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SOME RESULTS FOR ANTI-INVARIANT SUBMANIFOLD IN GENERALIZED SASAKIAN SPACE FORM

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Abstract. In this paper we prove some inequalities, relating R, the scalar curvature and H, the mean curvature vector field of an anti-invariant submanifold in a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Also, we obtain a necessary condition for such anti-invariant submanifolds, to admit a minimal manifold.

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1. Introduction

In [2], B.Y.Chen established in the following lemma the sharp inequality for submanifolds in Riemannian manifolds with constant sectional curvature.

Lemma 1.1. *Let $M^n(n > 2)$ be a submanifold of a Riemannian manifold $R^m(c)$ of constant sectional curvature c. Then*

$$\inf K \geq \frac{1}{2}\left\{R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)c\right\},$$

in which for any $p \in M$

$$(\inf K)(p) := \inf\{K(\pi)|\textit{plane sections } \pi \subset T_pM\}$$

and R is scalar curvature of M. Equality hold if and only if, with respect to suitable orthonormal frame $\{e_1, \dots, e_n, \dots, e_m\}$, the shape operators $A_{e_r}(r = n+1, \dots, e_m)$ of M in $R^m(c)$ take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}, a + b = \mu;$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{21}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r = n+2, \dots, m.$$

In present paper, we are going to establish the similar inequalities for anti-invariant submanifold M with $\dim M > 2$ in generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$, we will do this in two cases:
1) Structural vector field of $\overline{M}(f_1, f_2, f_3)$ be tangent to M,
2) Structural vector field of $\overline{M}(f_1, f_2, f_3)$ be normal to M.
Also, we establish the sharp relationships between the function f of an anti-invariant warped product submanifold $M_1 \times_f M_2$ in generalized Sasakian space form and squared mean curvature and scalar curvature of M.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

A $(2n+1)$-dimensional Riemannian manifold $(\overline{M}, g)$ is said to be *almost contact metric manifold* if there exist on $\overline{M}$ a (1,1)-tensor field ϕ, a vector field ξ(is called the structure vector field) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X,Y on $\overline{M}$. Also, it can be simply proved that in an almost contact metric manifold we have $\phi\xi = 0$, $\eta \circ \phi = 0$ and $\eta(X) = g(X, \xi)$ for any $X \in \tau(\overline{M})$(see for instance [1]).We denote an almost contact metric manifold by $(\overline{M}, \phi, \xi, \eta, g)$.

If in almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$,

$$2\Phi(X, Y) = d\eta(X, Y),$$

where $\Phi(X, Y) = g(Y, \phi X)$, then $(\overline{M}, \phi, \xi, \eta, g)$ is called the *contact metric manifold.* Also, if in an almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$,

$$\Big(\nabla_X \phi\Big)(Y) = \eta(Y)X - g(X, Y)\xi,$$

then $(\overline{M}, \phi, \xi, \eta, g)$ is called the *Sasakian manifold.* It is easy to see that every Sasakian manifold is contact metric manifold.

The submanifold M of almost contact metric manifold $(\overline{M}^{2n+1}, \phi, \xi, \eta, g)$ is called the *anti-invariant* submanifold if for any $p \in M$,

$$\phi_p(T_pM) \subset T_p^{\perp}M.$$

Also, a submanifold M in contact metric manifold $(\overline{M}^{2n+1}, \phi, \xi, \eta, g)$ is called the *Legendrian submanifold* if $\dim M = n$ and for any $p \in M$, $T_pM \subset Ker\eta_p$. It is easy to see that Legendrian submanifolds are anti-invariant.

Let $(\overline{M}, \phi, \xi, \eta, g)$ be an almost contact manifold. If $\pi_p \subset T_p\overline{M}$ is generated by $\{X, \phi X\}$ where $0 \neq X \in T_p\overline{M}$ is normal to ξ_p, is called the ϕ-section of $\overline{M}$ at p and $K(\pi_p)$ is the ϕ-sectional curvature of π_p. If in a Sasakian manifold, there exists $c \in \Re$ such that for any $p \in \overline{M}$ and for any ϕ-section π_p of $\overline{M}$, $K(\pi_p) = c$ then $\overline{M}$ is called the *Sasakian space form*. In [5] it is proved that in a Sasakian space form the curvature tensor is

$$\begin{aligned}\overline{R}(X,Y,)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+\frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi\}.\end{aligned}$$

Almost contact manifolds are said to be *Generalized Sasakian space form* if

$$\begin{aligned}\overline{R}(X,Y,)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi\}, \qquad (1)\end{aligned}$$

where f_1, f_2, f_3 are differentiable functions on $\overline{M}$. We denote this kind of manifolds by $\overline{M}(f_1, f_2, f_3)$.It is clear that every Sasakian space form is a generalized Sasakian space form, but the converse is not necessarily true.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The *warped product* of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

Where $g = g_1 + f^2 g_2$, f is called the *warped function.* (see, for instance [3] and [4]).

Let M^n be a submanifold of $\overline{M}^{2m+1}$ in which h is the second fundamental form of M and $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M respectively. The Gauss equation is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &+g\Big(h(X,W),h(Y,Z)\Big) - g\Big(h(X,Z),h(Y,W)\Big), \end{aligned} \tag{2}$$

for any vector fields X, Y, Z, W on M.

The normal vector field H is called the *mean curvature vector field* of M if for a local orthonormal frame $\{e_1,\cdots,e_n,\cdots,e_{2m+1}\}$ for $\overline{M}$ such that $e_1,\cdots,e_n$ restricted to M, are tangent to M, we have

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i),$$

thus

$$n^2\|H\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i,e_i),h(e_j,e_j)\Big). \tag{3}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^r = g\Big(h(e_i,e_j),e_r\Big),\ i,j\in\{1,\cdots,n\},\ r\in\{n+1,\cdots,2m+1\},$$

the coefficients of the second fundamental form h with respect to $\{e_1,\cdots,e_n,$ $\cdots,e_{2m+1}\}$, and

$$\|h\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i,e_j),h(e_i,e_j)\Big). \tag{4}$$

Now by (3) and (4) the gauss equation (2) can be rewritten as follows:

$$\sum_{1\leq i,j\leq n} \overline{R}_m(e_j,e_i,e_i,e_j) = R - n^2\|H\|^2 + \|h\|^2. \tag{5}$$

in which R is the scalar curvature of M. Let M^n be a Riemannian manfold and $\{e_1,\cdots,e_n\}$ be a local orthonormal frame of M. For a differentiable function f on M, the *Laplacian* Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n}\Big((\nabla_{e_j}e_j)f - e_j(e_jf)\Big). \tag{6}$$

We recall the following result of B.Y.Chen for later use.
Lemma 2.1.*([2]) Let $n \geq 2$ and $a_1, \cdots, a_n$ and b are real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big).$$

Then $2a_1a_2 \geq b$,with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

3.Submanifolds normal to structure vector field in generalized Sasakian space form

In this section, we are going to establish the inequalities for anti-invariant submanifold M with $\dim M > 2$ in generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ when Structural vector field of $\overline{M}(f_1, f_2, f_3)$ is normal to M.

Theorem 3.1.*Let $M_1 \times_f M_2$ be an anti-invariant submanifold in generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$ such that structure vector field of $\overline{M}^{2m+1}(f_1, f_2, f_3)$ be normal to $M_1 \times_f M_2$ and $\dim M_i = n_i (i = 1, 2)$ and $n_1 + n_2 = n > 2$ then*
a)

$$\begin{aligned} 2n_2\frac{\Delta f}{f} \quad \leq \quad & \Big(\frac{n(n-1)}{2} - n_1n_2\Big)\Big(\Big(\frac{n^2(n-2)}{n-1}\Big)\|H\|^2 + (n+1)(n-2)f_1\Big) \\ & + \Big(1 - \frac{n(n-1)}{2} + n_1n_2\Big)R \end{aligned} \tag{7}$$

b)

$$\frac{2\Delta f}{n_1 f} \geq R - (n-2)\Big(\frac{n^2}{n-1}\|H\|^2 + (n+1)f_1\Big), \tag{8}$$

in which H, R, Δ are mean curvature vector, scalar curvature and Laplacian operator of M,respectively.

Proof. **a)** In the warped product manifold $M_1 \times_f M_2$, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z,$$

for any vector fields X and Z tangent to M_1 and M_2, respectively (see [6]). If X and Z are unit vector fields, then the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\Big((\nabla_X X)f - X^2 f\Big). \tag{9}$$

We choose a local orthonomal fram $\{e_1, \ldots, e_{2m+1}\}$ for $\overline{M}$ such that $e_1, \ldots, e_{n_1}$ are tangent to M_1 and $e_{n_1+1}, \ldots, e_n$ are tangent to M_2 and e_{n+1} is parallel to H.

By using (6) and (9), we get

$$\frac{\Delta f}{f} = \sum_{i=1}^{n_1} K(e_i, e_j), \tag{10}$$

for any $j \in \{n_1 + 1, \ldots, n\}$. With simple computation on last equality we get

$$2n_2 \frac{\Delta f}{f} = R - \sum_{1 \le i \ne j \le n_1} K(e_j, e_i) - \sum_{n_1+1 \le i \ne j \le n} K(e_j, e_i). \tag{11}$$

From (3), with respect to this frame we have

$$n^2 \|H\|^2 \;=\; \sum_{i,j=1}^{n} g\Big(h(e_i, e_i), h(e_j, e_j)\Big) = \Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2, \tag{12}$$

from (1) and (5), we have

$$n^2 \|H\|^2 = R + \|h\|^2 - n(n-1)f_1. \tag{13}$$

We set

$$\delta := R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1. \tag{14}$$

Therefore (13), reduces to $n^2\|H\|^2 = (n-1)\Big(\delta + \|h\|^2 - 2f_1\Big)$.
From (4), (12) and above equality, we have

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 \;=\; (n-1)\Big(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1 \le i \ne j \le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - 2f_1\Big).$$

We set

$$b := \delta + \sum_{1 \le i \ne j \le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - 2f_1.$$

For $\alpha \neq \beta \in \{1, \ldots, n\}$, we let $a_1 = h^{n+1}_{\alpha\alpha}$ and $a_2 = h^{n+1}_{\beta\beta}$, then from Lemma.2.1, we have $a_1 a_2 \geq \frac{b}{2}$. Therefore

$$h^{n+1}_{\alpha\alpha} h^{n+1}_{\beta\beta} \geq \frac{\delta}{2} - f_1 + \sum_{1\leq i<j\leq n} (h^{n+1}_{ij})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h^r_{ii})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i<j\leq n} (h^r_{ij})^2. \quad (15)$$

On the other hand from Gauss equation (2) and (1), we have

$$f_1 = K(e_\beta, e_\alpha) - \sum_{r=n+1}^{2m+1} h^r_{\alpha\alpha} h^r_{\beta\beta} + \sum_{r=n+1}^{2m+1} (h^r_{\alpha\beta})^2,$$

therefore

$$f_1 + h^{n+1}_{\alpha\alpha} h^{n+1}_{\beta\beta} = K(e_\beta, e_\alpha) - \sum_{r=n+2}^{2m+1} h^r_{\alpha\alpha} h^r_{\beta\beta} + \sum_{r=n+1}^{2m+1} (h^r_{\alpha\beta})^2.$$

Then from (15) and the above equality, we have

$$K(e_\beta, e_\alpha) - \sum_{r=n+2}^{2m+1} h^r_{\alpha\alpha} h^r_{\beta\beta} + \sum_{r=n+1}^{2m+1} (h^r_{\alpha\beta})^2 \geq \frac{\delta}{2} + \sum_{1\leq i<j\leq n} (h^{n+1}_{ij})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h^r_{ii})^2 + \sum_{r=n+2}^{2m+1} \sum_{1\leq i<j\leq n} (h^r_{ij})^2.$$

After simplification we get

$$K(e_\beta, e_\alpha) - \sum_{r=n+2}^{2m+1} h^r_{\alpha\alpha} h^r_{\beta\beta} \geq \frac{\delta}{2} + \sum_{\substack{1\leq i<j\leq n \\ i\neq\alpha \vee j\neq\beta}} (h^{n+1}_{ij})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h^r_{ii})^2 + \sum_{r=n+2}^{2m+1} \sum_{\substack{1\leq i<j\leq n \\ i\neq\alpha \vee j\neq\beta}} (h^r_{ij})^2. \quad (16)$$

Since

$$\sum_{r=n+2}^{2m+1} h^r_{\alpha\alpha} h^r_{\beta\beta} = \frac{1}{2} \sum_{r=n+2}^{2m+1} (h^r_{\alpha\alpha} + h^r_{\beta\beta})^2 - \frac{1}{2} \sum_{r=n+2}^{2m+1} (h^r_{\alpha\alpha})^2 - \frac{1}{2} \sum_{r=n+2}^{2m+1} (h^r_{\beta\beta})^2,$$

therefore from (16) we get

$$K(e_\beta, e_\alpha) \geq \frac{\delta}{2} + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h^r_{\alpha\alpha} + h^r_{\beta\beta})^2 + \sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq\beta}}(h^{n+1}_{ij})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=1\\ i\neq\alpha,\beta}}^{n}(h^r_{ii})^2 + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq\beta}}(h^r_{ij})^2 \geq \frac{\delta}{2}. \tag{17}$$

From (11) and the above inequality we have

$$2n_2\frac{\Delta f}{f} \leq R - \Big(n_1(n_1-1) + n_2(n_2-1)\Big)\frac{\delta}{2} = R - \Big(\frac{n(n-1)}{2} - n_1n_2\Big)\delta.$$

By substituting δ in the above inequality, we get (7)

b) By (10) and (17), for any $\beta \in \{n_1+1, \ldots, n\}$, we have

$$\frac{\Delta f}{f} = \sum_{\alpha=1}^{n_1} K(e_\alpha, e_\beta) \geq \sum_{\alpha=1}^{n_1}\frac{\delta}{2}$$

in which δ is defined in (14). Therefore $\frac{\Delta f}{f} \geq n_1\frac{\delta}{2}$. By substituting δ in the above inequality, we get (8).

Corollary 3.2. *A necessary condition for an anti-invariant warped product submanifold $M_1 \times_f M_2$ in generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ such that structure vector field of $\overline{M}(f_1, f_2, f_3)$ be normal to $M_1 \times_f M_2$, to be minimal is*

a)

$$2n_2\frac{\Delta f}{f} \leq \Big(\frac{n(n-1)}{2} - n_1n_2\Big)(n^2 - n - 2)f_1 + \Big(1 - \frac{n(n-1)}{2} + n_1n_2\Big)R$$

b) $\frac{2\Delta f}{n_1 f} \geq R - (n-2)(n+1)f_1$, *in which* $\dim M_i = n_i (i = 1, 2)$, $n_1 + n_2 = n > 2$ *and R and Δ are the scalar curvature and Laplacian operator of M, respectively.*

In Theorem 3.1 the anti-invariant submanifold, was a warped product manifold. In the next theorem we remove this assumption and indeed we generalize the Chen's inequality, Lemma 1.1, for anti-invariant submanifolds $M^n (n > 2)$ of generalized Sasakian space forms.

Theorem 3.3. *If $M^n (n > 2)$ be an anti-invariant submanifold in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$ such that structure vector field of $\overline{M}^{2m+1}(f_1, f_2, f_3)$*

be normal to M then

$$\inf\mathcal{K} \geq \frac{1}{2}\left\{R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1\right\}, \tag{18}$$

in which

$$\mathcal{K} = \{K(\pi)|\ plane\ section\ fields\ \pi \subset TM\}$$

and R is the scalar curvature of M. Equality holds if and only if, with respect to an orthonormal frame $\{e_1, \ldots, e_n, \ldots, e_{2m+1}\}$, the shape operators $A_{e_r}(r = n+1, \ldots, 2m+1)$ of M in $\overline{M}^{2m+1}(f_1, f_2, f_3)$ take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} h_{11}^{n+1} & h_{12}^{n+1} & 0 & \ldots & 0 \\ h_{21}^{n+1} & h_{22}^{n+1} & 0 & \ldots & 0 \\ 0 & 0 & h_{33}^{n+1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & h_{nn}^{n+1} \end{pmatrix}, \tag{19}$$

in which $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}$ and

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \ldots & 0 \\ h_{21}^r & -h_{11}^r & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r = n+2, \ldots, 2m+1. \tag{20}$$

Proof. Let $\pi \subset TM$ be a 2-plane field. We choose a local orthonormal frame $\{e_1, \ldots, e_{2m+1}\}$ for $\overline{M}$ such that $e_1, \ldots, e_n$ are tangent to M, π generated by $\{e_1, e_2\}$ and e_{n+1} is parallel to H. With a similar computation as in theorem 3.1, we get $K(e_1, e_2) \geq \frac{\delta}{2}$, in which δ is defined in (14). Therefore we get (18).

If the equality sign of (18) holds, then for a local orthonormal frame, (17) becomes equality. with recursive computation, inequality (15) also change to equality. Therefore by (17)

$$h_{11}^r + h_{22}^r = 0 \quad n+2 \leq r \leq 2m+1,$$

$$h_{ii}^r = 0 \quad n+2 \leq r \leq 2m+1,\ 3 \leq i \leq n,$$

$$h_{1j}^r = h_{j1}^r = h_{2j}^r = h_{j2}^r = 0 \quad n+1 \leq r \leq 2m+1,\ 3 \leq j \leq n,$$

$$h_{ij}^r = 0 \quad n+1 \leq r \leq 2m+1,\ 3 \leq i \neq j \leq n,$$

from lemma 2.1 and (15), we have $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}$. Therefore we get (19) and (20). The converse statement is straightforward.

Corollary 3.4. *A necessary condition for anti-invariant submanifold $M^n (n > 2)$ in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$ such that structure vector field of $\overline{M}^{2m+1}(f_1, f_2, f_3)$ be normal to M, to be minimal, is $\inf \mathcal{K} \geq \frac{1}{2}\{R - (n+1)(n-2)f_1\}$, in which $\mathcal{K} := \{K(\pi)|$ plane section fields $\pi \subset TM\}$ and R is scalar curvature of M. Equality holds if and only if, with respect to an orthonormal frame $\{e_1, \ldots, e_n, \ldots, e_{2m+1}\}$, the shape operators $A_{e_r}(r = n+1, \ldots, e_{2m+1})$ of M in $\overline{M}^{2m+1}(f_1, f_2, f_3)$ take the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} h_{11}^{n+1} & h_{12}^{n+1} & 0 & \ldots & 0 \\ h_{21}^{n+1} & h_{22}^{n+1} & 0 & \ldots & 0 \\ 0 & 0 & h_{33}^{n+1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & h_{nn}^{n+1} \end{pmatrix},$$

in which $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}$ *and*

$$A_{e_r} = \begin{pmatrix} h_{11}^{r} & h_{12}^{r} & 0 & \ldots & 0 \\ h_{21}^{r} & -h_{11}^{r} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r = n+2, \ldots, 2m+1.$$

Remark 3.5. *Since the structure vector field in a generalized Sasakian space form is normal to Legendrian submanifolds and Legendrian submanifolds are anti-invariant, therefore Theorems (3.1) and (3.3) and corollaries (3.2) and (3.4) are satisfied when submanifolds in generalized Sasakian space form are a Legendrian.*

4. Submanifolds tangent to structure vector field in a generalized Sasakian space form

In this section, we are going to establish the inequalities for anti-invariant submanifold M with $\dim M > 2$ in generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ when Structural vector field of $\overline{M}(f_1, f_2, f_3)$ be tangent to M.

Theorem 4.1. *If $M_1 \times_f M_2$ is an anti-invariant warped product submanifold in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$ such that $\dim M_i = n_i (i = 1, 2)$ and $n_1 + n_2 = n > 2$, and the structure vector field of $\overline{M}(f_1, f_2, f_3)$ is tangent to M_2*

then

$$\frac{2\Delta f}{n_1 f} \geq R - (n-2)\Big(\frac{n^2}{n-1}\|H\|^2 + (n+1)f_1 - 2f_3\Big), \tag{21}$$

in which H, R and Δ are mean curvature vector, scalar curvature and Laplacian operator of M, respectively.

Proof. We choose local orthonormal frame $\{e_1, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_{n_1}$ are tangent to M_1, $e_{n_1}, \dots, e_n$ are tangent to M_2, $e_n = \xi$ and e_{n+1} is parallel to H.

From Gauss equation, similar to the proof of Theorem 3.1, we have

$$n^2\|H\|^2 = R - n(n-1)f_1 + 2(n-1)f_3 + \|h\|^2, \tag{22}$$

We set

$$\delta := R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 + 2(n-2)f_3, \tag{23}$$

then from (22) we have $n^2\|H\|^2 = (n-1)(\|h\|^2 + \delta - 2f_1 + 2f_3)$, and substituting (3) and (4) in the above equality, we get

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \delta - 2f_1 + 2f_3\Big).$$

Now we set $b := \delta - 2f_1 + 2f_3 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2$.
For $\alpha \in \{1, \dots, n-1\}$, we let $a_1 = h_{\alpha\alpha}^{n+1}$ and $a_2 = h_{nn}^{n+1}$, then from Lemma.2.1, we have $a_1 a_2 \geq \frac{b}{2}$. Therefore

$$h_{\alpha\alpha}^{n+1}h_{nn}^{n+1} \geq \frac{\delta}{2} - (f_1 - f_3) + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2.$$

Therefore

$$\begin{aligned} h_{\alpha\alpha}^{n+1}h_{nn}^{n+1} + (f_1 - f_3) \quad &\geq \quad \frac{\delta}{2} + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2 \\ &\quad + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2. \end{aligned} \tag{24}$$

On the other hand from (1) and the Gauss equation, for $\alpha \in \{1, \dots, n-1\}$ we have

$$f_1 - f_3 = K(e_\alpha, e_n) - \sum_{r=n+1}^{2m+1} h_{\alpha\alpha}^r h_{nn}^r + \sum_{r=n+1}^{2m+1}(h_{\alpha n}^r)^2.$$

By comparing the above equality with (24), we obtain

$$K(e_\alpha,e_n)-\sum_{r=n+2}^{2m+1}h^r_{\alpha\alpha}h^r_{nn}+\sum_{r=n+1}^{2m+1}(h^r_{\alpha n})^2$$

$$\geq\frac{\delta}{2}+\sum_{1\leq i<j\leq n}(h^{n+1}_{ij})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h^r_{ii})^2+\sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h^r_{ij})^2.$$

After simplification, we have

$$\begin{aligned} K(e_\alpha,e_n)-\sum_{r=n+2}^{2m+1}h^r_{\alpha\alpha}h^r_{nn} \ \geq\ & \frac{\delta}{2}+\sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq n}}(h^{n+1}_{ij})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h^r_{ii})^2 \\ & +\sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq n}}(h^r_{ij})^2. \end{aligned} \quad (25)$$

Since

$$\sum_{r=n+2}^{2m+1}h^r_{\alpha\alpha}h^r_{nn}=\frac{1}{2}\sum_{r=n+2}^{2m+1}(h^r_{\alpha\alpha}+h^r_{nn})^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}(h^r_{\alpha\alpha})^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}(h^r_{nn})^2,$$

therefore from (25) we get

$$\begin{aligned} K(e_\alpha,e_n) \ \geq\ & \frac{\delta}{2}+\sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq n}}(h^{n+1}_{ij})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=1\\ i\neq\alpha,n}}^{n}(h^r_{ii})^2 \\ & +\sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n\\ i\neq\alpha\vee j\neq n}}(h^r_{ij})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}(h^r_{\alpha\alpha}+h^r_{nn})^2. \end{aligned}$$

$$\Rightarrow K(e_\alpha,e_n)\geq\frac{\delta}{2}.$$

Therefore

$$2\sum_{\alpha=1}^{n_1}K(e_\alpha,e_n)\geq n_1\delta\overset{(10),(23)}{\Longrightarrow}2\frac{\Delta f}{n_1 f}\geq R-\frac{n^2(n-2)}{n-1}\|H\|^2-(n+1)(n-2)f_1+2(n-2)f_3.$$

Corollary 4.2. *A necessary condition for anti-invariant warped product submanifold* $M_1\times_f M_2$*, in a generalized Sasakian space form* $\overline{M}(f_1,f_2,f_3)$ *such that* $\dim M_i=$

$n_i (i = 1, 2)$ and $n_1 + n_2 = n > 2$ and the structure vector field of $\overline{M}(f_1, f_2, f_3)$ is tangent to M_2, to be minimal is

$$\frac{2\Delta f}{n_1 f} \geq R - (n-2)\Big((n+1)f_1 - 2f_3\Big),$$

in which R is the scalar curvature of M.

In Theorem 4.1 the anti-invariant submanifold, was a warped product manifold. In the next theorem we remove this assumption and indeed we generalize the Chen's inequality, Lemma 1.1, for anti-invariant submanifolds $M^n(n > 2)$ of generalized Sasakian space forms.

Theorem 4.3.*Let $M^n(n > 2)$ be an anti-invariant submanifold in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$ such that structure vector field of $\overline{M}(f_1, f_2, f_3)$ be tangent to M. Then*

$$\inf \mathcal{K} \geq \inf \left\{ \mathcal{A} + (n-2)f_3, \mathcal{A} + (n-1)f_3, \mathcal{A} + \frac{P}{2} f_3 - 2|f_3| \right\}, \tag{26}$$

where

$$\mathcal{K} := \{K(\pi)| \text{ plane section fields } \pi \subset TM\},$$

$$\mathcal{A} := \frac{1}{2}\left\{ R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 \right\},$$

$$P := \sum_{1 \leq i \neq j \leq n} \left(\Big(\eta(e_i)\Big)^2 + \Big(\eta(e_j)\Big)^2 \right),$$

in which $\{e_1, \ldots, e_{2m+1}\}$ is an orthonormal frame such that $e_1, \ldots, e_n$ are tangent to M and for any $i \in \{1, \ldots, n\}$, $\xi \neq e_i$ and R is the scalar curvature of M.

Proof. Let π be a 2-plane field in TM.

1) If ξ is tangent to π then:

we choose locale orthonormal frame $\{e_1, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1} is parallel to H, $e_1 = \xi$ and π generated by $\{e_1, e_2\}$. Therefore From Gauss equation, similar to the proof of theorem 4.1, we have

$$n^2\|H\|^2 = R - n(n-1)f_1 + 2(n-1)f_3 + \|h\|^2, \tag{27}$$

We defined δ as in (23)

$$\delta := R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 + 2(n-2)f_3,$$

then from (27) we have

$$n^2\|H\|^2 = (n-1)\Big(\|h\|^2 + \delta - 2f_1 + 2f_3\Big),$$

and substituting (3) and (4) in the above equality, we get

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta - 2f_1 + 2f_3\Big). \tag{28}$$

Now set

$$b := \delta - 2f_1 + 2f_3 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

From Lemma.2.1, we have

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{\delta}{2} - (f_1 - f_3) + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^{r})^2. \tag{29}$$

Therefore

$$h_{11}^{n+1}h_{22}^{n+1} + (f_1 - f_3) \geq \frac{\delta}{2} + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^{r})^2. \tag{30}$$

On the other hand from (1) and the Gauss equation, we have

$$f_1 - f_3 = K(e_1, e_2) - \sum_{r=n+1}^{2m+1} h_{11}^{r}h_{22}^{r} + \sum_{r=n+1}^{2m+1}(h_{12}^{r})^2.$$

By comparing the above equality with (30), we obtain

$$K(e_1, e_2) - \sum_{r=n+2}^{2m+1} h_{11}^{r}h_{22}^{r} + \sum_{r=n+1}^{2m+1}(h_{12}^{r})^2$$
$$\geq \frac{\delta}{2} + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^{r})^2.$$

After simplification, we have

$$\begin{aligned} K(e_1,e_2) \;\geq\; & \frac{\delta}{2} + \sum_{\substack{1\leq i<j\leq n \\ i\neq 1 \vee j\neq 2}} (h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\substack{i=1 \\ i\neq 1,2}}^{n} (h_{ii}^r)^2 \\ & + \sum_{r=n+2}^{2m+1}\sum_{\substack{1\leq i<j\leq n \\ i\neq 1 \vee j\neq 2}} (h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2. \end{aligned}$$

$$\Rightarrow K(e_1,e_2) \geq \frac{\delta}{2}.$$

By substituting δ in the above inequality, we have

$$K(e_1,e_2) \geq \mathcal{A} + (n-2)f_3. \tag{31}$$

2) If ξ is normal to π then:
we choose a locale orthonormal frame $\{e_1,\ldots,e_{2m+1}\}$ such that $e_1,\ldots,e_n$ are tangent to M and e_{n+1} is parallel to H, $e_n = \xi$ and π generated by $\{e_1,e_2\}$. Therefore from Gauss equation, similar to the proof of Theorem 4.1, we have (27). Therefore

$$n^2\|H\|^2 = (n-1)\Big(\|h\|^2 + \delta - 2f_1 + 2f_3\Big),$$

in which δ is defined in (23). By substituting (3) and (4) in the above equality, we get (28). From Lemma.2.1 we have (29) and then

$$\begin{aligned} h_{11}^{n+1}h_{22}^{n+1} + f_1 \;\geq\; & \frac{\delta}{2} + f_3 + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 \\ & + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2. \end{aligned} \tag{32}$$

On the other hand from (1) and the Gauss equation, we have

$$f_1 = K(e_1,e_2) - \sum_{r=n+1}^{2m+1} h_{11}^r h_{22}^r + \sum_{r=n+1}^{2m+1}(h_{12}^r)^2.$$

By comparing the above equality and (32), we obtain

$$\begin{aligned} & K(e_1,e_2) - \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r + \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\ & \geq \frac{\delta}{2} + f_3 + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2. \end{aligned}$$

By simple computation, we have

$$K(e_1, e_2) \geq \frac{\delta}{2} + f_3.$$

By substituting δ in the above inequality, we get

$$K(e_1, e_2) \geq \mathcal{A} + (n-1)f_3. \tag{33}$$

3) If ξ be neither tangent or normal to π then:
we choose locale orthonormal frame $\{e_1, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_n$ are tangent to M and e_{n+1} is parallel to H and π generated by $\{e_1, e_2\}$ and for any $i \in \{1, \dots, n\}$, $\xi \neq e_i$. Therefore from Gauss equation, similar to the proof of theorem 4.1, we have

$$n^2\|H\|^2 = R + \|h\|^2 - n(n-1)f_1 + Pf_3, \tag{34}$$

in which

$$P := \sum_{1\leq i\neq j\leq n} \Big(\Big(\eta(e_i)\Big)^2 + \Big(\eta(e_j)\Big)^2\Big).$$

We set

$$\delta := R - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n+1)(n-2)f_1 + Pf_3, \tag{35}$$

then from (34) we have

$$n^2\|H\|^2 = (n-1)(\|h\|^2 + \delta - 2f_1),$$

and substituting (3) and (4) in the above equality, we get

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta - 2f_1\Big).$$

Now set

$$b := \delta - 2f_1 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

From Lemma.2.1, we have

$$\begin{aligned} h_{11}^{n+1}h_{22}^{n+1} \quad &\geq \quad \frac{\delta}{2} - f_1 + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^2 + \sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^{r})^2. \end{aligned}$$

Therefore

$$h_{11}^{n+1}h_{22}^{n+1}+f_1\geq\frac{\delta}{2}+\sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2+\sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2. \quad (36)$$

On the other hand, from gauss equation we have

$$f_1=K(e_1,e_2)+\left(\left(\eta(e_1)\right)^2+\left(\eta(e_2)\right)^2\right)f_3-\sum_{r=n+1}^{2m+1}h_{11}^rh_{22}^r+\sum_{r=n+1}^{2m+1}(h_{12}^r)^2.$$

Then (36) becomes

$$\begin{aligned}&K(e_1,e_2)+\left(\left(\eta(e_1)\right)^2+\left(\eta(e_2)\right)^2\right)f_3-\sum_{r=n+2}^{2m+1}h_{11}^rh_{22}^r+\sum_{r=n+1}^{2m+1}(h_{12}^r)^2\\&\geq\frac{\delta}{2}+\sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h_{ii}^r)^2+\sum_{r=n+2}^{2m+1}\sum_{1\leq i<j\leq n}^{n}(h_{ij}^r)^2.\end{aligned}$$

After simplification we have

$$K(e_1,e_2)\geq\frac{\delta}{2}-\left(\left(\eta(e_1)\right)^2+\left(\eta(e_2)\right)^2\right)f_3. \quad (37)$$

On the other hand, for $i\in\{1,2\}$

$$\begin{aligned}0<g(\xi-e_i,\xi-e_i)&=g(\xi,\xi)-2g(\xi,e_i)+g(e_i,e_i)\\&\Rightarrow g(\xi,e_i)<1\\&\Rightarrow 0\leq\left(g(\xi,e_i)\right)^2<1.\\&\Rightarrow 0\leq\left(\eta(e_1)\right)^2+\left(\eta(e_2)\right)^2<2.\end{aligned}$$

Therefore (37) can be rewriten as

$$\begin{aligned}K(e_1,e_2)&\geq\frac{\delta}{2}-2|f_3|.\\&\geq\frac{1}{2}\left\{R-\frac{n^2(n-2)}{n-1}\|H\|^2-(n+1)(n-2)f_1+Pf_3\right\}-2|f_3|\end{aligned}$$

From (31) and (33) and the above inequality, we get (26).

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