ON SOME CONDITIONS FOR n-STARLIKENESS

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Abstract. In this paper we obtain a sufficient condition for n-starlikeness of the form: $(2\alpha-1)\left(\frac{D^{n+1}f(z)}{D^n f(z)}-1\right)+(1-\alpha)\frac{D^{n+2}f(z)}{D^n f(z)} \prec h(z)$where $h(z)$ is an univalent function in the unit disc U and D^n is the Sălăgean differential operator.

1. Introduction

Let $\mathcal{A}_n$, $n \in N^*$ denote the class of functions of the form: $f(z) = z + \sum\limits_{k=n+1}^{\infty} a_k z^k$ wich are analytic in the unit disc $U = \{z; z \in C,\ |z| < 1\}$ and $\mathcal{A}_1 = \mathcal{A}$.

We note $S^* = \left\{f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}$the class of functions $f \in \mathcal{A}$ which are *starlike* in the unit disc.

We denote by K *the* class of functions $f \in \mathcal{A}$ which are *convex* in the unit disc U, that is K=$\left\{f \in \mathcal{A} : \mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\right\}$.

For $f \in \mathcal{A}_n$ we define the Sălăgean differential operator D^n ([2]) by

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \end{aligned}$$

and $D^{n+1}f(z) = D(D^n f(z))$; $\quad n \in N^* \cup \{0\}$.

Let $\alpha \in [0,1)$ and let $n \in N$. The class $S_n(\alpha)$ named the class of $n-$*starlike function of order* α is defined by $S_n(\alpha) = \left\{f \in \mathcal{A} : \mathrm{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \alpha, z \in U\right\}$.

Theorem 1.[1] *Let q be a univalent function in U and let the functions θ, ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike in U*

(ii)$\mathrm{Re}\frac{zh'(z)}{Q(z)}=\mathrm{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right]>0$,$z\in U$.

If p is analytic in U, with $p(0)=q(0)$, $p(U)\subset D$, and$\theta\left(p(z)\right)+zp'(z)\phi\left(p(z)\right)\prec\theta\left(q(z)\right)+zq'(z)\phi\left(q(z)\right)=h(z)$ then $p\left(z\right)\prec q\left(z\right)$ and q is the best dominant.

2. Main results

Theorem 2.*Let $\alpha\in[0,1]$, $n\in N$, $f(z)\in\mathcal{A}$ and let q be a convex function in U with $q\left(0\right)=1$ and $\mathrm{Re}q\left(z\right)>\frac{1}{2}$, $z\in U$.If*

$$(2\alpha-1)\left(\frac{D^{n+1}f\left(z\right)}{D^{n}f\left(z\right)}-1\right)+(1-\alpha)\frac{D^{n+2}f\left(z\right)}{D^{n}f\left(z\right)} \tag{1}$$

$$\prec\ (1-\alpha)\,q^{2}\left(z\right)+(2\alpha-1)\left(q\left(z\right)-1\right)+(1-\alpha)\,zq'\left(z\right)\equiv h\left(z\right), \tag{2}$$

then

$$\frac{D^{n+1}f(z)}{D^{n}f\left(z\right)}\prec q\left(z\right),z\in U. \tag{3}$$

and q is the best dominant of (2).

Proof. For $\alpha=1$ it is evident. Suppose that $0\leq\alpha<1$. In Theorem 1 we choose

$$\begin{aligned}\theta\left(w\right)&=(1-\alpha)\,w^{2}+(2\alpha-1)w-\alpha\\ \phi\left(w\right)&=1-\alpha\end{aligned}$$

and we have in (i) $Q\left(z\right)=(1-\alpha)\,zq'(z)$ is starlike in U, because q is convex.

$$(ii)\mathrm{Re}\frac{zh'\left(z\right)}{Q\left(z\right)}=\mathrm{Re}\left[\frac{\theta'\left(q\left(z\right)\right)}{\phi\left(q\left(z\right)\right)}+\frac{zQ'\left(z\right)}{Q\left(z\right)}\right]=\mathrm{Re}\left[2q\left(z\right)+\frac{2\alpha-1}{1-\alpha}+\frac{zQ'\left(z\right)}{Q\left(z\right)}\right]>$$

$$>2\frac{1}{2}+\frac{2\alpha-1}{1-\alpha}+\mathrm{Re}\left[\frac{zQ'\left(z\right)}{Q\left(z\right)}\right]=\frac{\alpha}{1-\alpha}+\mathrm{Re}\left[\frac{zQ'\left(z\right)}{Q\left(z\right)}\right]>0,\ z\in U.$$

The conditions of Theorem 1 are satisfied and for $p(z) = 1 + p_1 z + ...$ which satisfies
$(1-\alpha)\, p^2(z) + (2\alpha - 1)(p(z) - 1) + (1-\alpha)\, zp'(z) \prec h(z)$
we have $p(z) \prec q(z)$ and q is the best dominant.
Let $p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}$ then

$$\begin{aligned} & (1-\alpha)\, p^2(z) + (2\alpha-1)(p(z)-1) + (1-\alpha)\, zp'(z) \\ = \ & (2\alpha-1)\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right) + (1-\alpha)\frac{D^{n+2}f(z)}{D^n f(z)} \prec \\ \prec \ & (1-\alpha)\, q^2(z) + (2\alpha-1)(q(z)-1) + (1-\alpha)\, zq'(z) \end{aligned}$$

which implies that

$$\tfrac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

Remark. For $n = 0$ we obtain the result given in [3].

Corollary 1. Let $\alpha \in [0,1]$ and let $f(z) \in \mathcal{A}$, that satisfy

$$\mathrm{Re}\left[(2\alpha-1)\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right) + (1-\alpha)\frac{D^{n+2}f(z)}{D^n f(z)}\right] > -\frac{1}{2}, \quad z \in U, \qquad (4)$$

then

$$f \in S_n^*\left(\frac{1}{2}\right). \qquad (5)$$

Proof. If we take $q(z) = \frac{1}{1-z}$ in Theorem 2, then the function h is equal to $h(z) = \frac{z(2-\alpha-z)}{(1-z)^2}$ and we deduce

$$\mathrm{Re} h\left(e^{i\theta}\right) = -\tfrac{1}{2} - \tfrac{1-\alpha}{2} ctg^2 \tfrac{\theta}{2} \leq -\tfrac{1}{2},\ \theta \in [0, 2\pi).$$

Now, if the relation (3) is satisfied, then (1) is true and from Theorem 2 we get $\frac{D^{n+1}f(z)}{D^n f(z)} \prec \frac{1}{1-z}$ which is equivalent to (4).
If $\alpha = 0$ we get

COROLLARY 2. If $f(z) \in \mathcal{A}$ satisfies

$$\mathbf{Re}\left[\frac{D^{n+2}f\left(z\right)}{D^{n}f\left(z\right)}\right] > -\frac{1}{2}, \quad z \in U, \tag{6}$$

then

$$f \in S_n^*\left(\frac{1}{2}\right). \tag{7}$$

References

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