THE 8-PUZZLE PROBLEM (MATRIX THEORY IN A. I.)

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Abstract. In the mathematical aspects of Fuzzy Theory, we can find new ways of approximation by heuristic procedures. Also, by the matrix theory we can reach very interesting applications to A. I. For instance, the 8-puzzle problem will be a very illustrative example, showing the action of two different strategies until reaching the final state through the adequate sequence of steps.

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1.Introduction

The *problems* analyzed *in A. I.* can be classified according to their level. In a first level, the problems of: *decision, learning, perception, planning* and *reasoning.* In a second level, the tasks of: *classification, representation* and *search.*

When we formulate a problem, we depart of the statement (explanation) of it, in natural language. Fundamentally, its treatment is based in the *"level of knowledge"*, introduced by Newell in 1981, as *"abstract level of interpretation of systems, in A. I."* Also is basic the assert called *"Rationality Principle"*, according to which: *"if a system has the knowledge according to which one of its actions leads to one of its goals, then such action is carried out".*

The problems in A. I. can be classified in two types: *Search Problems and Representation Problems.*

For the application of the Search Procedures, we need the *characteristics:*

- the relationship between situations of the Domain and states

- the existence of one or more initial states, where it departs the Search Process

- the existence of operators, allowing the path between states

- the existence of final state for the process, signifying that the solution is reached

So, in the Search Process, we associate *states* with *nodes* and *arcs* or links with *operators*. The process consists of a progressive description: how, departing from the initial node, and selecting in each step the most plausible link, we can reach the final node.

We distinguish between:

- the *Blind Search* (without information of domain, which obliges, therefore, to an exhaustive exploration of the nodes, in the case of a graph).

- the *Heuristic Search* (with information of the domain, which allows the possibility of election between different paths, in the searching tree, because we have at our disposal information about the domain).

2.Heuristic search

Now, we can show an illustrative example, with two ways of application of the algorithm A^*, according their heuristic. We suppose an uniform cost (equal to 1), for each available operator.

Such heuristics can be, for instance:

i) In the first case, h_1, reflects the addition of the "Manhattan distances", of all the tokens of some concrete state on the board, respect to the final case.

ii) Whereas h_2 give us the number of tokens in bad position. Counting only one for each bad placed token.

The problem in question would be the so called *8-puzzle problem:* We have eight movable pieces (horizontal and vertically), each displacing with their moves the adjacent tokens, on a 3 x 3 board. Therefore, nine positions, one free, which can be denoted:

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The situation can be assimilated to a squared matrix of dimension 3:

$$\begin{pmatrix} & \uparrow & \\ \leftarrow & \blacksquare & \rightarrow \\ & \downarrow & \end{pmatrix}$$

But with lateral restrictions:

$$\begin{pmatrix} & \rceil \rightarrow \\ & \rceil \rightarrow \\ & \rceil \rightarrow \end{pmatrix}$$

$$\begin{pmatrix} \rceil \uparrow & \rceil \uparrow & \rceil \uparrow \\ & & \end{pmatrix}$$

$$\begin{pmatrix} \rceil \leftarrow & \\ \rceil \leftarrow & \\ \rceil \leftarrow & \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & & \\ \rceil \downarrow & \rceil \downarrow & \rceil \downarrow \end{pmatrix}$$

Respectively, not trespassing the last (third) column, the first row, the first column and the last (third) row.

We can suppose the initial state (corresponding to the root node):

$$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & \blacksquare \\ 8 & 7 & 5 \end{pmatrix}$$

until the final state, or node:

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & \blacksquare & 4 \\ 7 & 6 & 5 \end{pmatrix}$$

through the least number of steps.

The *rules of the 8-puzzle* could be:

a) If $\blacksquare \notin$ *first row* $\Rightarrow$ *we list up one position.*
Their operator denoted by :

$$\circ \uparrow R_1$$

b) If $\blacksquare \notin$ *third row* $\Rightarrow$ *we list down one position.*
Operator denoted by :

$$\circ \downarrow R_2$$

c) If $\blacksquare \notin$ *third col.* $\Rightarrow$ *we list right one position.*
Operator denoted by :

$$\circ \rightarrow R_3$$

d) If $\blacksquare \notin$ *first col.* $\Rightarrow$ *we list left one position.*
Operator denoted by :

$$R_4 \leftarrow \circ$$

In the first case (with the heuristic h_1), we have:

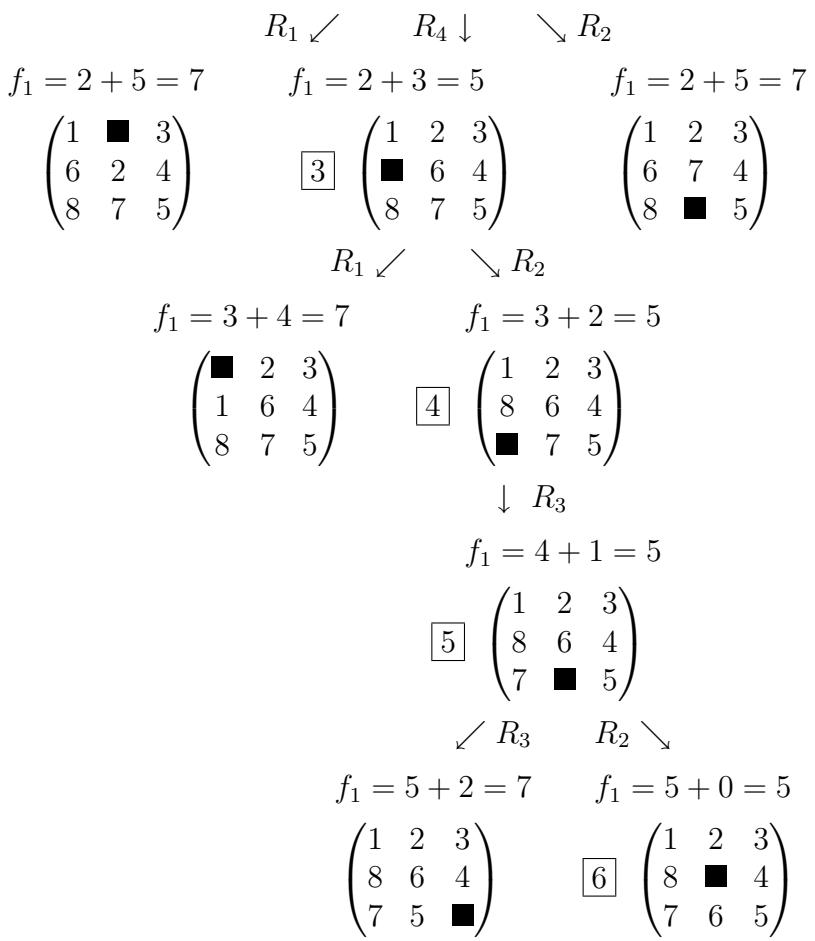
$$f_1 = g + h_1 = 0 + 5 = 5$$

$$\boxed{1} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & \blacksquare \\ 8 & 7 & 5 \end{pmatrix}$$

$$R_1 \swarrow \qquad R_4 \downarrow \qquad \searrow R_2$$

$$f_1 = 1 + 6 = 7 \qquad f_1 = 1 + 4 = 5 \qquad f_1 = 1 + 6 = 7$$

$$\begin{pmatrix} 1 & 2 & \blacksquare \\ 6 & 4 & 3 \\ 8 & 7 & 5 \end{pmatrix} \qquad \boxed{2} \begin{pmatrix} 1 & 2 & 3 \\ 6 & \blacksquare & 4 \\ 8 & 7 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 8 & 7 & \blacksquare \end{pmatrix}$$



If an entry is bad placed in row and column, it must be counted twice, in the heuristic h_1. The first summand, for f_1, indicates the level (g), whereas the second (h_1) the addition of distances, in both senses, horizontal and vertical.

The sequence of labels shows the steps to give until reaching the solution, with the first heuristic. So:

$$\{R_4,\ R_4,\ R_2,\ R_3,\ R_1\}$$

would be the adequate sequence of operators necessary to obtain such solution.

In the *second case* (ii), for the heuristic h_2, which reflects the number of tokens wrongly located, respect to the initial configuration, but counting only one each time, independent if the bad location is in row, in column, or in both. The g continues as the level counter. Then, the graph of searching could be:

$$f_2 = g + h_2 = 0 + 4 = 4$$

$$\boxed{1}\ \begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & \blacksquare \\ 8 & 7 & 5 \end{pmatrix}$$

$$R_1 \swarrow \qquad R_4 \downarrow \qquad \searrow R_2$$

$f_2 = 1 + 5 = 6$ $\begin{pmatrix} 1 & 2 & \blacksquare \\ 6 & 4 & 3 \\ 8 & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 1 + 3 = 4$ $\boxed{2}\ \begin{pmatrix} 1 & 2 & 3 \\ 6 & \blacksquare & 4 \\ 8 & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 1 + 5 = 6$ $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 8 & 7 & \blacksquare \end{pmatrix}$

$$R_1 \swarrow \qquad R_4 \downarrow \qquad \searrow R_2$$

$f_2 = 2 + 4 = 6$ $\begin{pmatrix} 1 & \blacksquare & 3 \\ 6 & 2 & 4 \\ 8 & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 2 + 3 = 5$ $\boxed{4}\ \begin{pmatrix} 1 & 2 & 3 \\ \blacksquare & 6 & 4 \\ 8 & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 2 + 3 = 5$ $\boxed{3}\ \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 8 & \blacksquare & 5 \end{pmatrix}$

$$R_1 \swarrow \qquad R_2 \searrow \qquad\qquad R_3 \swarrow \qquad \searrow R_4$$

$f_2 = 3 + 4 = 7$ $\begin{pmatrix} \blacksquare & 2 & 3 \\ 1 & 6 & 4 \\ 8 & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 3 + 2 = 5$ $\boxed{5}\ \begin{pmatrix} 1 & 2 & 3 \\ 8 & 6 & 4 \\ \blacksquare & 7 & 5 \end{pmatrix}$ $\qquad$ $f_2 = 3 + 4 = 7$ $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ 8 & 5 & \blacksquare \end{pmatrix}$ $\qquad$ $f_2 = 3 + 3 = 6$ $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 4 \\ \blacksquare & 8 & 5 \end{pmatrix}$

$$\downarrow R_3$$

$$f_2 = 4 + 1 = 5$$

$$\boxed{6}\ \begin{pmatrix} 1 & 2 & 3 \\ 8 & 6 & 4 \\ 7 & \blacksquare & 5 \end{pmatrix}$$

$$\swarrow R_3 \qquad R_2 \searrow$$

$$f_2 = 5 + 2 = 7 \qquad f_2 = 5 + 0 = 5$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & 6 & 4 \\ 7 & 5 & \blacksquare \end{pmatrix} \qquad \boxed{7} \begin{pmatrix} 1 & 2 & 3 \\ 8 & \blacksquare & 4 \\ 7 & 6 & 5 \end{pmatrix}$$

If an entry is bad situated in row and column, must be counted only once, when the heuristic is h_2. The first summand, for f_2, indicates the level (g), whereas the second (h_2) reflects the number of tokens wrongly placed, only counted once, it does not matter whether it fails horizontally or vertically.

We can see, through the labelling of steps, that the algorithm A^**is more efficient with the first heuristic function,* h_1.

As the heuristic function h_1 is *monotonic*:

$$h_1(n) \leq c(n,\ n') + h_1(n'),\ \textit{for each pair of nodes}\ (n,\ n')$$

where c indicates the cost of the path between the nodes n and n', we do not need reorientations.

Also is true:

$$\forall\ \textit{node}\ n,\ h_1(n)\ \leq\ h_1^*(n)$$

that is, the heuristic function is *admissible*. Therefore, we have the best solution.

Here, h_1^* would be the real distance until the final node. The value of the estimation $h_1(n)$, in each node, must not exceed the effective value, $h_1^*(n)$.

The number of tokens badly disposed must be constantly referred to the last node.

We will think such situation as a problem of game with two adversaries, each one with an 8-puzzle and having previously selected the first or the second heuristic, as respective strategies.

Then, the winner must be the player with the first strategy: it suffices with fewer steps, because their algorithm is more efficient.

Also we can increase the dimension of the puzzle, to a *15-puzzle* (4 x 4 board, with one free position), and so on... In general, until a (n^2-1)-puzzle, provided with a board in form of square *n x n,* with n^2-1 movable pieces and one free position.

I hope (with this illustrative example) shows clearly the way of application of Matrix Theory in some typical problems of A. I.

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