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ON WEAK CONCIRCULAR SYMMETRIES OF KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study weakly concircular symmetric and weakly concircular Ricci symmetric Kenmotsu manifolds.

Keywords and phrases: weakly symmetric manifold, weakly concircular symmetric manifold, weakly Ricci symmetric manifold, concircular Ricci tensor, weakly concircular Ricci symmetric manifold, Kenmotsu manifold.

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1. Introduction

The notion of weakly symmetric manifolds were introduced by Tamássy and Binh [10]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ &+ H(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ &+ E(V)R(Y,Z,U,X)\end{aligned} \tag{1}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 De and Bandyopadhyay [3] studied a $(WS)_n$ and proved that in such a manifold the associated 1-forms $B = H$ and $D = E$. Hence (1) reduces to the following:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ &+ B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ &+ D(V)R(Y,Z,U,X).\end{aligned} \tag{2}$$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [13]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [13]

$$\tilde{C}(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{n(n-1)}\big[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big], \quad (3)$$

where r is the scalar curvature of the manifold.

Recently Shaikh and Hui [8] introduced the notion of weakly concircular symmetric manifolds. A Riemannian manifold $(M^n,g)(n>2)$ is called weakly concircular symmetric manifold if its concircular curvature tensor $\tilde{C}$ of type (0,4) is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X\tilde{C})(Y,Z,U,V) &= A(X)\tilde{C}(Y,Z,U,V) + B(Y)\tilde{C}(X,Z,U,V) \\ &+ H(Z)\tilde{C}(Y,X,U,V) + D(U)\tilde{C}(Y,Z,X,V) \\ &+ E(V)\tilde{C}(Y,Z,U,X)\end{aligned} \quad (4)$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and an n-dimensional manifold of this kind is denoted by $(W\tilde{C}S)_n$. Also it is shown that [8], in a $(W\tilde{C}S)_n$ the associated 1-forms $B=H$ and $D=E$, and hence the defining condition (4) of a $(W\tilde{C}S)_n$ reduces to the following form:

$$\begin{aligned}(\nabla_X\tilde{C})(Y,Z,U,V) &= A(X)\tilde{C}(Y,Z,U,V) + B(Y)\tilde{C}(X,Z,U,V) \\ &+ B(Z)\tilde{C}(Y,X,U,V) + D(U)\tilde{C}(Y,Z,X,V) \\ &+ D(V)\tilde{C}(Y,Z,U,X),\end{aligned} \quad (5)$$

where A, B and D are 1-forms (not simultaneously zero).

Again Tamássy and Binh [11] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold (M^n,g) $(n>2)$ is called weakly Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(Y,X), \quad (6)$$

where A, B and D are three non-zero 1-forms, called the associated 1-forms of the manifold, and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y,V) = \sum_{i=1}^{n} \tilde{C}(Y, e_i, e_i, V), \tag{7}$$

then from (3), we get

$$P(Y,V) = S(Y,V) - \frac{r}{n} g(Y,V). \tag{8}$$

The tensor P is called the concircular Ricci symmetric tensor [4], which is a symmetric tensor of type (0,2). In [4] De and Ghosh introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly concircular Ricci symmetric manifold [4] if its concircular Ricci tensor P of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = A(X)P(Y,Z) + B(Y)P(X,Z) + D(Z)P(Y,X), \tag{9}$$

where A, B and D are three 1-forms (not simultaneously zero).

In [12] Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He proved that they could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $R \times_f C^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [5] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors.

As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [6] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0,0), $(\alpha, 0)$ and $(0, \beta)$ are called the cosympletic, α-Sasakian and β-Kenmotsu manifolds respectively, α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively.

Tamássy and Binh [11] studied weakly symmetric and weakly Ricci symmetric Sasakian manifolds and proved that in such a manifold the sum of the associated 1-forms vanishes everywhere. Again Özgür [7] studied weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds and proved that in such a manifold the sum of the associated 1-forms is zero everywhere and hence such a manifold does not exist unless the sum of the associated 1-forms is everywhere zero. In this connection Shaikh and Hui [9] studied weakly symmetric and weakly Ricci symmetric trans-Sasakian manifolds and proved that the sum of the associated 1-forms of a weakly symmetric and also of a weakly Ricci symmetric trans-Sasakian manifold of non-vanishing ξ-sectional curvature are non-zero everywhere and hence such two structure exists, provided that the manifold is of non-vanishing ξ-sectional curvature.

The object of the present paper is to study *weakly concircular symmetric and weakly concircular Ricci symmetric Kenmotsu manifolds.* Section 2 deals with preliminaries of Kenmotsu manifolds. In section 3 of the paper we have obtained all the 1-forms of a weakly concircular symmetric Kenmotsu manifold and hence such a structure exist, provided that $r \neq -n(n-1)$. Again in section 4 we study weakly concircular Ricci symmetric Kenmotsu manifolds and obtained all the 1-forms of a weakly concircular Ricci symmetric Kenmotsu manifold and consequently such a structure exist, provided that $r \neq -n(n-1)$. Also it is proved that the sum of the associated 1-forms of a weakly concircular Ricci symmetric Kenmotsu manifold is non-vanishing, provided that $r \neq -n(n-1)$.

2. Kenmotsu manifolds

A smooth manifold (M^n, g) (where $n = 2m+1, m > 1$) is said to be an almost contact metric manifold [1] if it admits a (1,1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{10}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{11}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{12}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$) is said to be Kenmotsu manifold if the following condition holds [5]:

$$\nabla_X \xi = X - \eta(X)\xi \tag{13}$$

and

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{14}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold, the following relations hold [5]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{15}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{16}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{17}$$

$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z), \tag{18}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{19}$$

$$S(\xi,\xi) = -(n-1),\ i.e.,\ Q\xi = -(n-1)\xi \tag{20}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0, 2) such that $g(QX,Y) = S(X,Y)$.

3. Weakly concircular symmetric Kenmotsu manifolds

Definition 1. *A Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ (where $n = 2m+1, m > 1$) is said to be weakly concircular symmetric if its concircular curvature tensor $\tilde{C}$ of type* (0,4) *satisfies* (5).
Setting $Y = V = e_i$ in (5) and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}
&(\nabla_X S)(Z,U) - \frac{dr(X)}{n}g(Z,U) \\
&= A(X)\Big[S(Z,U) - \frac{r}{n}g(Z,U)\Big] + B(Z)\Big[S(X,U) - \frac{r}{n}g(X,U)\Big] \\
&+D(U)\Big[S(X,Z) - \frac{r}{n}g(X,Z)\Big] + B(R(X,Z)U) + D(R(X,U)Z) \\
&-\frac{r}{n(n-1)}\Big[\{B(X)+D(X)\}g(Z,U) - B(Z)g(X,U) - D(U)g(Z,X)\Big].
\end{aligned} \tag{21}$$

Plugging $X = Z = U = \xi$ in (21) and then using (16) and (20), we obtain

$$A(\xi) + B(\xi) + D(\xi) = \frac{dr(\xi)}{r+n(n-1)}, \quad r + n(n-1) \neq 0. \tag{22}$$

This leads to the following:

Theorem 1. *In a weakly concircular symmetric Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ (where $n = 2m+1, m > 1$), the relation* (22) *holds.*
Next, substituting X and Z by ξ in (21) and then using (16), (17) and (19), we obtain

$$\begin{aligned}
&\big[A(\xi) + B(\xi)\big]\Big[\frac{r}{n} + n - 1\Big]\eta(U) \\
&+\Big[\frac{r}{n(n-1)} + 1\Big]\big[(n-2)D(U) + \eta(U)D(\xi)\big] - \frac{dr(\xi)}{n}\eta(U) = 0.
\end{aligned} \tag{23}$$

By virtue of (22), it follows from (23) that

$$D(U) = \Big[D(\xi) + \frac{r+n(n-2)}{n^2(n-1)(n-2)}dr(\xi)\Big]\eta(U), \quad r+n(n-1) \neq 0. \tag{24}$$

Next, setting $X = U = \xi$ in (21) and proceeding in a similar manner as above, we get

$$B(Z) = \Big[B(\xi) + \frac{r+n(n-2)}{n^2(n-1)(n-2)}dr(\xi)\Big]\eta(Z), \quad r+n(n-1) \neq 0. \tag{25}$$

Again, setting $Z = U = \xi$ in (21) and using (16) and (20), we get

$$\begin{aligned} A(X) &= \frac{dr(X)}{r+n(n-1)} - \frac{1}{n-1}\big[B(X) + D(X)\big] \\ &- \frac{n-2}{n-1}\big[B(\xi) + D(\xi)\big]\eta(X), \quad r+n(n-1) \neq 0. \end{aligned} \tag{26}$$

This leads to the following:

Theorem 2. *In a weakly concircular symmetric Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$), the associated 1-forms D, B and A are given by* (24), (25) *and* (26)*, respectively.*

4. Weakly concircular Ricci symmetric Kenmotsu manifolds

Definition 2. *A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$) is said to be weakly concircular Ricci symmetric if its concircular Ricci tensor P of type* (0,2) *satisfies* (9).

In view of (8), (9) yields

$$\begin{aligned} (\nabla_X S)(Y,Z) - \frac{dr(X)}{n}g(Y,Z) &= A(X)\left[S(Y,Z) - \frac{r}{n}g(Y,Z)\right] \\ &+ B(Y)\left[S(X,Z) - \frac{r}{n}g(X,Z)\right] \\ &+ D(Z)\left[S(X,Y) - \frac{r}{n}g(X,Y)\right]. \end{aligned} \tag{27}$$

Setting $X = Y = Z = \xi$ in (27), we get the relation (22) and hence we can state the following:

Theorem 3. *In a weakly concircular Ricci symmetric Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$), the relation* (22) *holds.*

Next, substituting X and Y by ξ in (27) and using (19) and (22), we obtain

$$D(Z) = D(\xi)\eta(Z), \quad r+n(n-1) \neq 0. \tag{28}$$

Again putting $X = Z = \xi$ in (27) and proceeding in a similar manner as above we get

$$B(Y) = B(\xi)\eta(Y), \quad r + n(n-1) \neq 0. \tag{29}$$

Again, setting $Y = Z = \xi$ in (27) and using (20) and (22), we get

$$A(X) = \frac{dr(X)}{r + n(n-1)} + \Big[A(\xi) - \frac{dr(\xi)}{r + n(n-1)}\Big]\eta(X), \quad r + n(n-1) \neq 0. \tag{30}$$

This leads to the following:

Theorem 4. *If in a weakly concircular Ricci symmetric Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$), $r + n(n-1) \neq 0$ then the associated 1-forms D, B and A are given by* (28)*,* (29) *and* (30)*, respectively.*

Adding (28), (29) and (30) and using (22), we get

$$A(X) + B(X) + D(X) = \frac{dr(X)}{r + n(n-1)} \quad \forall\, X. \tag{31}$$

This leads to the following:

Theorem 5. *If in a weakly concircular Ricci symmetric Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ (where $n = 2m+1, m > 1$), $r + n(n-1) \neq 0$, the sum of the associated 1-forms is given by* (31).

Also from (31), we can state the following:

Corollary 1. *There exist no weakly concircular Ricci symmetric Kenmotsu manifold of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero.*

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