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ON CERTAIN RESULTS FOR SAKAGUCHI TYPE FUNCTIONS

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Abstract. In this paper, we obtain certain sufficient conditions for normalized analytic functions to belong to the class of Sakaguchi type functions of order β. Using the technique of differential subordination, certain known results are extended. We also use the dual concept of differential subordination and superordination to obtain some sandwich type results. Mathematica 7.0 is used to show the extended regions of the complex plane, pictorially.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

To derive certain sandwich-type results, we, here, use the dual concept of differential subordination and superordination:

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p(z) \prec q(z)$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z),\ \ h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2), if $q(z) \prec p(z)$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2), is said to be the best subordinant of (2).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\beta, t)$ if it satisfies the condition

$$\Re\left(\frac{(1-t)zf'(z)}{f(z) - f(tz)}\right) > \beta,\ \ z \in \mathbb{E},$$

for some $\beta(0 \leq \beta < 1)$ and $|t| \leq 1,\ t \neq 1$.

Recently Goyal et al. [2] studied the above class and proved the following result:

Theorem 1.1. *If $f \in \mathcal{A}$, satisfies*

$$\Re\left[\frac{(1-t)^2 zf'(z)}{f(z) - f(tz)}\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z) - f(tz)} + 1\right)\right] > \alpha\beta\left(\beta - \frac{1-t}{2}\right) + \left(\beta - \frac{\alpha}{2}\right)(1-t),$$

for $z \in \mathbb{E},\ 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ |t| \leq 1$ and $t \neq 1$, then

$$\Re\left(\frac{(1-t)zf'(z)}{f(z) - f(tz)}\right) > \beta,\ \ z \in \mathbb{E}, \quad i.e. \ \ f \in \mathcal{S}(\beta, t).$$

In this paper, we extend the region of the complex plane in which the differential operator $\dfrac{(1-t)^2 zf'(z)}{f(z) - f(tz)}\left(\dfrac{\alpha zf''(z)}{f'(z)} + \dfrac{\alpha tzf'(tz)}{f(z) - f(tz)} + 1\right)$ for $f \in \mathcal{A}$, may take values for being a member of the class $\mathcal{S}(\beta, t)$. Consequently, the results like above are extended. Using the dual concept of differential subordination and superordination, we derive some sandwich-type results. Mathematica 7.0 is used to plot the image of the unit disk under certain analytic functions.

2 Preliminaries

We shall use the following definition and lemmas to prove our main results.

Definition 2.1. *([4], p.21, Definition 2.2b) We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where*

$$\mathbb{B}(p) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 2.1. *([4], p.132, Theorem 3.4 h). Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

(i) h is convex, or

(ii) Q_1 is starlike.

In addition, assume that

(iii) $\Re \frac{zh'(z)}{Q_1(z)} > 0, \; z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2. *([1]). Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*

(i) Q_1 is starlike in $\mathbb{E}$ and

(ii) $\Re \frac{\theta'(q(z))}{\phi(q(z))} > 0, \; z \in \mathbb{E}$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3 Main Results

Theorem 3.1. *Let $\alpha, \; t$ be complex numbers such that $\alpha \neq 0$ and $|t| \leq 1, \; t \neq 1$. Let $q, \; q(z) \neq 0$ be univalent convex function in $\mathbb{E}$ such that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + \frac{2}{1-t}q(z) + \frac{1-\alpha}{\alpha}\right) > 0.$$

If $f \in \mathcal{A}, \; \dfrac{(1-t)zf'(z)}{f(z)-f(tz)} \neq 0, \; z \in \mathbb{E}$, satisfies the differential subordination

$$\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z)-f(tz)} + 1\right)$$

$$\prec (1-\alpha)(1-t)q(z) + \alpha(q(z))^2 + \alpha(1-t)zq'(z), \tag{3}$$

then

$$\frac{(1-t)zf'(z)}{f(z)-f(tz)} \prec q(z),$$

and q is the best dominant.

Proof. On writing $\dfrac{(1-t)zf'(z)}{f(z)-f(tz)} = p(z)$, a little calculation yields:

$$\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left(\frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha tzf'(tz)}{f(z)-f(tz)} + 1\right)$$
$$= (1-\alpha)(1-t)p(z) + \alpha(p(z))^2 + \alpha(1-t)zp'(z). \tag{4}$$

Define the functions θ and ϕ as under:

$$\theta(w) = (1-\alpha)(1-t)w + \alpha w^2 \quad \text{and} \quad \phi(w) = \alpha(1-t).$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$. Setting the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha(1-t)zq'(z),$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = (1-\alpha)(1-t)q(z) + \alpha(q(z))^2 + \alpha(1-t)zq'(z).$$

A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)},$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \frac{2}{1-t}q(z) + \frac{1-\alpha}{\alpha}.$$

Therefore, we have: Q_1 is starlike in $\mathbb{E}$ and $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0,\ z \in \mathbb{E}$. Thus conditions (ii) and (iii) of Lemma 2.1, are satisfied. In view of (3) and (4), we have

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),\ z \in \mathbb{E}.$$

Hence, the proof now follows from Lemma 2.1.

Theorem 3.2. *Let $\alpha,\ t$ be complex numbers such that $\alpha \neq 0$ and $|t| \leq 1,\ t \neq 1$. Let $q,\ q(z) \neq 0$ be univalent convex function in $\mathbb{E}$ such that $\Re\left(\frac{2}{1-t}q(z) + \frac{1-\alpha}{\alpha}\right) > 0$. If $f \in \mathcal{A},\ \frac{(1-t)zf'(z)}{f(z)-f(tz)} \in \mathcal{H}[q(0),1] \cap Q$ with $\frac{(1-t)zf'(z)}{f(z)-f(tz)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential superordination*

$$(1-\alpha)(1-t)q(z) + \alpha(q(z))^2 + \alpha(1-t)zq'(z)$$

$$\prec \frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left(\frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha t z f'(tz)}{f(z)-f(tz)} + 1\right) = h(z), \tag{5}$$

where h is univalent in $\mathbb{E}$, then

$$q(z) \prec \frac{(1-t)zf'(z)}{f(z)-f(tz)},$$

and q is the best subordinant.

Proof. Setting $\frac{(1-t)zf'(z)}{f(z)-f(tz)} = p(z)$ and by defining the functions $\theta,\ \phi$ and Q_1 same as in case of Theorem 3.1 and observing that

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{2}{1-t}q(z) + \frac{1-\alpha}{\alpha}.$$

The use of Lemma 2.2 along with (4) and (5), completes the proof on the same lines as in case of Theorem 3.1.

On combining Theorem 3.1 and Theorem 3.2, we obtain the following sandwich-type theorem.

Theorem 3.3. *Suppose $\alpha,\ t$ are complex numbers such that $\alpha \neq 0$ and $|t| \leq 1,\ t \neq 1$ and suppose that $q_1, q_2,\ (q_1(z) \neq 0, q_2(z) \neq 0,\ z \in \mathbb{E})$ are univalent convex function in $\mathbb{E}$ such that*

(i) $\Re\left(1 + \frac{zq_2''(z)}{q_2'(z)} + \frac{2}{1-t}q_2(z) + \frac{1-\alpha}{\alpha}\right) > 0$ *and*

(ii) $\Re\left(\frac{2}{1-t}q_1(z) + \frac{1-\alpha}{\alpha}\right) > 0.$

If $f \in \mathcal{A},\ \frac{(1-t)zf'(z)}{f(z)-f(tz)} \in \mathcal{H}[q_1(0),1] \cap Q$ with $\frac{(1-t)zf'(z)}{f(z)-f(tz)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential sandwich-type condition

$$(1-\alpha)(1-t)q_1(z) + \alpha(q_1(z))^2 + \alpha(1-t)zq_1'(z)$$

$$\prec \frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left(\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right)=h(z)$$
$$\prec (1-\alpha)(1-t)q_2(z)+\alpha(q_2(z))^2+\alpha(1-t)zq_2'(z),$$

where h is univalent in $\mathbb{E}$, then

$$q_1(z) \prec \frac{(1-t)zf'(z)}{f(z)-f(tz)} \prec q_2(z).$$

Moreover q_1 and q_2 are the best subordinant and the best dominant respectively.

4 Deductions

We start this section with the justification of our claim that Theorem 3.1 extends the result of Goyal et al. [2] stated in Theorem 1.1. Select the dominant $q(z) = \dfrac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$. A little calculation yields that

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)=\Re\left(\frac{1+z}{1-z}\right)>0.$$

For $0 < \alpha \leq 1$, we have

$$\Re\left(1+\frac{zq''(z)}{q'(z)}+q(z)+\frac{1-\alpha}{\alpha}\right)=\Re\left(\frac{1+z}{1-z}+\frac{1+(1-2\beta)z}{1-z}+\frac{1-\alpha}{\alpha}\right)>0.$$

By selecting $t=-1$ in Theorem 3.1, we immediately get the following result.

Corollary 4.1. *If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)-f(-z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$\frac{zf'(z)}{f(z)-f(-z)}\left(\frac{\alpha zf''(z)}{f'(z)}-\frac{\alpha zf'(-z)}{f(z)-f(-z)}+1\right)$$
$$\prec (1-\alpha)\frac{1+(1-2\beta)z}{2(1-z)}+\alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2+\alpha\frac{(1-\beta)z}{(1-z)^2},$$

where $0<\alpha\leq 1$, then

$$\frac{zf'(z)}{f(z)-f(-z)} \prec \frac{1+(1-2\beta)z}{2(1-z)},\ z\in\mathbb{E}.$$

Taking $\alpha=1$ and $\beta=0$ in above corollary, we obtain:

Corollary 4.2. *If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)-f(-z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$\frac{zf'(z)}{f(z)-f(-z)}\left(\frac{zf''(z)}{f'(z)} - \frac{\alpha zf'(-z)}{f(z)-f(-z)} + 1\right) \prec \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 + \frac{z}{(1-z)^2} = F(z), \tag{6}$$

then

$$\frac{zf'(z)}{f(z)-f(-z)} \prec \frac{1+z}{2(1-z)}, \quad i.e.\ \Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0,\ z \in \mathbb{E}.$$

Remark 4.1. *For $\alpha = 1$, $\beta = 0$ and $t = -1$, Theorem 1.1 gives the following result:*
If $f \in \mathcal{A}$, satisfies

$$\Re\left[\frac{zf'(z)}{f(z)-f(-z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(-z)}{f(z)-f(-z)} + 1\right)\right] > -\frac{1}{4},\ z \in \mathbb{E}, \tag{7}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0,\ z \in \mathbb{E}.$$

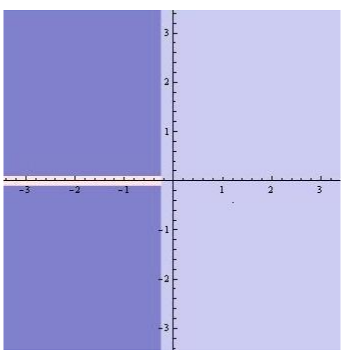

Figure 1: Figure 4.1

We notice that the image of the unit disk $\mathbb{E}$ under the function F (given in (6)) is the entire complex plane except the slit $-\infty < x \leq -1/4$. Therefore, the operator $\dfrac{zf'(z)}{f(z)-f(-z)}\left(\dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(-z)}{f(z)-f(-z)} + 1\right)$ in (6) may take the values in the entire complex plane except the slit $-\infty < x \leq -1/4$ whereas according to (7), the same operator can take values only in the portion of the complex plane right to the line $y = -1/4$. Thus the result in Corollary 4.2 extends the above stated result. In Figure 4.1, the dark portion shows the claimed extension.

Remark 4.2. *We notice that results in Corollary 1 and Corollary 2 of Goyal et al. [2] are extended in the sense same as in above corollary. We also remark that the result in Corollary 3 of Goyal et al. [2] which is initially due to Ravichandran et al. [6] and hence the results in Corollary 4 and Corollary 5 of Goyal et al. [2] which are due to Li and Owa [3] can be looked upon extended in the sense same as above.*

We, now, apply Theorem 3.3 to find certain sandwich-type results. Select the subordinant $q_1(z) = 1 + az$ and the dominant $q_2(z) = 1 + bz,\ 0 < a < b$ and taking $t = -1$ in Theorem 3.3, we obtain:

Corollary 4.3. *Suppose $\alpha > 0$ and $a,\ b\ (a < b)$ are real numbers such that $0 < a < \dfrac{1}{\alpha}$ and $0 < b < 1 + \dfrac{1}{\alpha}$. If $f \in \mathcal{A}$ is such that $\dfrac{zf'(z)}{f(z) - f(-z)} \in \mathcal{H}[1, 1] \cap Q$ with $\dfrac{zf'(z)}{f(z) - f(-z)} \neq 0,\ z \in \mathbb{E}$ and satisfies the condition*

$$\frac{2 - \alpha + 2(1 + \alpha)az + \alpha a^2 z^2}{4} \prec \frac{zf'(z)}{f(z) - f(-z)}\left(\frac{\alpha z f''(z)}{f'(z)} - \frac{\alpha z f'(-z)}{f(z) - f(-z)} + 1\right) \prec \frac{2 - \alpha + 2(1 + \alpha)bz + \alpha b^2 z^2}{4},\ z \in \mathbb{E},$$

where $\dfrac{zf'(z)}{f(z) - f(-z)}\left(\dfrac{\alpha z f''(z)}{f'(z)} - \dfrac{\alpha z f'(-z)}{f(z) - f(-z)} + 1\right)$ is univalent in $\mathbb{E}$, then

$$\frac{1 + az}{2} \prec \frac{zf'(z)}{f(z) - f(-z)} \prec \frac{1 + bz}{2},\ z \in \mathbb{E}.$$

Taking $\alpha = 1,\ a = 1/4$ and $b = 19/20$ in above corollary, we get:

Example 4.1. *Suppose $a,\ b\ (a < b)$ are real numbers such that $0 < a < 1$ and $0 < b < 2$. If $f \in \mathcal{A}$ is such that $\dfrac{zf'(z)}{f(z) - f(-z)} \in \mathcal{H}[1, 1] \cap Q$ with $\dfrac{zf'(z)}{f(z) - f(-z)} \neq 0,\ z \in \mathbb{E}$ and satisfies the condition*

$$\frac{1}{4} + \frac{1}{4}z + \frac{1}{64}z^2 \prec \frac{zf'(z)}{f(z) - f(-z)}\left(\frac{z f''(z)}{f'(z)} - \frac{z f'(-z)}{f(z) - f(-z)} + 1\right) \prec \frac{1}{4} + \frac{19}{20}z + \frac{361}{1600}z^2, \tag{8}$$

where $\dfrac{zf'(z)}{f(z) - f(-z)}\left(\dfrac{z f''(z)}{f'(z)} - \dfrac{z f'(-z)}{f(z) - f(-z)} + 1\right)$ is univalent in $\mathbb{E}$, then

$$\frac{1}{2} + \frac{1}{8}z \prec \frac{zf'(z)}{f(z) - f(-z)} \prec \frac{1}{2} + \frac{19}{40}z,\ z \in \mathbb{E}. \tag{9}$$

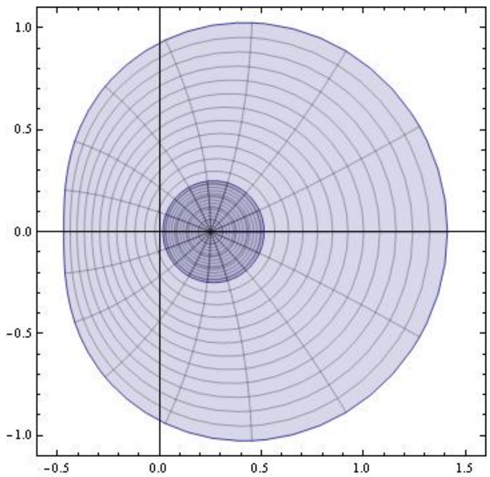

Figure 2: Figure 4.2

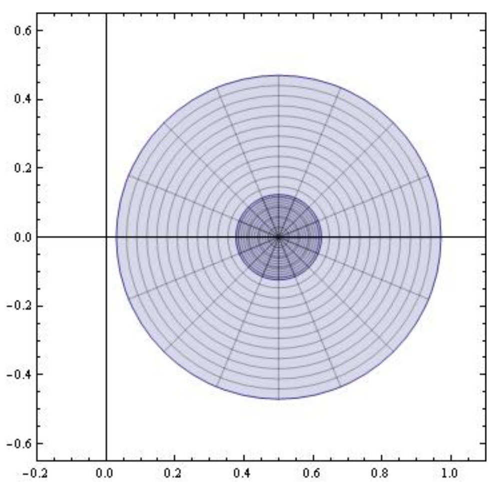

Figure 3: Figure 4.3

Using Mathematica 7.0, *we plot the image of the unit disk* $\mathbb{E}$ *under the dominant and subordinant of (8) in Figure 4.2 and that of under the dominant and subordinant of (9) in Figure 4.3.*

This shows that if the operator $\frac{zf'(z)}{f(z)-f(-z)}\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}+1\right)$ *takes values in the light shaded portion of Figure 4.2, then the operator* $\frac{zf'(z)}{f(z)-f(-z)}$ *takes values in the light shaded portion of Figure 4.3.*

Taking the subordinant $q_1(z)=1+az$ and the dominant $q_2(z)=1+bz,\ 0<a<b$, and $t=0$ in Theorem 3.3, we obtain:

Corollary 4.4. *Let* $\alpha>0$ *and* $a,\ b\ (a<b)$ *be real numbers such that* $0<a<\frac{1}{2}+\frac{1}{2\alpha}$ *and* $0<b<1+\frac{1}{2\alpha}$. *If* $f\in\mathcal{A}$ *is such that* $\frac{zf'(z)}{f(z)}\in\mathcal{H}[1,1]\cap Q$ *with* $\frac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$ *and satisfies the condition*

$$1+(1+2\alpha)az+\alpha a^2z^2\prec\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec 1+(1+2\alpha)bz+\alpha b^2z^2,\ z\in\mathbb{E},$$

where $\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}$ *is univalent in* $\mathbb{E}$, *then*

$$1+az\prec\frac{zf'(z)}{f(z)}\prec 1+bz,\ z\in\mathbb{E}.$$

Remark 4.3. *The subordination part of above corollary was proved by Mocanu and Oros [5].*

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