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INCLUSION AND NEIGHBORHOOD PROPERTIES OF SOME ANALYTIC AND MULTIVALENT FUNCTIONS ASSOCIATED WITH AN EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR

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Abstract. By means of a certain extended fractional differintegral operator $\Omega_z^{(\lambda,p)}(-\infty < \lambda < p+1; p \in N)$, the authors introduce and investigate two new subclasses of p-valently analytic functions of complex order. The various results obtained here for each of these function classes include coefficient inequalities and the consequent inclusion relationships involving the neighborhoods of the p-valently analytic functions.

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1.Introduction, Definitions and Preliminaries

Let $A_p(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0;\ n,p \in \mathbb{N} := \{1,2,3,...\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Following the works of Goodman [4] , Ruscheweyh [12], Altintas et al. [3] and Raina and Srivastava [13], we define the $(n, \delta)-$ neigborhood of a function $f(z) \in A_p(n)$ by (see also [2], [7] and [15]),

$$N_{n,\delta}(f) := \left\{ g(z) \in A_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{2}$$

It follows from (2) that, if

$$h(z) = z^p \quad (p \in \mathbb{N}), \tag{3}$$

then

$$N_{n,\delta}(h)=\left\{g(z)\in A_p(n):g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}k\left|b_k\right|\leq\delta\right\}. \tag{4}$$

The above concept of (n,δ)-neighborhoods was extended and applied recently to families of analytically multivalent functions by Altintas et al. [3] and to families of meromorphically multivalent functions by Liu and Srivastava ([5] and [6]). The main object of the present paper is to investigate the (n,δ)-neighborhoods of several subclasses of the class $A_p(n)$ of p-valent analytic functions in U with negative and missing coefficients, which are introduced below by making use an extended fractional differintegral operator (see[11]).

We say that a function $f(z)\in A_p(n)$ is starlike of complex order γ ($\gamma\in\mathbb{C}\backslash\{0\}$), that is, $f(z)\in S^*_{n,p}(\gamma)$, if it also satisfies the following inequality :

$$Re\left(1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right)>0 \qquad (z\in U;\gamma\in\mathbb{C}\backslash\{0\}). \tag{5}$$

Furthermore, a function $f(z)\in A_p(n)$ is said to be convex of complex order γ ($\gamma\in\mathbb{C}\backslash\{0\}$), that is, $f(z)\in C_{n,p}(\gamma)$, if it also satisfies the following inequality :

$$Re\left(1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in U;\gamma\in\mathbb{C}\backslash\{0\}). \tag{6}$$

The classes $S^*_{n,p}(\gamma)$ and $C_{n,p}(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [8] and Wiatrowski [16], respectively (see also [2]).

The Hadamard product (or convolution) of the function $f(z)\in A_p(n)$ given by (1) and the function $g(z)\in A_p(n)$ given by

$$g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \quad (b_k\geq 0;n,p\in\mathbb{N}) \tag{7}$$

is defined (as usual) by

$$(f*g)(z):=z^p+\sum_{k=n+p}^{\infty}a_kb_kz^k=(g*f)(z). \tag{8}$$

In [11] Patel and Mishra define the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}: A_p \to A_p$ for a function $f(z)$ of the form (1) and for a real number $\lambda(-\infty < \lambda < p+1)$ by

$$\Omega_z^{(\lambda,p)} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k z^k. \tag{9}$$

We note that

$$\Omega_z^{(0,p)} f(z) = f(z), \quad \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p},$$

and, in general

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) (-\infty < \lambda < p+1; p \in \mathbb{N}; z \in U),$$

where $D_z^{\lambda} f(z)$ is respectively, the fractional integral of $f(z)$ of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of $f(z)$ of order λ when $0 \leq \lambda < p+1$ (see [9],[10] and [14]). By using the operator $\Omega_z^{(\lambda,p)} f(z)$ $(-\infty < \lambda < p+1, p \in \mathbb{N})$ given by (9), we now introduce a new subclass $H_{n,m}^{p}(\lambda, b)$ of p-valently analytic function class $A_p(n)$, which includes funcions $f(z)$ satisfying the following inequality :

$$\left| \frac{1}{b} \left(\frac{z(\Omega_z^{(\lambda,p)} f(z))^{(m+1)}}{(\Omega_z^{(\lambda,p)} f(z))^{(m)}} - (p-m) \right) \right| < 1, \tag{10}$$

$(z \in U, p \in \mathbb{N}, m \in \mathbb{N}_0 = \{0,1,2,...\}, -\infty < \lambda < p+1, p > \max(m, -\lambda); b \in \mathbb{C} \backslash \{0\})$.

Also we denote by $L_{n,m}^{p}(\lambda, b, \mu)$, the subclass of $A_p(n)$ consisting of functions $f(z)$ with satisfing the inequality (11) below:

$$\left| \frac{1}{b} \left(p(1-\mu) \left(\frac{\Omega^{\lambda,p} f(z)}{z} \right)^m + \mu (\Omega^{\lambda,p} f(z))^{(m+1)} - (p-m) \right) \right| < p-m, \tag{11}$$

$(z \in U, p \in \mathbb{N}, m \in \mathbb{N}_0, -\infty < \lambda < p+1, p > \max(m, -\lambda); \mu \geq 0, b \in \mathbb{C} \backslash \{0\})$.

The object of the present paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses

$$H_{n,m}^{p}(\lambda, b) \text{ and } L_{n,m}^{p}(\lambda, b; \mu),$$

which we have introduced here. A part from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (n,δ)-neighorhoods of analytic p-valent functions (with negative and missing coefficients) belonging to these subclasses.

Our definitions of the function classes

$$H^p_{n,m}(\lambda, b) \text{ and } L^p_{n,m}(\lambda, b; \mu)$$

are motivated essentially by two earlier investigations [2] and [7], in each of which further details and references to other closely - related subclasses can be found. In particular, in our definition of the function class $L^p_{n,m}(\lambda, b; \mu)$ involving the inequality (1.9), we have relaxed the parametric constraint $0 \leq \mu \leq 1$, which was imposed earlier by Murugusundaramoorthy and Srivastava [7, p.3, Equation (1.14)].

2. A Set of Coefficient Bounds

In this section, we prove the following results which yield the coefficient inequalities for functions in the subclasses

$$H^p_{n,m}(\lambda, b) \quad \text{and} \quad L^p_{n,m}(\lambda, b; \mu).$$

Theorem 1. Let $f(z) \in A_p(n)$ be given by (1.1). Then $f(z) \in H^p_{n,m}(\lambda, b)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \binom{k}{m} (k+|b|-p)\, a_k \leq |b| \binom{p}{m}. \qquad (12)$$

Proof. Let $f(z)$ of the form (1) belongs to the class $H^p_{n,m}(\lambda, b)$. Then, in view of (9) and (10) yields the following inequality :

$$\mathbb{R}\left(\frac{\sum\limits_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \binom{k}{m} (p-k) z^{k-p}}{\binom{p}{m} - \sum\limits_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \binom{k}{m} z^{k-p}} \right) > -|b| \quad (z \in U). \qquad (13)$$

Putting $z = r (0 \leq r < 1)$ in (13), we observe that the expression in the denominator on the left-hand side of (13) is positive for $r = 0$ and also for all

$r(0<r<1)$. Thus, by letting $r\to 1^-$ through real values, (13) leads us to the desired assertion (12) of Theorem 1.
Conversely, by applying (12) and setting $|z|=1$, we find by using (9) that

$$\left|\frac{z\left(\Omega^{\lambda,p}f(z)\right)^{m+1}}{\left(\Omega^{\lambda,p}f(z)\right)^{m}}-(p-m)\right|$$

$$=\left|\frac{\sum\limits_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\binom{k}{m}(p-k)a_k z^{k-m}}{\binom{p}{m}z^{p-m}-\sum\limits_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\binom{k}{m}a_k z^{k-m}}\right|$$

$$\leq\frac{|b|\left[\binom{p}{m}-\sum\limits_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\binom{k}{m}a_k\right]}{\binom{p}{m}-\sum\limits_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\binom{k}{m}a_k}=|b|\,.$$

Hence, by the maximum modulus principle, we infer that $f(z)\in H^p_{n,m}(\lambda,b)$, which completes the proof of Theorem 1.

By using the same argument as in the proof of Theorem 1, we can establish Theorem 2 below

Theorem 2. *Let $f(z)\in A_p(n)$ be given by (1.1). Then $f(z)\in L^p_{n,m}(\lambda,b;\mu)$ if and only if*

$$\sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\binom{k-1}{m}\left[\mu(k-1)+1\right]a_k\leq(p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]. \tag{14}$$

3. Inclusion Relationships Involving (n,δ)-Neighborhoods

In this section, we establish several inclusion relationships for the function classes.

$$H^p_{n,m}(\lambda,b)\quad\text{and}\quad L^p_{n,m}(\lambda,b;\mu)$$

involving the (n,δ)-neighborhoods defined by (4).
Theorem 3. If

$$\delta=\frac{(n+p)\,|b|\binom{p}{m}}{(n+|b|)\left[\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\binom{n+p}{m}\right]}(p>|b|), \tag{15}$$

then

$$H^p_{n,m}(\lambda, b) \ \subset \ N_{n,\delta}(h). \tag{16}$$

Proof. Let $f(z) \in H^p_{n,m}(\lambda, b)$. Then, in view of the assertion (12) of Theorem 1, we have

$$(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \le |b|\binom{p}{m}. \tag{17}$$

This yields

$$\sum_{k=n+p}^{\infty} a_k \le \frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}}. \tag{18}$$

Applying the assertion (12) of Theorem 1, agian, in conjuction with (18), we obtain

$$\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} k a_k$$

$$\le |b|\binom{p}{m} + (p-|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k$$

$$\le |b|\binom{p}{m} + (p-|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}$$

$$\cdot\frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}}$$

$$= |b|\binom{p}{m}\left(\frac{n+p}{n+|b|}\right).$$

Hence

$$\sum_{k=n+p}^{\infty} k a_k \le \frac{|b|\,(n+p)\binom{p}{m}}{(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)(n+p+1-\lambda)}\right)\binom{n+p}{m}} = \delta, \quad (p > |b|), \tag{19}$$

which, by virtue of (1), establishes the inclusion relation (16) of Theorem 3.

In an analogous manner, by applying assertion (14) of Theorem 2 instead of the assertion (12) of Theorem 1 to functions in the class $L^p_{n,m}(\lambda, b; \mu)$, we can prove the following inclusion relationship.

Theorem 4. If

$$\delta = \frac{(p-m)(n+p)\left[\frac{|b|-1}{m!} + \binom{p}{m}\right]}{[\mu(n+p-1)+1]\left[\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\binom{n+p}{m}\right]}(\mu > 1), \tag{20}$$

then

$$H^p_{n,m}(\lambda, b; \mu) \subset N_{n,\delta}(h).$$

4. Further Neighborhood Properties

In this last section, we determine the neighborhood properties for each of the following (slightly modified) function classes :

$$H^{p,\alpha}_{n,m}(\lambda, b) \text{ and } L^{p,\alpha}_{n,m}(\lambda, b; \mu).$$

Here the class $H^{p,\alpha}_{n,m}(\lambda, b)$ consists of functions $f(z) \in A_p(n)$ for which there exists another function $g(z) \in H^p_{n,m}(\lambda, b)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \qquad (z \in U, 0 \le \alpha < p). \tag{21}$$

Analogously, the class $L^{p,\alpha}_{n,m}(\lambda, b; \mu)$ consists of functions $f(z) \in A_p(n)$ for which there exists another function $g(z) \in L^p_{n,m}(\lambda, b, \mu)$ satisfying the inequality (21).

The proofs of the following results involving the neighborhood properties for the classes

$$H^{p,\alpha}_{n,m}(\lambda, b) \text{ and } L^{p,\alpha}_{n,m}(\lambda, b; \mu)$$

are similar to those given in [2], [7] and [12].

Theorem 5. *Let $g(z) \in H^p_{n,m}(\lambda, b)$. Suppose also that*

$$\alpha = p - \frac{\delta(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m}}{(n+p)\left[(n+|b|)\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p}{m} - |b|\binom{p}{m}\right]}. \tag{22}$$

Then

$$N_{n,\delta}(g(z)) \subset H^{p,\alpha}_{n,m}(\lambda, b). \tag{23}$$

Theorem 6. *Let $g(z) \in L^{p}_{n,m}(\lambda, b, \mu)$. Suppose also that*

$$\alpha = p - \frac{\delta\left[\mu(n+p)+1\right]\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p-1}{m}}{(n+p)\left[\left[\mu(n+p)+1\right]\left(\frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\right)\binom{n+p-1}{m} - (p-m)\left\{\frac{|b|-1}{m!} + \binom{p}{m}\right\}\right]}. \tag{24}$$

Then

$$N_{n,\delta}(g) \subset L^{p,\alpha}_{n,m}(\lambda, b; \mu). \tag{25}$$

Remark 1. In the special case when $\lambda = m = 0$, $p = 1$ and $b = \alpha, (0 \leq \alpha < 1)$, the result due to O. Altintas and S. Owa [1].

References

[1] O. Altintas and S. Owa, Neighborhoods of certain analytic functions with negative coefficients, Internat J. Math. and Math. Sci. 19(1996), no.4, 797-800.

[2] O. Altintas, Ö.Ozkan and H. M. Srivastava, Neighborhoods of a class of analyic functions with negative coefficients, Appl. Math. Lett., 13(3)(2000), 63-67.

[3] O. Altintas, Ö.Ozkan and H. M. Srivastava, Neighborhoods of certain family of multivalent functions with negative coefficients, Comput. Math. Appl., 47 (2004), 1667-1672.

[4] A. W. Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8(1957), 598-601.

[5] J.-L.Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, Math. Comput. Modelling, 39(2004), 21-34.

[6] J.-L.Liu and H. M. Srivastava, Subclasses of meromorphically multivalent functions associated with a certain linear operator, Math. Comput. Modelling, 39 (2004), 35-44.

[7] G. Murugusundaramoorthy and H. M. Srivastava, Neighborhoods of certain classes of analytic functions of complex order, J. Inequal. Pure. Appl. Math. 5(2)(2004), Art. 24.

[8] M. A. Nasr and M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math., 25(1985), 1-12.

[9] S.Owa, On the distortion theorem.I, Kyungpook Math. J. 18(19978),53-59.

[10] S.Owa and H. M. Srivastava,Univalent and starlike generalized hypergeometric functions, Canad. J. Math. 39(1987),1057-1077.

[11] J. Patel and A. K. Mishra, On certain subclasses of multivalent functions associated with an extended fractional differintegral operator, J. Math. Anal. Appl. 332(2007), 109-122.

[12] S. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc. 81(1981), 521-527.

[13] H. M. Srivastava and S.Owa (Eds.), Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester),Wiley, New York, 1989.

[14] H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey. London and Hong Kong, 1992.

[15] R. K. Raina and H. M. Srivastava, Inclusion and Neighborhood Properties of Some Analytic and Multivalent Functions, J. Inequalities in Pure and Applied Mathematics, Volume 7,Issue 1, Article 5,2006.

[16] P. Wiatrowski, On the coefficients of some family of holomorphic functions, Zeszyty Nauk. Univ. Lodz Nouk. Mat. -Przyrod. (2), 39(1970), 75-85.

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