } n \to \infty.$$

Hence there exists a positive integer $n_0$ such that

$sup_k \left\{ M\left(\frac{\left(k!\left|x_k^{(n)} - x_k\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} < \epsilon$ for all $n \geq n_0$. In particular, we have
$\left\{ M\left(\frac{\left(k!\left|x_k^{(n)} - x_k\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} < \epsilon$. Now
$\left\{ M\left(\frac{(k!|x_k|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} \leq \left\{ M\left(\frac{\left(k!\left|x_k - x_k^{(n_0)}\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} + \left\{ M\left(\frac{\left(k!\left|x_k^{(n_0)}\right|\right)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} < \epsilon + 0\, as\, k \to \infty$. Thus

$$\left\{ M\left(\frac{(k!\left|x_k\right|)^{1/k}}{\pi_k^{1/k}\rho}\right)\right\} < \epsilon\, as\, k \to \infty.$$

That is $x \in \chi^{\pi}_M$. Therefore, $\chi^{\pi}_M$ is a complete metric space. This completes the proof.

## References

[1] A.K.Snyder and A.Wilansky, *Inclusion Theorems and semi conservative FK-spaces*, Rocky Mountain Journal of Math., 2 (1972), 595-603.

[2] K.Chandrasekhara Rao and T.G.Srinivasalu, *The Hahn sequence space-II*, Journal of Faculty of Education, 1(2)(1996), 43-45.

[3] A.K.Snyder, *Consistency theory in semi conservative spaces*, Studia Math., 5(1982), 1-13.

[4] W.Orlicz, *Über Raume* $\left(L^M\right)$ Bull. Int. Acad. Polon. Sci. A, 1936, 93-107.

[5] J.Lindenstrauss and L.Tzafriri, *On Orlicz sequence spaces*, *Israel J. Math.*, **10** (1971), 379-390.

[6] S.D.Parashar and B.Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math., 25(4) (1994), 419-428.

[7] M.Mursaleen,M.A.Khan and Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Math., Vol. XXXII (1999), 145-150.

[8] C.Bektas and Y.Altin, *The sequence space* $\ell_M\left(p,q,s\right)$ *on seminormed spaces*, Indian J. Pure Appl. Math., 34(4) (2003), 529-534.

[9] B.C.Tripathy,M.Etand Y.Altin, *Generalized difference sequence spaces defined by Orlicz function in a locally convex space*, J. Analysis and Applications, 1(3) (2003), 175-192.

[10] K.Chandrasekhara Rao and N.Subramanian, *The Orlicz space of entire sequences*, Int. J. Math. Math. Sci., 68 (2004), 3755-3764.

[11] M.A.Krasnoselskii and Y.B.Rutickii, *Convex functions and Orlicz spaces*, Gorningen, Netherlands, 1961.

[12] W.H.Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, *Canad. J. Math.*, **25**(1973), 973-978.

[13] I.J.Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc, 100(1) (1986), 161-166.

[14] H.I.Brown, *The summability field of a perfect* $\ell-\ell$ *method of summation*, J. Anal. Math., 20 (1967), 281-287.

[15] A.Wilansky, *Summability through Functional Analysis*, North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam, Vol.85 (1984).

[16] P.K.Kamthan and M.Gupta, *Sequence spaces and series. Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker Inc. New York, 65 (1981).

[17] K.Chandrasekhara Rao and N.Subramanian, *Semi analytic spaces*, Science Letters, 26 (2003), 146-149.

[18] Nakano, *Concave modulars*, J. Math. Soc. Japan, 5 (1953), 29-49.

[19] P.K.Kamthan, *Bases in a certain class of Frechet space*, Tamkang J. Math., 1976, 41-49.

[20] S.Sridhar, *A matrix transformation between some sequence spaces*, Acta Ciencia Indica, 5 (1979), 194-197.

[21] S.M.Sirajiudeen, *Matrix transformations of* $c_0(p), \ell_\infty(p), \ell^p$ *and* $\ell$ *into* $\chi$, Indian J. Pure Appl. Math., 12(9) (1981), 1106-1113.

[22] K.Chandrasekhara Rao and N.Subramanian, *The semi Orlicz space of analytic sequences*, Thai Journal of Mathematics, Vol.2 (No.1), (2004), 197-201.

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