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TANGENT DEVELOPABLE OF GENERAL HELICES IN THE SOL SPACE

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Abstract. In this paper, we study tangent developable of general helices in the $\mathfrak{Sol}^3$. Finally, we find explicit parametric equations of tangent developable of general helices in the $\mathfrak{Sol}^3$.

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Introduction

In elementary differential geometry [3] that under the assumption of sufficient differentiability, a developable surface is either a plane, conical surface, cylindrical surface or tangent surface of a curve or a composition of these types. Thus a developable surface is a ruled surface, where all points of the same generator line share a common tangent plane.

Design using free-form developable surfaces plays an important role in the manufacturing industry. Currently most commercial systems can only support converting free-form surfaces into approximate developable surfaces. Direct design using developable surfaces by interpolating descriptive curves is much desired in industry.

In this paper, we study tangent developable of general helices in the $\mathfrak{Sol}^3$. Finally, we find explicit parametric equations of tangent developable of general helices in the $\mathfrak{Sol}^3$.

2.Preliminaries

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{2.1}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i, \tag{2.2}$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz, \tag{2.3}$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.4}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathfrak{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to \left(e^{-c}x, e^{c}y, z + c\right). \end{aligned}$$

3. General Helices in Sol Space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B}, 3.1 \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{N}) = 1,\ g_{\mathfrak{Sol}^3}(\mathbf{B},\mathbf{B}) = 1,\ 3.2 \\ g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{B}) = g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

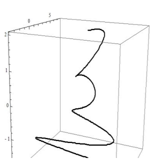
$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3, 3.3 \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3. \end{aligned} \tag{3}$$

Theorem 3.1. ([14]) *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4, \\ y(s) &= \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s+\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5, 3.4 \\ z(s) &= \cos\mathfrak{P}s+\mathfrak{C}_3, \end{aligned} \tag{4}$$

where $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4,\mathfrak{C}_5$ are constants of integration.

The obtained parametric equations for Eq. (3.4) is illustrated in Fig. 1:



4. Tangent Developable Surfaces of General Helices in $\mathfrak{Sol}^3$

The purpose of this section is to study tangent developable surfaces of general helices in $\mathfrak{Sol}^3$.

Developable surfaces are defined as the surfaces on which the Gaussian curvature is 0 everywhere. The developable surfaces are useful since they can be made out of sheet metal or paper by rolling a flat sheet of material without stretching it. Most large-scale objects such as airplanes or ships are constructed using un-stretched sheet metals, since sheet metals are easy to model and they have good stability and vibration properties. Moreover, sheet metals provide good fluid dynamic properties. In ship or airplane design, the problems usually stem from engineering concerns and in engineering design there has been a strong interest in developable surfaces.

The tangent developable of γ is a ruled surface

$$\mathfrak{W}_{(\gamma,\gamma')}(s,u) = \gamma(s) + u\gamma'(s). \tag{4.1}$$

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ is a unit speed non-geodesic general helix in $\mathfrak{Sol}^3$. Then, the parametric equations of tangent developable of γ are*

$$\begin{aligned}
x_{\mathfrak{W}}(s,u) &= \frac{\sin\mathfrak{P}e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]] \\
&\quad +u\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}+\mathfrak{C}_4, \\
y_{\mathfrak{W}}(s,u) &= \frac{\sin\mathfrak{P}e^{\cos\mathfrak{P}s+\mathfrak{C}_3}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]4.2 \quad (5) \\
&\quad +u\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]e^{\cos\mathfrak{P}s+\mathfrak{C}_3}+\mathfrak{C}_5, \\
z_{\mathfrak{W}}(s,u) &= \cos\mathfrak{P}s+u\cos\mathfrak{P}+\mathfrak{C}_3,
\end{aligned}$$

where $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4,\mathfrak{C}_5$ are constants of integration.

Proof. By the Frenet–Serret formula, we have the following equation

$$\mathbf{T} = \sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\mathbf{e}_1+\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\mathbf{e}_2+\cos\mathfrak{P}\mathbf{e}_3. \tag{4.3}$$

Using (2.4) in (3.9), we obtain

$$\mathbf{T} = (\sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{-z}, \sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{z}, \cos\mathfrak{P}). \tag{4.4}$$

In terms of Eqs. (2.4) and (3.4), we may give:

$$\mathbf{T} = (\sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}, \sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{\cos\mathfrak{P}s+\mathfrak{C}_3}, \cos\mathfrak{P}). \tag{4.5}$$

Consequently, the parametric equations of $\mathfrak{W}$ can be found from Eqs. (4.1), (4.5). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{W}$ its tangent developable surface in Sol space. Then the equation of tangent developable is*

$$\begin{aligned}
\mathfrak{W}(s,u) \quad = \quad & [\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \mathfrak{C}_1\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]] \\
& +\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3} + u\sin\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}]\mathbf{e}_1 4.6 \qquad (6) \\
& +[\frac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \cos\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]] \\
& +\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3} + u\sin\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] e^{\cos\mathfrak{P}s+\mathfrak{C}_3}]\mathbf{e}_2 \\
& +[\cos\mathfrak{P}s + u\cos\mathfrak{P} + \mathfrak{C}_3]\mathbf{e}_3,
\end{aligned}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

Proof. We assume that γ is a unit speed general helix.
Substituting (4.3) to (4.1), we have (4.6). Thus, the proof is completed.

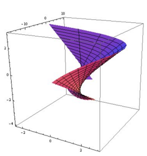
Thus, we proved the following:

Corollary 4.3. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{W}$ its tangent developable surface in Sol space. Then, unit normal of tangent* *developable of γ is*

$$
\begin{aligned}
\mathbf{n}_{\mathfrak{W}} \;\; = \;\; & [\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1 \\
& -[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2 \\
& +[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] \\
& -\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_3,
\end{aligned}
$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

We may use Mathematica in Theorem 4.1, yields



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