RESIDUAL TRANSCENDENTAL EXTENSIONS OF A VALUATION

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Abstract. Let K be a field and v a valuation on K. We discuss some properties of extensions w of v to the field $K(X_1, \ldots, X_n)$ of rational functions in n variables over K, for which the residue field of w has transcendence degree n over the residue field of v.

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1.Introduction

Let K be a field and v a valuation on K. Residual transcendental extensions of v to the field $K(X)$ of rational functions in one variable over K have been studied in a number of articles, including [8], [9], [10], [11], [13], [1], [2], [3] and [14]. Nagata [8] conjectured that if w is a residual transcendental extension of v to $K(X)$, then the residue field k_w of w is a simple transcendental extension of a finite algebraic extension of the residue field k_v of v. The problem was solved by Ohm [10] and by Popescu [13]. Further questions on residual transcendental extensions of a valuation have been considered by Ohm, and in the process he stated in [11] three conjectures concerning some natural numbers like ramification index and residual degree. These three problems were later solved in [1]. The main ingredient used in [1] to investigate these and other related questions was a theorem of characterization of residual transcendental extensions of v to $K(X)$. The situation is a lot more complicated if one replaces $K(X)$ by the field $K(X_1, \ldots, X_n)$ of rational functions in n variables over K and one attempts to describe all the extensions w of v to $K(X_1, \ldots, X_n)$ for which the residue field k_w of w has transcendence degree n over the residue field k_v of v. One could of course use the characterization theorem from [1] mentioned above in order to describe the valuation w on $K(X_1, \ldots, X_n)$ in n steps, by taking a residual transcendental extension of v to $K(X_1)$, followed by a residual transcendental extension of this valuation to $K(X_1, X_2)$, followed by a residual transcendental extension from $K(X_1, X_2)$ to $K(X_1, X_2, X_3)$, and so on, until one obtains the valuation w on $K(X_1, \dots, X_n)$. This description however is not satisfactory for the following reason. In the characterization theorem from [1] one expresses various important aspects of the behavior of the valuation w in terms of a so called minimal pair of definition. Now, in order to successfully apply this theorem in practice one needs some sort of knowledge of these minimal pairs of definition. If we assume that one knows well enough the given valuation v on K, and a fixed extension $\bar{v}$ of v to a fixed algebraic closure $\bar{K}$ of K, then we have a pretty good knowledge of minimal pairs over K with respect to $\bar{v}$. See for example further developments of the subject in the case of a local field, in [15], [7], [16], [4], [5], [6], [12]. Returning to the general problem concerned with the description of the extension w of the valuation v to $K(X_1, \dots, X_n)$, the first step in this extension, namely from K to $K(X_1)$ is well understood, by the theory from [1] mentioned above. But the next steps, from $K(X_1)$ to $K(X_1, X_2)$, from $K(X_1, X_2)$ to $K(X_1, X_2, X_3)$, and so on, are not well understood, because of our lack of knowledge of minimal pairs over the intermediate fields $K(X_1)$, $K(X_1, X_2), \dots, K(X_1, \dots, X_{n-1})$ which would appear in such a description of the valuation w. There are however some meaningful things one can say in this full generality. We consider below a fundamental inequality obtained by Ohm [11], which was also proved in [1] as a consequence of the characterization theorem for residual transcendental extensions of a valuation v from K to $K(X)$, and we discuss an analogue of this inequality in the case when $K(X)$ is replaced by a field of rational functions over K in several variables.

2. The case $n = 1$

As was mentioned in the introduction, this case is better understood than the case of a general n.

Let K be a field and v a valuation on K. Denote by k_v, Γ_v and O_v the residue field, the value group and the valuation ring of v respectively. If x is in O_v we denote by x^* the canonical image of x in k_v.

Let w be an extension of v to the field $K(X)$ of rational functions in one variable over K, and denote by k_w, Γ_w and O_w the residue field, the value group and the valuation ring of w respectively. We shall canonically identify k_v with a subfield of k_w and Γ_v with a subgroup of Γ_w.

The valuation w on $K(X)$ is said to be a residual transcendental extension of v provided that k_w is a transcendental extension of k_v. In this case the transcendence degree of k_w over k_v equals 1.

Nagata's conjecture mentioned in the introduction, proved by Ohm and Popescu, states that if w is a residual transcendental extension of v to $K(X)$, then k_w is a simple transcendental extension of a finite algebraic extension of k_v.

For the remaining part of this section we will assume that w is a residual transcendental extension of v to $K(X)$. It is then easy to see that Γ_v is a subgroup of finite index in Γ_w. We denote this index by $e(w/v)$.

Let k be the algebraic closure of k_v in k_w. It is easy to see that k is a finite extension of k_v. We denote the degree of this extension by $f(w/v)$.

Since w is a residual transcendental extension of v, there are elements r of O_w for which r^* is transcendental over k_v. We denote by $\deg(w/v)$ the smallest positive integer d for which there exists an element r in O_w such that $[K(X) : K(r)] = d$ and r^* is transcendental over k_v.

As was proved by Ohm, the natural numbers $e(w/v)$, $f(w/v)$ and $\deg(w/v)$ satisfy the fundamental inequality

$$e(w/v)f(w/v) \leq \deg(w/v).$$

This holds true for any residual transcendental extension w of v to $K(X)$. There are cases when we have equality,

$$e(w/v)f(w/v) = \deg(w/v).$$

As was shown in [1], three situations when the above equality holds true, are the following:

1) v Henselian and char $k_v = 0$;

2) v of rank one and char $k_v = 0$;

3) v of rank one and discrete.

This answers three conjectures raised by Ohm. The main tool used in [1] to investigate these and other, related problems is a theorem of characterization of residual transcendental extensions of v to $K(X)$. Before we state the theorem we need to introduce some more notation and terminology.

Let $\bar{K}$ be a fixed algebraic closure of K, and let $\bar{v}$ be a fixed extension of v to $\bar{K}$. If w is an extension of v to $K(X)$, then there exists an extension $\bar{w}$ of w to $\bar{K}(X)$ such that $\bar{w}$ is also an extension of $\bar{v}$. If w is a residual transcendental extension of v to $K(X)$, then $\bar{w}$ is a residual transcendental extension of $\bar{v}$ to $\bar{K}(X)$. One shows in this case that there exists a pair (α, δ), with $\alpha \in \bar{K}$ and $\delta \in \Gamma_{\bar{v}}$ such that $\bar{w}$ is the valuation on $\bar{K}(X)$ defined by inf, $\bar{v}$, α and δ. By this one means that for any polynomial $F(X)$ in $\bar{K}[X]$, if one uses Taylor's expansion to write $F(X)$ in the form

$$F(X) = c_0 + c_1(X-\alpha) + \cdots + c_m(X-\alpha)^m,$$

then one has

$$\bar{w}(F(X)) = \inf_i \{\bar{v}(c_i) + i\delta\}.$$

Next, for any rational function $R(X) = F(X)/G(X)$, with $F(X), G(X)$ in $\bar{K}[X]$, one has

$$\bar{w}(R(X)) = \bar{w}(F(X)) - \bar{w}(G(X)).$$

Therefore, with $\bar{v}$ fixed, $\bar{w}$ is uniquely determined by the pair (α, δ), which is then called a pair of definition for $\bar{w}$. One shows that two pairs (α_1, δ_1) and (α_2, δ_2) define the same valuation $\bar{w}$ if and only if $\delta_1 = \delta_2$ and $\bar{v}(\alpha_1 - \alpha_2) \geq \delta_1$.

By a minimal pair of definition for $\bar{w}$ with respect to K one means a pair of definition (α, δ) for $\bar{w}$, for which the degree of α over K is minimal.

Thus for every residual transcendental extension w of v to $K(X)$, there is a minimal pair of definition for $\bar{w}$, and, if (α, δ) and (α', δ) are two minimal pairs, then $[K(\alpha) : K] = [K(\alpha') : K]$.

For any α in $\bar{K}$ and any γ in $\Gamma_{\bar{v}}$ let us denote by $e(\gamma, K(\alpha))$ the smallest positive integer e for which $e\gamma$ belongs to the value group $\Gamma_{K(\alpha)}$ of the restriction of $\bar{v}$ to $K(\alpha)$.

We can now state the following theorem of characterization of residual transcendental extensions of v to $K(X)$ from [1].

Theorem 1. *Let v be a valuation on a field K and let w be a residual transcendental extension of v to $K(X)$. Let $\alpha \in \bar{K}$ and $\delta \in \Gamma_{\bar{v}}$ such that (α, δ) is a minimal pair of definition for $\bar{w}$ with respect to K. Then:*

(a) If we denote $[K(\alpha) : K] = n$, then for every polynomial $g(X)$ in $K[X]$ such that $\deg g(X) < n$*, one has*

$$w(g(X)) = \bar{v}(g(\alpha)).$$

(b) For the monic minimal polynomial $f(X)$ of α over K, let $\gamma = w(f(X))$ and $e = e(\gamma, K(\alpha))$. Then there exists $l(X)$ in $K[X]$ with $\deg l < n$ such that for $r = f^e/l$ one has $w(r) = 0$, and r^* is transcendental over k_v.

(c) If v_1 denotes the restriction of $\bar{v}$ to $K(\alpha)$, then

$$\deg(w/v) = ne, \quad e(w/v) = e(v_1/v)e.$$

(d) The field k_{v_1} can be canonically identified with the algebraic closure of k_v in k_w, and

$$f(w/v) = f(v_1/v).$$

3.The case of a general n

Let K be a field and v a valuation on K. As in the previous section we denote by k_v, Γ_v and O_v the residue field, the value group and the valuation ring of v respectively.

Let w be an extension of v to the field $K(X_1, \ldots, X_n)$ of rational functions in n variables $X_1, \ldots, X_n$ over K. Denote by k_w, Γ_w and O_w the residue field, the value group and respectively the valuation ring of w.

We shall canonically identify k_v with a subfield of k_w and Γ_v with a subgroup of Γ_w. For any x in O_w we denote by x^* the canonical image of x in k_w. In what follows we will only work with extensions w of v to $K(X_1, \ldots, X_n)$ for which the transcendence degree of k_w over k_v equals n.

As we shall see below, in this case Γ_v will be a subgroup of finite index in Γ_w, and we denote this index by $e(w/v)$.

Let k be the algebraic closure of k_v in k_w. Then k is a finite extension of k_v, and we denote the degree of this extension by $f(w/v)$.

By analogy with the definition of $\deg(w/v)$ from the previous section, we now denote by $\deg(w/v)$ the smallest positive integer d for which there exist elements $r_1, \ldots, r_n$ in O_w such that $[K(X_1, \ldots, X_n) : K(r_1, \ldots, r_n)] = d$ and $r_1^*, \ldots, r_n^*$ are algebraically independent over k_v.

Then we have the following analogue of the fundamental inequality from the previous section involving the natural numbers $e(w/v)$, $f(w/v)$ and $\deg(w/v)$.

THEOREM 2. *Let K be a field and v a valuation on K. Let w be an extension of v to $K(X_1, \ldots, X_n)$ such that the residue field k_w of w has transcendence degree n over the residue field k_v of v. Then*

$$\deg(w/v) \geq e(w/v)f(w/v).$$

Proof. Let K be a field, v a valuation on K, and w an extension of v to $K(X_1, \dots, X_n)$ such that the residue field k_w of w has transcendence degree n over the residue field k_v of v. Choose $r_1, \dots, r_n$ such that

$$[K(X_1, \dots, X_n) : K(r_1, \dots, r_n)] = \deg(w/v),$$

and such that $r_1^*, \dots, r_n^*$ are algebraically independent over k_v. Denote by k the algebraic closure of k_v in k_w. Next, choose elements $u_1, \dots, u_m$ in O_w such that their images $u_1^*, \dots, u_m^*$ in k_w belong to k and are linearly independent over k_v. We also choose $v_1, \dots, v_s$ in O_w such that the elements $w(v_1), \dots, w(v_s)$ of Γ_w belong to distinct cosets of Γ_w modulo Γ_v, in other words the images of $w(v_1), \dots, w(v_s)$ in the quotient group Γ_w/Γ_v are distinct. Let now S be a sum of the form

$$S = \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq m}} c_{ij} v_i u_j,$$

with c_{ij} in $K(r_1, \dots, r_n)$, for $1 \leq i \leq s$ and $1 \leq j \leq m$. We claim that for any such sum S one has

$$w(S) = \min_{\substack{1 \leq i \leq s \\ 1 \leq j \leq m}} w(c_{ij} v_i u_j).$$

Indeed, let us denote

$$\gamma = \min_{\substack{1 \leq i \leq s \\ 1 \leq j \leq m}} w(c_{ij} v_i u_j).$$

We write S in the form

$$S = S_1 + S_2,$$

where in S_1 we collect all the terms $c_{ij} v_i u_j$ with $w(c_{ij} v_i u_j) = \gamma$, and in S_2 we put all the terms $c_{ij} v_i u_j$ for which $w(c_{ij} v_i u_j) > \gamma$. Then $w(S_2) > \gamma$, so clearly the claim will be proved if we show that $w(S_1) = \gamma$.

At this point we note that $w(c_{ij})$ belongs to Γ_v for any $1 \leq i \leq s$ and any $1 \leq j \leq m$. For, fix i, j and write c_{ij} as a quotient of two polynomials in $r_1, \dots, r_n$ with coefficients in O_v, say

$$c_{ij} = \frac{P(r_1, \ldots, r_n)}{Q(r_1, \ldots, r_n)}$$

with $P(r_1, \ldots, r_n)$, $Q(r_1, \ldots, r_n)$ in $O_v[r_1, \ldots, r_n]$. If $w(c_{ij})$ does not belong to Γ_v, then at least one of $w(P(r_1, \ldots, r_n))$ or $w(Q(r_1, \ldots, r_n))$ does not belong to Γ_v. Say $w(P(r_1, \ldots, r_n)) \notin \Gamma_v$.

Let $b \in O_v$ be one of the coefficients of the polynomial $P(r_1, \ldots, r_n)$ for which $v(b)$ is minimal. Then $P(r_1, \ldots, r_n)/b$ is a polynomial in $r_1, \ldots, r_n$ with coefficients in O_v, and at least one of these coefficients is a unit. Then the image of $P(r_1, \ldots, r_n)/b$ in k_w will be a polynomial in $r_1^*, \ldots, r_n^*$ with coefficients in k_v, and not all these coefficients vanish in k_v. Since $r_1^*, \ldots, r_n^*$ are algebraically independent over k_v, it follows that the image of $P(r_1, \ldots, r_n)/b$ in k_w is not the zero element of k_w. Therefore

$$w(P(r_1, \ldots, r_n)/b) = 0,$$

which implies that

$$w(P(r_1, \ldots, r_n)) = w(b) = v(b) \in \Gamma_v,$$

contrary to our assumption that $w(P(r_1, \ldots, r_n))$ does not belong to Γ_v. We conclude that all the elements $w(c_{ij})$ of Γ_w belong to Γ_v. Note also that since $u_1^*, \ldots, u_m^*$ are nonzero elements of k, we have $w(u_j) = 0$ for any $1 \leq j \leq m$.

We deduce that for any $1 \leq i \leq s$ and any $1 \leq j \leq m$, the image of $w(c_{ij}v_iu_j)$ in Γ_w/Γ_v coincides with the image of $w(v_i)$ in Γ_w/Γ_v. For terms belonging to S_1, this image further coincides with the image of γ in Γ_w/Γ_v. It follows that all the terms $c_{ij}v_iu_j$ which appear in S_1 correspond to the same value of i, call it i_0, which is uniquely determined such that $w(v_{i_0})$ and γ have the same image in Γ_w/Γ_v.

Therefore S_1 has the form

$$S_1 = \sum_{j \in J} c_{i_0 j} v_{i_0} u_j,$$

for some nonempty subset J of the set $\{1, \ldots, m\}$. Here

$$w(c_{i_0 j}) = \gamma - w(v_{i_0}) - w(u_j) = \gamma - w(v_{i_0})$$

for any j in J. Fix now an element j_0 in J. Then the required equality $w(S_1) = \gamma$ will follow if we prove that

$$w\left(\sum_{j\in J} c_{i_0 j} u_j\right) = w(c_{i_0 j_0}).$$

This is equivalent to

$$w\left(\sum_{j\in J} a_j u_j\right) = 0,$$

where for any j in J, $a_j = c_{i_0 j}/c_{i_0 j_0}$ is an element of $K(r_1, \ldots, r_n)$ for which $w(a_j) = 0$. Let us assume that

$$w\left(\sum_{j\in J} a_j u_j\right) > 0.$$

Then one has

$$\sum_{j\in J} a_j^* u_j^* = 0$$

in k_w. Here each a_j^* is a rational function of $r_1^*, \ldots, r_n^*$ with coefficients in k_v, and each u_j^* belongs to k. Multiplying the above equality by a suitable element of $k_v[r_1^*, \ldots, r_n^*]$ we obtain an equality of the form

$$\sum_{j\in J} F_j(r_1^*, \ldots, r_n^*) u_j^* = 0,$$

where each $F_j(r_1^*, \ldots, r_n^*)$ belongs to $k_v[r_1^*, \ldots, r_n^*]$. Since $r_1^*, \ldots, r_n^*$ are algebraically independent over k_v, and therefore also over k, it follows that in the above equality the corresponding coefficients to any given monomial in $r_1^*, \ldots, r_n^*$ must cancel. This produces nontrivial linear combinations of the u_j^*, $j \in J$, with coefficients in k_v, which vanish, contradicting our assumption that the u_j^* are linearly independent over k_v. This proves our claim that

$$w(S) = \min_{\substack{1\le i\le s\\ 1\le j\le m}} w(c_{ij} v_i u_j).$$

As a consequence it follows that $S = 0$ if and only if all the coefficients c_{ij} are zero. In other words this says that the elements $v_i u_j$, $1 \le i \le s$, $1 \le j \le m$ of $K(X_1, \ldots, X_n)$ are linearly independent over the field $K(r_1, \ldots, r_n)$.

Let us assume now that at least one of $e(w/v)$, $f(w/v)$ is infinite, or that both are finite and their product is strictly larger than $\deg(w/v)$. Then we can find positive integers m, s such that $ms > \deg(w/v)$ and we can find elements $u_1, \ldots, u_m, v_1, \ldots, v_s$ of O_w such that the images $u_1^*, \ldots, u_m^*$ of $u_1, \ldots, u_m$ in k_w belong to k and are linearly independent over k_v, and the elements $w(v_1), \ldots, w(v_s)$ of Γ_w have distinct images in the quotient group Γ_w/Γ_v. Then we know that the elements $v_i u_j$, $1 \leq i \leq s$, $1 \leq j \leq m$ of $K(X_1, \ldots, X_n)$ are linearly independent over $K(r_1, \ldots, r_n)$. But

$$[K(X_1, \ldots, X_n) : K(r_1, \ldots, r_n)] = \deg(w/v) < ms,$$

which implies that the elements $v_i u_j$, $1 \leq i \leq s$, $1 \leq j \leq m$ can not be linearly independent over $K(r_1, \ldots, r_n)$. The contradiction obtained shows that both $e(w/v)$ and $f(w/v)$ are finite, and

$$e(w/v)f(w/v) \leq \deg(w/v),$$

which completes the proof of the theorem.

We now consider the cases when the inequality from the statement of Theorem 2 becomes an equality. We show that in such cases an analogue of Nagata's conjecture holds true.

THEOREM 3. *Let K be a field and v a valuation on K. Let w be an extension of v to $K(X_1, \ldots, X_n)$ such that the residue field k_w of w has transcendence degree n over the residue field k_v of v. Assume that*

$$\deg(w/v) = e(w/v)f(w/v).$$

Then k_w is isomorphic to the field of rational functions in n variables over a finite extension of k_v.

Proof. Let K be a field, v a valuation on K, and w an extension of v to $K(X_1, \ldots, X_n)$ such that the residue field k_w of w has transcendence degree n over the residue field k_v of v. Choose $r_1, \ldots, r_n$ such that

$$[K(X_1, \ldots, X_n) : K(r_1, \ldots, r_n)] = \deg(w/v),$$

and such that $r_1^*, \ldots, r_n^*$ are algebraically independent over k_v. Denote by k the algebraic closure of k_v in k_w. The theorem will be proved if we show that

$$k_w = k(r_1^*, \ldots, r_n^*).$$

As in the proof of Theorem 2, choose elements $u_1, \ldots, u_m$ in O_w such that their images $u_1^*, \ldots, u_m^*$ in k_w belong to k and are linearly independent over k_v. Also, choose $v_1, \ldots, v_s$ in O_w such that the elements $w(v_1), \ldots, w(v_s)$ of Γ_w have distinct images in the quotient group Γ_w/Γ_v. Here we take $m = f(w/v)$ and $s = e(w/v)$. Note that then exactly one of the elements $w(v_1), \ldots, w(v_s)$, say $w(v_1)$, belongs to Γ_v. We may then choose for simplicity $v_1 = 1$. We know from the proof of Theorem 2 that the elements $v_i u_j$, $1 \leq i \leq s$, $1 \leq j \leq m$ of $K(X_1, \ldots, X_n)$ are linearly independent over $K(r_1, \ldots, r_n)$. Since their number is $ms = e(w/v)f(w/v)$, which by the assumption from the statement of the theorem equals $\deg(w/v)$, which further equals the degree of $K(X_1, \ldots, X_n)$ over $K(r_1, \ldots, r_n)$, it follows that the elements $v_i u_j$, $1 \leq i \leq s$, $1 \leq j \leq m$ form a basis of $K(X_1, \ldots, X_n)$ over $K(r_1, \ldots, r_n)$.

Let us now take any element t of k_w, and choose a representative z of t in the valuation ring O_w. Express z in terms of the above basis, say

$$z = \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq m}} c_{ij} v_i u_j,$$

with c_{ij} in $K(r_1, \ldots, r_n)$, for $1 \leq i \leq s$ and $1 \leq j \leq m$. We know from the proof of Theorem 2 that

$$w(z) = \min_{\substack{1 \leq i \leq s \\ 1 \leq j \leq m}} w(c_{ij} v_i u_j).$$

We write $z = S_1 + S_2$, where S_1 and S_2 have the same meaning as in the proof of Theorem 2. Then we know that

$$S_1 = \sum_{j \in J} c_{i_0 j} v_{i_0} u_j,$$

for some integer i_0 in $\{1, \ldots, s\}$ and some subset J of $\{1, \ldots, m\}$. Since $w(S_1) = w(z) = 0$, this forces $i_0 = 1$, $v_{i_0} = v_1 = 1$, hence

$$S_1 = \sum_{j \in J} c_{1j} u_j.$$

On the other hand we know that

$$w(z - S_1) = w(S_2) > w(z) = 0,$$

therefore the image of S_1 in the residue field k_w coincides with the image of z in k_w, that is, it coincides with t. We also know that $w(c_{1j})$ has the same value for any j in J. In our case this value is zero, so each c_{1j} is an element of $K(r_1, \ldots, r_n)$ which also belongs to O_w.

Lastly, by taking the image of S_1 in the residue field k_w, we find that

$$t = S_1^* = \sum_{j \in J} c_{1j}^* u_j^*.$$

Here each c_{1j}^* belongs to $k_v(r_1^*, \ldots, r_n^*)$, and each u_j^* belongs to k. It follows that t belongs to $k(r_1^*, \ldots, r_n^*)$. Since t was an arbitrary element of k_w, we conclude that

$$k_w = k(r_1^*, \ldots, r_n^*),$$

which completes the proof of the theorem.

References

[1] V. Alexandru, N. Popescu, A. Zaharescu, *A theorem of characterization of residual transcendental extensions of a valuation*, J. Math. Kyoto Univ. **28** (1988), no. 4, 579–592.

[2] V. Alexandru, N. Popescu, A. Zaharescu, *Minimal pairs of definition of a residual transcendental extension of a valuation*, J. Math. Kyoto Univ. **30** (1990), no. 2, 207–225.

[3] V. Alexandru, N. Popescu, A. Zaharescu, *All valuations on K(X)*, J. Math. Kyoto Univ. **30** (1990), no. 2, 281–296.

[4] V. Alexandru, N. Popescu, A. Zaharescu, *On the closed subfields of* C_p, J. Number Theory **68** (1998), no. 2, 131–150.

[5] V. Alexandru, N. Popescu, A. Zaharescu, *The generating degree of* C_p, Canad. Math. Bull. **44** (2001), no. 1, 3–11.

[6] V. Alexandru, N. Popescu, A. Zaharescu, *Trace on* C_p, J. Number Theory **88** (2001), no. 1, 13–48.

[7] A. Ioviţă, A. Zaharescu, *Completions of r.a.t.-valued fields of rational functions*, J. Number Theory **50** (1995), no. 2, 202–205.

[8] M. Nagata, *A theorem on valuation rings and its applications*, Nagoya Math. J. **29** (1967), 85–91.

[9] J. Ohm, *Simple transcendental extensions of valued fields*, J. Math. Kyoto Univ. **22** (1982), 201–221.

[10] J. Ohm, *The ruled residue theorem for simple transcendental extensions of valued fields*, Proc. Amer. Math. Soc. **89** (1983), 16–18.

[11] J. Ohm, *Simple transcendental extensions of valued fields. II: A fundamental inequality*, J. Math. Kyoto Univ. **25** (1985), 583–596.

[12] A. Popescu, N. Popescu, M. Vâjâitu, A. Zaharescu, *Chains of metric invariants over a local field*, Acta Arith. **103** (2002), no. 1, 27–40.

[13] N. Popescu, *On a problem of Nagata in valuation theory*, Rev. Roumaine Math. Pures Appl. **31** (1986), 639–641.

[14] N. Popescu, A. Zaharescu, *On a class of valuations on* $K(X)$, XIth National Conference of Algebra (Constanţa, 1994). An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **2** (1994), 120–136.

[15] N. Popescu, A. Zaharescu, *On the structure of the irreducible polynomials over local fields*, J. Number Theory **52** (1995), no. 1, 98–118.

[16] N. Popescu, A. Zaharescu, *On the main invariant of an element over a local field*, Portugal. Math. **54** (1997), no. 1, 73–83.

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