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ON SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION STRUCTURE

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Abstract. In this paper we derive some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disc. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ denotes the subclass of the functions $f \in H(U)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots . (a \in \mathbb{C}). \tag{1.1}$$

Also, let A be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

For $f, g \in H(U)$, we say that the function $f\ (z)$is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad or \quad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [14]; see also [15, p.4]):

$$f(z) \prec g\ (z) \Leftrightarrow f(0) \prec g(0) \ \ and \ \ f(U) \subset g(U).$$

Supposing that p and h are two analytic functions in U, let

$$\varphi(r,s,t;z):\mathbb{C}^3\times U\to\mathbb{C}.$$

If p and $\varphi(p(z),zp^{'}(z),z^2p^{''}(z);z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z)\prec\varphi(p(z),zp^{'}(z),z^2p^{''}(z);z), \tag{1.3}$$

then p is called to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordination to f). An analytic function q is called a subordinant of (1.3), if $q(z)\prec p(z)$ for all the functions p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all of the subordinants q of (1.3), is called the best subordinant (cf., e.g.,[14], see also [15]).

Recently, Miller and Mocanu [16] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$k(z)\prec\varphi(p(z),zp^{'}(z),z^2p^{''}(z);z)\Rightarrow q(z)\prec p(z). \tag{1.4}$$

Using the results Miller and Mocanu [16], Bulboaca [6] considered certain classes of first-order differential superordinations as well as superordination preserving integral operators [5]. Ali et al. [1], have used the results of Bulboaca [6] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z), \tag{1.5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1$. Shanmugam et al. [24] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1$ and $q_2(0)=1$.

For functions $f(z)\in A$, given by (1.1), and $g(z)\in A$ defined by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k\,, \tag{1.6}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k\;=(g*f)(z)\;. \tag{1.7}$$

For functions $f, g \in A$, we define the linear operator $D_\lambda^n : A \to A \quad (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$by:

$$\begin{aligned} D_\lambda^0(f*g)(z) &= (f*g)(z), 1.8 && (1) \\ D_\lambda^1(f*g)(z) &= D_\lambda(f*g) = (1-\lambda)(f*g)(z) + \lambda z(f*g)'(z) \ (\lambda \geq 0), && (1.9) \end{aligned}$$

and (in general)

$$\begin{aligned} D_\lambda^n(f*g)(z) &= D(D_\lambda^{n-1}(f*g) \\ &= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n a_k b_k z^k \ (\lambda \geq 0; n \in \mathbb{N}_0). \\ & \quad 1.10 && (2) \end{aligned}$$

From (1.10) it is easy to verify that

$$\lambda z(D_\lambda^n(f*g)(z))' = D_\lambda^{n+1}(f*g)(z) - (1-\lambda)D_\lambda^n(f*g)(z) \ (\lambda > 0; n \in \mathbb{N}_0). \quad (1.11)$$

The operator $D_\lambda^n(f*g)(z)$ was introduced by Aouf and Seoudy [3].

We observe that the linear operator $D_\lambda^n(f*g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of n, λ and the function $g(z)$:

(i) For $g(z) = \dfrac{z}{1-z}$, we have $D_\lambda^n(f*g)(z) = D_\lambda^n f(z)$ reduces to

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n a_k z^k,$$

where D_λ^n is the generalized Sălăgean operator (or Al-Oboudi operator [2]) which yield Sălăgean operator D^n for $\lambda = 1$ introduced and studied by Sălăgean [22];

(ii) For $n = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k \quad (1.12)$$

$$\left(\alpha_i \in \mathbb{C}; i = 1, ..., q; \beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, ..., s; \right.$$
$$\left. q \leq s+1; q, s \in \mathbb{N}_0; z \in U \right),$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(1)_{k-1}} \tag{1.13}$$

and

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta-1)...(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}) \ , \end{cases}$$

we have $D_\lambda^0(f*g)(z) = (f*g)(z) = H_{q,s}(\alpha_1;\beta_1) f(z)$, where the operator $H_{q,s}(\alpha_1;\beta_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [8] (see also [9] and [10]). The operator $H_{q,s}(\alpha_1;\beta_1)$, contains in turn many interesting operators such as, Hohlov linear operator (see [11]), the Carlson-Shaffer linear operator (see [7] and [21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [4], [12] and [13]) and Owa-Srivastava fractional derivative operator (see [19]);

(*iii*) For $g(z)$ of the form (1.12), the operator $D_\lambda^n(f*g)(z) = D_\lambda^n(\alpha_1,\beta_1)f(z)$, inroduced and studied by Selvaraj and Karthikeyan [23].

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in U such that $(f*\Psi)(z) \neq 0$ and $f(z)$ to satisfy

$$q_1(z) \prec \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U and $\Phi(z) = z + \sum\limits_{k=2}^{\infty} \lambda_k z^k$, $\Psi(z) = z + \sum\limits_{k=2}^{\infty} \mu_k z^k$ are analytic functions in U with $\lambda_k \geq 0,\ \mu_k \geq 0$ and $\lambda_k \geq \mu_k \ \ (k \geq 2)$. Also, we obtain the number of known results as their special case

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and lemmas.

Definition 1 [16]. *Denote by Q the set of all functions $f(z)$that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta : \zeta \in \partial U \ \ \textit{and} \ \ \lim_{z\to\zeta} f(z) = \infty\}, \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 [15]. *Let the function $q(z)$ be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\mathrm{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$, and q is the best dominant .

Lemma 2 [6]. *Let $q(z)$ be a convex univalent function in the unit disc U and let ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\left\{\dfrac{\vartheta'(q(z))}{\varphi(q(z))}\right\} > 0$ for $z \in U$;

(ii) $zq'(z)\varphi(q(z))$ is starlike in U.

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$, and q is the best subordinant.

3. Subordination and superordination results

Unless otherwise mentioned we shall assume throughout the paper that $\lambda > 0, \gamma_4 \neq 0, \gamma_1, \gamma_2, \gamma_3$ be the complex numbers and $n \in \mathbb{N}_0$.

Theorem 1. *Let* $\Phi, \Psi \in A$ and *q be convex univalent in U, with $q(0) = 1$. Suppose that*

$$\mathrm{Re}\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in U). \tag{3.1}$$

If $f(z) \in A$ satisfies the subordination:

$$\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 zq'(z), \tag{3.2}$$

where

$$\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) =$$

$$\gamma_1 + \gamma_2\left(\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right)^2 + \gamma_3\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} +$$
$$\frac{\gamma_4}{\lambda}\left[\frac{D_\lambda^{n+1}(f*\Phi)(z)}{D_\lambda^n(f*\Phi)(z)} - \frac{D_\lambda^{n+2}(f*\Psi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right]\left(\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right), 3.3 \tag{3}$$

then

$$\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \prec q(z) \tag{3.4}$$

and $q(z)$ is the best dominant of (3.2).

Proof. Define the function $p(z)$ by

$$p(z) = \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \qquad (z \in U). \tag{3.5}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, by making use of (3.5), we have

$$\gamma_1 + \gamma_2 \left(\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right)^2 + \gamma_3 \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} +$$

$$\frac{\gamma_4}{\lambda}\left[\frac{D_\lambda^{n+1}(f*\Phi)(z)}{D_\lambda^n(f*\Phi)(z)} - \frac{D_\lambda^{n+2}(f*\Psi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right]\left(\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}\right),$$

$$= \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \tag{3.6}$$

By using (3.6) in (3.2), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z)\ . \tag{3.7}$$

By setting

$$\theta(w) = \gamma_1 w^2(z) + \gamma_3 w \qquad \textit{and} \qquad \phi(w) = \gamma_4,$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}^*$ and that $\phi(w) \neq 0$. Hence the result now follows by an application of Lemma 1 .

Putting $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let* $\Phi, \Psi \in A$ and q *be convex univalent in* U*, with* $q(0) = 1$ *and suppose that* $q(z)$ *satisfy the condition (3.1). If* $f(z) \in A$ *satisfies the subordination:*

$$\zeta(f, \Phi, \Psi, n, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z),$$

where

$$\zeta(f, \Phi, \Psi, n, \gamma_1, \gamma_2, \gamma_3, \gamma_4) =$$

$$\gamma_1+\gamma_2\left(\frac{D^n(f*\Phi)(z)}{D^{n+1}(f*\Psi)(z)}\right)^2+\gamma_3\frac{D^n(f*\Phi)(z)}{D^{n+1}(f*\Psi)(z)}+$$
$$\gamma_4\left[\frac{D^{n+1}(f*\Phi)(z)}{D^n(f*\Phi)(z)}-\frac{D^{n+2}(f*\Psi)(z)}{D^{n+1}(f*\Psi)(z)}\right]\left(\frac{D^n(f*\Phi)(z)}{D^{n+1}(f*\Psi)(z)}\right), 3.8 \tag{4}$$

then

$$\frac{D^n(f*\Phi)(z)}{D^{n+1}(f*\Psi)(z)}\prec q(z)$$

and $q(z)$ is the best dominant.

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let q be convex univalent in U, with $q(0)=1$ and suppose that $q(z)$ satisfy the condition (3.1). If $f(z)\in A$ satisfies the subordination:*

$$\gamma_1+\gamma_2\left(\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}\right)^2+\gamma_3\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}+\frac{\gamma_4}{\lambda}\left[1-\frac{D_\lambda^n f(z).D_\lambda^{n+2}f(z)}{\left[D_\lambda^{n+1}f(z)\right]^2}\right]$$
$$\prec\ \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{D_\lambda^n f(z)}{D_\lambda^{n+1}f(z)}\prec q(z)$$

and $q(z)$ is the best dominant.

Remark 1. (i) *Putting $\gamma_1=\gamma_2=0,\gamma_3=1$ and $\gamma_4=\gamma$ in Corollary 2, we obtain the result obtained by Nechita [18, Theorem 5];*

(ii) *Putting $\gamma_1=\gamma_2=0,\gamma_3=1,\gamma_4=\gamma$ and $\lambda=1$ in Corollary 2, we obtain the result obtained by Nechita [18, Corollary 7] and improve the result obtained by Shanmugam et al. [24, Theorem 5.1];*

(iii) *Putting $\lambda=1$ and $n=0$ in Theorem 1, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.7];*

(iv) *Putting $\lambda=1,n=0$ and $\Phi(z)=\Psi(z)$ in Theorem 1, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.8];*

(v) *Putting $\lambda=1,n=0$ and $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 1, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.9];*

(vi) *Putting $\lambda=1,n=0$, $\Phi(z)=\Psi(z)=\frac{z}{1-z},\gamma_1=\gamma_2=0,\gamma_3=1$ and $\gamma_4=\gamma$ in Theorem 1, we obtain the result obtained by Nechita [18, Corollary 8] and improve the result obtained by Shanmugam et al. [24, Theorem 3.1].*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Suppose that*

$$\operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}\frac{1+Az}{1+Bz} + \left(\frac{1-Bz}{1+Bz}\right)\right\} > 0.$$

If $f(z) \in A$ satisfies the subordination

$$\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2\left(\frac{1+Az}{1+Bz}\right)^2 + \gamma_3\frac{1+Az}{1+Bz} + \gamma_4\frac{(A-B)z}{(1+Bz)^2},$$

where $\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (3.3), then

$$\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \prec \frac{1+Az}{1+Bz}$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant.

Remark 2. *Putting* $\gamma_1 = \gamma_2 = 0, \gamma_3 = 1, \gamma_4 = \gamma$ and $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ *in Corollary 3, we obtain the result obtained by Nechita [18, corollary 9].*

By fixing $\Phi(z) = \Psi(z) = z + \sum\limits_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, where $\Gamma_k(\alpha_1)$ is given by (1.13), $n = 0$ and $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 4. *Let q be convex univalent in U, with $q(0) = 1$ and suppose that $q(z)$ satisfy the condition (3.1). If $f(z) \in A$ satisfies the subordination:*

$$\zeta(f, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z),$$

where

$$\zeta(f, \gamma_1, \gamma_2, \gamma_3, \gamma_4) =$$

$$\gamma_1 + \gamma_2\left(\frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'}\right)^2 + \gamma_3\frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'} +$$
$$\gamma_4\left[1 - \frac{H_{q,s}(\alpha_1;\beta_1)f(z).(H_{q,s}(\alpha_1;\beta_1)f(z))''}{[(H_{q,s}(\alpha_1;\beta_1)f(z))']^2} - \frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'}\right],$$

3.11 (5)

then

$$\frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'} \prec q(z)$$

and $q(z)$ is the best dominant.

Remark 3. *Putting $\gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ and $\gamma_4 = \gamma$ in Corollary 4, we obtain the result obtained by Aouf and Seoudy [3, Corollary 3].*

Theorem 2. *Let $\Phi, \Psi \in A$ and $q(z)$ be convex univalent in U with $q(0) = 1$. Suppose that*

$$\operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z)\right\} > 0 \,. \tag{3.12}$$

If $f(z) \in A$ such that $\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \in H[q(0), 1] \cap Q$ and $\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is univalent in U and satisfies the superordination:

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \,, \tag{3.13}$$

where $\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (3.3), then

$$q(z) \prec \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}$$

and $q(z)$ is the best subordinant.

Proof. Let $p(z)$ be given by (3.5). Simple computations from (3.5), we get,

$$\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z),$$

then

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z).$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) = \gamma_4$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C}^*$ and that $\phi(w) \neq 0$.

Since $q(z)$ is convex univalent function, it follows that

$$\operatorname{Re}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z)\right\} > 0 \quad (z \in U).$$

Now Theorem 2 follows by applying Lemma 2.

Putting $\lambda = 1$ in Theorem 2, we obtain the following corollary.

Corollary 5. *Let $\Phi, \Psi \in A$, q be convex univalent in U, with $q(0) = 1$ and suppose that $q(z)$ satisfy the condition (3.12). If $f(z) \in A$ such that $\frac{D^n(f*\Phi)(z)}{D^{n+1}(f*\Psi)(z)} \in H[q(0), 1] \cap Q$ and $\zeta(f, \Phi, \Psi, n, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is univalent in U and satisfies the superordination:*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \zeta(f, \Phi, \Psi, n, \gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

where $\zeta(f, \Phi, \Psi, n, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (3.8), then

$$q(z) \prec \frac{D^n(f * \Phi)(z)}{D^{n+1}(f * \Psi)(z)}$$

and $q(z)$ is the best subordinant.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let q be convex univalent in U, with $q(0) = 1$ and suppose that $q(z)$ satisfy the condition (3.12). If $f(z) \in A$ such that $\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \in H[q(0), 1] \cap Q$ and*

$$\gamma_1 + \gamma_2 \left(\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \right)^2 + \gamma_3 \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma_4}{\lambda} \left[1 - \frac{D_\lambda^n f(z).D_\lambda^{n+2} f(z)}{\left[D_\lambda^{n+1} f(z) \right]^2} \right]$$

is univalent in U and satisfies the superordination:

$$\begin{aligned} & \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \\ \prec \quad & \gamma_1 + \gamma_2 \left(\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \right)^2 + \gamma_3 \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma_4}{\lambda} \left[1 - \frac{D_\lambda^n f(z).D_\lambda^{n+2} f(z)}{\left[D_\lambda^{n+1} f(z) \right]^2} \right], \end{aligned}$$

then

$$q(z) \prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)}$$

and $q(z)$ is the best subordinant.

Remark 4. (i) *Putting $\lambda = 1, n = 0, \gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ and $\gamma_4 = \gamma$ in Corollary 6, we obtain a result which improves the result obtained by Shanmugam et al. [24, Theorem 5.2];*

(ii) *Putting $\lambda = 1$ and $n = 0$ in Theorem 2, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.14];*

(iii) *Putting* $\lambda = 1, n = 0$ *and* $\Phi(z) = \Psi(z)$ *in Theorem 2, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.15];*

(iv) *Putting* $\lambda = 1, n = 0$ *and* $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ *in Theorem 2, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 2.16];*

(v) *Putting* $\lambda = 1, n = 0,$ $\Phi(z) = \Psi(z) = \frac{z}{1-z}, \gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ *and* $\gamma_4 = \gamma$ *in Theorem 2, we obtain a result which improves the result obtained by Shanmugam et al. [24, Theorem 3.2].*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we obtain the following corollary.

Corollary 7. *Suppose that*

$$\operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}\frac{1+Az}{1+Bz}\right\} > 0.$$

If $f(z) \in A$ *such that* $\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \in H[q(0),1] \cap Q$ *and* $\zeta(f,\Phi,\Psi,n,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ *is univalent in* U *and satisfies the superordination:*

$$\gamma_1 + \gamma_2\left(\frac{1+Az}{1+Bz}\right)^2 + \gamma_3\frac{1+Az}{1+Bz} + \gamma_4\frac{(A-B)z}{(1+Bz)^2} \prec \zeta(f,\Phi,\Psi,n,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4),$$

where $\zeta(f,\Phi,\Psi,n,\lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ is given by (3.3), then

$$\frac{1+Az}{1+Bz} \prec \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)}$$

and $\dfrac{1+Az}{1+Bz}$ is the best subordinant.

By fixing $\Phi(z) = \Psi(z) = z + \sum\limits_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, where $\Gamma_k(\alpha_1)$ is given by (1.13), $n = 0$ and $\lambda = 1$ in Theorem 2, we obtain the following corollary.

Corollary 8. *Let* q *be convex univalent in* U*, with* $q(0) = 1$ *and suppose that* $q(z)$ *satisfy the condition (3.12). If* $f(z) \in A$ *such that* $\frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'} \in H[q(0),1] \cap Q$ *and* $\zeta(f,\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ *is univalent in* U *and satisfies the superordination:*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \zeta(f,\gamma_1,\gamma_2,\gamma_3,\gamma_4)\ ,$$

where $\zeta(f,\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ is given by (3.11), then

$$q(z) \prec \frac{H_{q,s}(\alpha_1;\beta_1)f(z)}{z(H_{q,s}(\alpha_1;\beta_1)f(z))'}$$

and $q(z)$ is the best subordinant.

Remark 5. *Putting $\gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ and $\gamma_4 = \gamma$ in Corollary 8, we obtain the result obtained by Aouf and Seoudy [3, Corollary 8].*

4. Sandwich results

We conclude this paper by stating the following sandwich results.
Combining Theorem 1 and Theorem 2, we get the following sandawich theorem.

Theorem 3. *Let q_1, q_2 be convex in U. Suppose q_1 satisfies (3.1) and q_2 satisfies (3.12). If $f(z) \in A$ such that $\frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \in H[q(0), 1] \cap Q$ and $\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is univalent in U and satisfies*

$$\begin{aligned} \gamma_1 + \gamma_2 q_1^2(z) + \gamma_3 q_1(z) + \gamma_4 z q_1'(z) &\prec \zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \\ &\prec \gamma_1 + \gamma_2 q_2^2(z) + \gamma_3 q_2(z) + \gamma_4 z q_2'(z), \\ & 4.1 \end{aligned} \tag{6}$$

where $\zeta(f, \Phi, \Psi, n, \lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (3.3), then

$$q_1(z) \prec \frac{D_\lambda^n(f*\Phi)(z)}{D_\lambda^{n+1}(f*\Psi)(z)} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Remark 6. *By taking* $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ $(-1 \le B_1 < A_1 \le 1)$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$ $(-1 \le B_2 < A_2 \le 1)$, $\lambda = 1$ and $n = 0$ in Theorem 3, we obtain the result obtained by Murugusundarmoorthy and Magesh [17, Corollary 3.2].

Combining Corollary 2 and Corollary 6, we obtain the following corollary.

Corollary 9. *Let q_1, q_2 be convex in U. Suppose q_1 satisfies (3.1) and q_2 satisfies (3.12). If $f(z) \in A$ such that $\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \in H[q(0), 1] \cap Q$ and*

$$\gamma_1 + \gamma_2 \left(\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)}\right)^2 + \gamma_3 \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} + \frac{\gamma_4}{\lambda}\left[1 - \frac{D_\lambda^n f(z).D_\lambda^{n+2} f(z)}{\left[D_\lambda^{n+1} f(z)\right]^2}\right]$$

is univalent in U and satisfies

$$\begin{aligned}
& \gamma_1+\gamma_2 q_1^2(z)+\gamma_3 q_1(z)+\gamma_4 z q_1'(z) \\
\prec\ & \gamma_1+\gamma_2\left(\frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)}\right)^2+\gamma_3 \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)}+\frac{\gamma_4}{\lambda}\left[1-\frac{D_\lambda^n f(z) . D_\lambda^{n+2} f(z)}{\left[D_\lambda^{n+1} f(z)\right]^2}\right] \\
\prec\ & \gamma_1+\gamma_2 q_2^2(z)+\gamma_3 q_2(z)+\gamma_4 z q_2'(z),
\end{aligned}$$

then

$$q_1(z) \prec \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ ar, respectively, the best subordinant and best dominant

Remark 7. *Putting $\lambda=1, \gamma_1=\gamma_2=0, \gamma_3=1$ and $\gamma_4=\gamma$ in Theorem 3, we obtain a result which improves the result obtained by Shanmugam et al. [24, Theorem 5.3].*

Combining Corollary 4 and Corollary 8, we get the following corollary.

Corollary 10. *Let q_1, q_2 be convex in U. Suppose q_1 satisfies (3.1) and q_2 satisfies (3.12). If $f(z) \in A$ such that $\frac{H_{q,s}(\alpha_1;\beta_1) f(z)}{z(H_{q,s}(\alpha_1;\beta_1) f(z))'} \in H[q(0),1] \cap Q$ and $\zeta(f, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is univalent in U and satisfies*

$$\begin{aligned}
\gamma_1+\gamma_2 q_1^2(z)+\gamma_3 q_1(z)+\gamma_4 z q_1'(z) & \prec \zeta(f, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \\
& \prec \gamma_1+\gamma_2 q_2^2(z)+\gamma_3 q_2(z)+\gamma_4 z q_2'(z),
\end{aligned}$$

where $\zeta(f, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (3.11), then

$$q_1(z) \prec \frac{H_{q,s}(\alpha_1;\beta_1) f(z)}{z(H_{q,s}(\alpha_1;\beta_1) f(z))'} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ ar, respectively, the best subordinant and best dominant.

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