DETERMINING OF AN EXTREMAL DOMAIN FOR THE FUNCTIONS FROM THE S-CLASS

by
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Abstract. Let S be the class of analytic functions of the form f(z) = z + $a_2 z^2$ +…, f(0)= 0, f′(0)=1 defined on the unit disk $|z|$<1. Petru T. Mocanu [2] raised the question of the determination max Re f(z) when Rez f′(z)=0, $|z|$= r, r>0 given. For solving the problem we shall use the variational method of Schiffer-Goluzin [1].

Key words: olomorf functions, variational method, extremal functions.

1. Let S the class of functions f(z) = z + $a_2 z^2$ +…, f(0)= 0, f′(0)=1 holomorf and univalent in the unit disk $|z|$<1.

For the first time Petru T. Mocanu [2] brought into discussion the problem of determination the max Re f(z) when Rez f′(z)=0, $|z|$= r, r>0 existed.

Geometrically this is expressed like in the figure below:

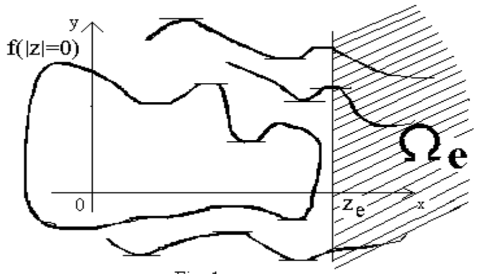

Fig. 1.

In region (Ω_e) any parallel (Re$|z|$>Re$|z_e|$) to Ox, is intersected f($|z|$= r), in one point. Because the class S is compact, exists this region. In this paper we will resolve this problem with the variational method of Schiffer-Goluzin [1].

2.Let $|z| = r$ and let $f \in S$ with Rez f'(z)=0, extremal function for exists the maximum max Re f(z), $f \in S$. We consider a variation $f^*(z)$ for the function f(z) given by Schiffer-Goluzin formula [1],

$$(1) \qquad f^*(z) = f(z) + \lambda V(z;\zeta;\psi) + O(\lambda^2), |\zeta| < 1, \lambda > 0$$

ψ real, where

$$(2) \qquad \begin{cases} V(z;\zeta;\psi) = e^{i\psi} \dfrac{f^2(z)}{f(z) - f(\zeta)} - e^{i\psi} f(z) \left[\dfrac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 - \\ - e^{i\psi} \cdot \dfrac{zf'(z)}{z-\zeta} \cdot \zeta \cdot \left[\dfrac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 + e^{-i\psi} \cdot \dfrac{z^2 f'(z)}{1-\overline{\zeta} z} \cdot \overline{\zeta \left[\dfrac{f(\zeta)}{\zeta f'(\zeta)}\right]^2}. \end{cases}$$

Is known that for λ sufficiently small, the function $f^*(z)$ is in the class S. We consider a variation z^* for z:

$$z^* = z + \lambda h + O(\lambda^2), \quad h = \left.\frac{\partial z^*}{\partial \lambda}\right|_{\lambda=0}$$

where satisfy the conditions:

$$(3) \qquad |z^*| = r \text{ şi } \operatorname{Re} z^* f^{*'}(z^*) = 0$$

Observing that :

$$|z^*|^2 = |z|^2 + 2\lambda \operatorname{Re}(\overline{z}\, h) + O(\lambda^2) = r^2.$$

Because $|z| = r$ from relation (3) we obtain :

$$(4) \qquad \operatorname{Re}(\overline{z}\, h) = 0.$$

Replacing z with z^* in $f^*(z)$ we have : $z^* f^{*'}(z^*) = A + B\lambda + O(\lambda^2)$ where :

$$\begin{cases} A = hf'(z) \\ B = hf'(z) + zhf''(z) + zV'(z;\zeta;\psi) \end{cases}$$

The condition $\operatorname{Re} z^* f^{*'}(z^*) = 0$ from relation (3) become:

$$(5) \qquad \text{Re } \{h(f'(z)+zf''(z))+zV'(z;\zeta;\psi)\}=0\,.$$

Because f(z) is extremes we have:

$$\text{Re } f^*(z^*)\le \text{Re } f(z)$$

where is equivalent with :

$$\text{Re}\{f(z)+\lambda hf'(z)+\cdots+\lambda V(z;\zeta:\psi)+\cdots\}\le \text{Re } f(z)$$

or

$$(6) \qquad \text{Re}\{hf'(z)+V(z;\zeta;\psi)\}\le 0.$$

From (5) ($\overline{h}=-\dfrac{\overline{z}}{z}h$) and (6) we obtain:

$$h(f'(z)+zf''(z))+zV'(z;\zeta;\psi)-\frac{\overline{z}}{z}h(\overline{f'(z)}+\overline{z}\cdot\overline{f''(z)})+\overline{z}\cdot\overline{V'}(z;\xi;\psi)=0$$

from where:

$$(7) \qquad h=\frac{z\overline{z}\cdot\overline{V}'(z;\zeta;\psi)+z^2V'(z;\zeta;\psi)}{-zf'(z)-z^2f''(z)+\overline{z}\cdot\overline{f'(z)}+\overline{z^2}\cdot\overline{f''(z)}}\,.$$

We will use the next denotations:

f= f(z), w=f(ζ) , $l=f'(z)$, m = $f''(z)$,V = (z; $\zeta;\psi$), $V'=V_Z'(z;\zeta;\psi)$.

With previous denotations, the relations (6) and (7) can be writhed as follows:

$$(8) \qquad \text{Re } \{pzV'+V\}\le 0$$

where $p=\dfrac{zl-\overline{z}\cdot\overline{l}}{-zl-z^2m+\overline{z}\cdot\overline{l}+\overline{z^2}\cdot\overline{m}}$ (p real).

I .We suppose that Im (zl + z^2m) $\neq 0$ (-zl + z^2m + $\overline{z}\cdot\overline{l}+\overline{z^2}\cdot\overline{m}\neq 0$). From the relation (2) obtain:

$$V = e^{i\psi} \cdot \frac{f^2}{f-w} - e^{i\psi} \cdot f\left(\frac{w}{\zeta \cdot w'}\right)^2 - e^{i\psi} \cdot \frac{zl}{z-\zeta} \cdot \zeta \cdot \left(\frac{w}{\zeta \cdot w'}\right)^2 + e^{-i\psi} \cdot \frac{z^2 l}{1-\overline{\zeta} \cdot z} \cdot \overline{\xi} \cdot \overline{\left(\frac{w}{\zeta \cdot w'}\right)^2}$$

and

$$V' = e^{i\psi} \cdot \frac{fl(f-2w)}{(f-w)^2} - e^{i\psi} \cdot l \cdot \left(\frac{w}{\zeta \cdot w'}\right)^2 - e^{i\psi} \frac{z(z-\zeta)\cdot m - \zeta \cdot l}{(z-\zeta)^2} \cdot \zeta \cdot \left(\frac{w}{\zeta \cdot w'}\right)^2 +$$

$$+ e^{-i\psi} \frac{z^2(1-\overline{\zeta} \cdot z)m + zl(2-\overline{\zeta} z)}{(1-\overline{\zeta} z)^2} \cdot \overline{\zeta} \cdot \overline{\left(\frac{w}{\zeta \cdot w'}\right)^2} .$$

Replacing in relation (8) the expression of V and V$^{/}$ we obtain:

$$\text{(9)} \qquad \operatorname{Re}\left[e^{i\psi}(E - GF)\right] \le 0,$$

where:

$$\begin{cases} E = \dfrac{f\left[(-f - 2pzl)w + f^2 + pzlf\right]}{(f-w)^2} \\ G = f + \dfrac{zl}{z-\zeta}\zeta - \dfrac{\overline{z}^2 \cdot \overline{l}}{(1-\overline{z}\zeta)^2}\zeta + pzl + \\ + \dfrac{pz\left[z(z-\zeta)m - \zeta \cdot l\right]\cdot \zeta}{(z-\zeta)^2} - \dfrac{pz\left[\overline{z}^2(1-\zeta \cdot \overline{z}) \cdot \overline{m} + \overline{z} \cdot \overline{l} \cdot (2 - \zeta \cdot \overline{z})\right]\cdot \zeta}{(1-\overline{z}\cdot)^2} \\ F = \left(\dfrac{w}{\zeta w'}\right)^2 . \end{cases}$$

Because ψ is arbitrary, from relation (8) is result that the function $w = f(\xi)$ has to satisfy the differential equation :

(10) $$\left(\frac{\zeta\cdot w'}{w}\right)^2\cdot\frac{f\left[(-f-2pzl)w+f^2+pzlf\right]}{(f-w)^2}=\frac{\sum_{k=0}^{4}t_k\zeta^k}{(z-\zeta)^2(1-\overline{z}\cdot\zeta)^2}$$

where

$$\begin{cases} t_0=z^2(f+pzl)+f \\ t_1=-2zf(1+r^2)+z^2l-\overline{z}^2\overline{l}-2plz^2(1-r^2)+pz^3m-pr^2(\overline{mz}+2\overline{l}), \\ t_2=f(r^4+4r^2+1)-zl(2r^2+1)-\overline{z}\cdot\overline{l}(2r^2+r^4)+pzl(r^4+4r^2+1)- \\ -pz(2r^2\cdot zm+mz+l)+pz\left[2r^2(\overline{m}\cdot\overline{z}+2\cdot\overline{l})+r^4(\overline{m}\cdot\overline{z}+\overline{l})\right] \\ t_3=-2f\overline{z}(1-2\overline{r})+r^2zl(\overline{z}+2)-\overline{z}^2\overline{l}(1+2r^2)+pr^2(-2r^2+mr^2z+2mz-\overline{mz}-2\overline{l}-2r^2\overline{zm}-2r^2\overline{l}) \\ t_4=f\overline{z}^2+p\left[\overline{m}(z^4-\overline{z}^4)+\overline{z}^2(lz-\overline{l}\cdot\overline{z})\right]. \end{cases}$$

The extremal function transforms the unit disk in the domain without external points. To justify this thing is sufficient to suppose that the transformed domain by an external function $w=f(\zeta)$ has an external point w_0 and to consider the function the variation:

$$f^*(z)=f(z)+\lambda e^{i\psi}\frac{f^2(z)}{f(z)-w_0}\ ,\ \lambda>0,\ \psi\ \text{real},\ f^*\in S$$

3.Is known that the extrema function $w=f(\zeta)$ transform the unit disk $|\zeta|<$ 1, in whole plane, cutted lengthwise of a finite number of analytically arc. Let q $=e^{i\theta}$, the point of the circle $|\zeta|=1$ where corresponding the extremity of this kind of section in which $w'(q)=0$ and $\zeta=q$ is double root for the polynom $\sum_{k=0}^{4}t_k\zeta^k$. Because $\zeta=q$ is double root for this polynom , we can write :

$$\sum_{k=0}^{4}t_k\zeta^k=\left(1-\overline{q}\cdot\zeta\right)^2\left(a_0+a_1\zeta+a_2\zeta^2\right).$$

From the relation about the coefficients t_k, $k=\overline{0,4}$ results that we can take $a_0 = t_0, a_1 = -2kq, a_2 = q^2 \cdot t_4$.

The differential equation (10) can be write:

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f\left[(-f-2pzl)w + f^2 + pzlf\right]}{(f-w)^2} = \frac{\left(1-\overline{q}\zeta\right)^2\left(t_0 - 2kq\zeta + q^2 t_4 \zeta^2\right)}{(z-\zeta)^2\left(1-\overline{z}\zeta\right)^2}. \tag{11}$$

4.After radical extraction in (11) we obtain:

$$\frac{\sqrt{f\left[(-f-2pzl)w + f^2 + pzlf\right]}}{w(f-w)}dw = \frac{(1-\overline{q}\zeta)\sqrt{t_0 - 2kq\zeta + q^2 t_4 \zeta^2}}{\zeta(z-\zeta)(1-\overline{z}\zeta)}.$$

From double integration:

$$\int_0^w \frac{\sqrt{f\left[(-f-2pzl)w + f^2 + pzlf\right]}}{w(f-w)}dw = \int_0^\zeta \frac{(1-\overline{q}\zeta)\sqrt{t_0 - 2kq\zeta + q^2 t_4 \zeta^2}}{\zeta(z-\zeta)(1-\overline{z}\zeta)}. \tag{12}$$

For calculation the integral from left side of (12) we denote:

$$I_1 = \int \frac{\sqrt{f\left[(-f-2pzl)w + f^2 + pzlf\right]}}{w(f-w)}dw.$$

We observe that :

$$I_1 = \sqrt{f(-f-2pzl)}\int \frac{\sqrt{w+a^2}}{w(f-w)}dw \text{ where we denoted } \frac{f^2 + pzlf}{-f-2pzl} = a^2.$$

For the calculation of I_1 we make the substitution : $w = u^2 - a^2, dw = 2udu$. We obtain $I_1 = \sqrt{f(-f-2pzl)}\int \frac{-2u^2 du}{(u^2-a^2)(u^2-b^2)}, b^2 = a + f$.

Observe that :

$$\frac{-2u^2}{(u^2-a^2)(u^2-b^2)}=\frac{2a^2}{b^2-a^2}\cdot\frac{1}{u^2-a^2}-\frac{2b^2}{b^2-a^2}\cdot\frac{1}{u^2-b^2}\ .$$

So:

$$I_1=\sqrt{f(-f-2pzl)}\cdot\left[\frac{a}{b^2-a^2}\ln\frac{u-a}{u+a}-\frac{b}{b^2-a^2}\ln\frac{u-b}{u+b}\right]$$

or

$$\text{(13)}\qquad I_1=\frac{\sqrt{f-(-f-2pzl)}}{b^2-a^2}\cdot\ln\left[\left(\frac{u-a}{u+a}\right)^a\cdot\left(\frac{u+b}{u-b}\right)^b\right]$$

where :

$$u=\sqrt{w+a^2}\quad .$$

For calculation the integral from right side of (12) we denote:

$$I_2=\int\frac{(1-\bar{q}\zeta)\sqrt{t_0-2kq\zeta+q^2t_4\zeta^2}}{\zeta(z-\zeta)(1-\bar{z}\zeta)}d\zeta\ .$$

We have: $q^2t_4\zeta^2-2kq\zeta+t_0=q^2t_4\cdot(\zeta-\zeta_1)(\zeta-\zeta_2)$

where $\zeta_{1,2}=\dfrac{k\pm\sqrt{k^2-t_0t_4}}{t_4}\bar{q}$. If we denotation $k-\sqrt{k^2-t_0t_4}=\delta$ observe that $\zeta_1=\dfrac{\delta}{t_4}\bar{q}$ and $\zeta_2=\dfrac{t_0}{\delta}\bar{q}$.

with this denotations, $\sqrt{q^2t_4-2kq\zeta+t_0}=\sqrt{q^2t_4}\sqrt{(\zeta-\dfrac{\delta}{t_4}\bar{q})(\zeta-\dfrac{t_0}{\delta}\bar{q})}$. For the calculate the integral I_2 make the substitution :

$$\text{(14)}\qquad \sqrt{(\zeta-\frac{\delta}{t_4}\bar{q})(\zeta-\frac{t_0}{\delta}\bar{q})}=v(\zeta-\frac{\delta}{t_4}\bar{q})\ .$$

From (14) obtain :

(15) $$\zeta = \sigma \cdot \frac{v^2 - \alpha^2}{v^2 - 1} \text{ with } \sigma = \frac{\delta}{t_4}\bar{q} \text{ and } \alpha^2 = \frac{t_0 t_4}{\delta^2}.$$

By an elementary calculation from (15) obtained successively:

(16) $$\begin{cases} d\zeta = \dfrac{2v\sigma(\alpha^2-1)}{(v^2-1)^2}dv, z-\zeta = (z-\sigma)\cdot\dfrac{v^2-\beta^2}{v^2-1} \text{ cu } \beta^2 = \dfrac{\sigma\alpha^2 - z}{\sigma - z}, \\ 1-\bar{z}\zeta = (1-\bar{z}\sigma)\cdot\dfrac{v^2-\gamma^2}{v^2-1} cu\gamma^2 = \dfrac{1-\bar{z}\sigma\alpha^2}{1-\overline{z\sigma}}, 1-\bar{q}\zeta = (1-\bar{q}\sigma)\dfrac{v^2-\delta^2}{v^2-1} \\ \text{cu } \delta^2 = \dfrac{1-\bar{q}\sigma\alpha^2}{1-\bar{q}\sigma} \text{ și } \sqrt{(\zeta - \dfrac{\delta}{t_4}\bar{q})(\zeta - \dfrac{t_0}{\delta}\bar{q})} = \dfrac{\sigma(1-\alpha^2)v}{v^2-1}. \end{cases}$$

By using previous relations we obtain:

(17) $$I_2 = \frac{2\sigma q(1-\bar{q}\sigma)(1-\alpha^2)^2\sqrt{t_4}}{(\sigma - z)(1-\bar{z}\sigma)}\int\frac{v^2(v^2-\delta^2)}{(v^2-1)(v^2-\alpha^2)(v^2-\beta^2)(v^2-\gamma^2)}dv.$$

Let:

$$F(v) = \frac{v^2(v^2-\delta^2)}{(v^2-1)(v^2-\alpha^2)(v^2-\beta^2)(v^2-\gamma^2)};$$

we are looking for a decomposition

(18) $$F(v) = \frac{A_1}{v-1} + \frac{A_2}{v+1} + \frac{A_3}{v-\alpha} + \frac{A_4}{v+\alpha} + \frac{A_5}{v-\beta} + \frac{A_6}{v+\beta} + \frac{A_7}{v-\gamma} + \frac{A_8}{v+\gamma}.$$

From (1) we obtain the next values for the coefficients:

(19)
$$\begin{cases} A_1 = A_2 = \dfrac{1-\delta^2}{2(1-\alpha^2)(1-\beta^2)(1-\gamma^2)} \stackrel{not}{=} \tau_1 \\ A_3 = -A_4 = \dfrac{\alpha(\alpha^2-\delta^2)}{2(\alpha^2-1)(\alpha^2-\beta^2)(\alpha^2-\gamma^2)} = \tau_2 \\ A_5 = -A_6 = \dfrac{\beta(\beta^2-\delta^2)}{2(\beta^2-1)(\beta^2-\alpha^2)(\beta^2-\gamma^2)} = \tau_3 \\ A_7 = -A_8 = \dfrac{\gamma(\gamma^2-\delta^2)}{2(\gamma^2-1)(\gamma^2-\alpha^2)(\gamma^2-\beta^2)} = \tau_4 \cdot \end{cases}$$

We denote, $\mu = \dfrac{2\sigma q(1-\overline{q}\sigma)(1-\alpha^2)^2 \cdot \sqrt{t_4}}{(\sigma - z)(1-\overline{z}\sigma)}$; from (18) and (19) we obtain for I_2 the expression:

(20)
$$I_2 = \mu\left[\tau_1 \ln\frac{v-1}{v+1} + \tau_2 \ln\frac{v-\alpha}{v+\alpha} + \tau_3 \ln\frac{v-\beta}{v+\beta} + \tau_4 \ln\frac{v-\gamma}{v+\gamma}\right].$$

From the relation (14) observe that:

(21)
$$v(\zeta) = \sqrt{\frac{\zeta - \frac{t_0}{\delta}\overline{q}}{\zeta - \frac{\delta}{t_4}\overline{q}}}, \quad v(0) = \pm\alpha$$

and from $w = u^2 - a^2$ we obtain:

(22)
$$u(\zeta) = \sqrt{w(\zeta) - \frac{f^2 + 2pzlf}{f + 2pzl}}, \quad u(0) = \pm a \ .$$

With the relations (13) and (20) the relation (12) becomes:

$$(12') \qquad I_1\Big|_0^w = I_2\Big|_0^\zeta$$

For $\zeta = 0$ from (13), (20) and ($12'$) obtained the constant (which is obtained from that two members of relations (13) and (20) corresponding to $\frac{u-a}{u+a}$ from $\frac{v-\alpha}{v+\alpha}$ (from left)): $\ln(-1)^{\frac{a\sqrt{f(-f-2pzl)}}{b^2-a^2}+\mu\tau_2}$, obtained in left member of equality ($12'$). Thus, ($12'$) can be writhed:

$$\frac{\sqrt{f(-f-2pzl)}}{b^2-a^2}\ln\left[\left(\frac{u(\zeta)-a}{u(\zeta)+a}\right)^2\cdot\left(\frac{u(\zeta)+b}{u(\zeta)-b}\right)^2\right]+\frac{\sqrt{f(-f-2pzl)}}{b^2-a^2}\ln\left(\frac{a+b}{a-b}\right)^b$$

$$+\ln(-1)^{\frac{a\sqrt{f(-f-2pzl)}}{b^2-a^2}+\mu\tau_2} =$$

$$=\mu\left[\ln\left(\frac{v(\zeta)-1}{v(\zeta)+1}\right)^{\tau_1}+\ln\left(\frac{v(\zeta)-\alpha}{v(\zeta)+\alpha}\right)^{\tau_2}+\ln\left(\frac{v(\zeta)-\beta}{v(\zeta)+\beta}\right)^{\tau_3}+\ln\left(\frac{v(\zeta)-\gamma}{v(\zeta)+\gamma}\right)^{\tau_4}\right]-$$

$$-\mu\ln\left[\left(\frac{\alpha-1}{\alpha+1}\right)^{\tau_1}\cdot\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{\tau_3}\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)^{\tau_4}\right].$$

With calculus the previsious equality can be writhed:

$$(23)\quad \begin{cases} \left[\left(\frac{a-u(\zeta)}{a+u(\zeta)}\right)^a\cdot\left(\frac{u(\zeta)+b}{u(\zeta)-b}\cdot\frac{a+b}{a-b}\right)^b\right]^{\frac{\sqrt{f(-f-2pzl)}}{b^2-a^2}} = \\ =\left[\left(\frac{v(\zeta)-1}{v(\zeta)+1}\right)^{\tau_1}\cdot\left(\frac{\alpha-v(\zeta)}{\alpha+v(\zeta)}\right)^{\tau_2}\cdot\left(\frac{v(\zeta)-\beta}{v(\zeta)+\beta}\cdot\frac{\alpha+\beta}{\alpha-\beta}\right)^{\tau_3}\left(\frac{v(\zeta)-\gamma}{v(\zeta)+\gamma}\cdot\frac{\alpha-\gamma}{\alpha+\gamma}\right)^{\tau_4}\right]^{\mu}. \end{cases}$$

where $u(\zeta)$ and $v(\zeta)$ is obtained by the relations (22) and (21).

The relation (23) represent under the implicitly equation where is verify the extremal function $w = w(\zeta)$, where realized $\max\limits_{f \in S} \operatorname{Re} f(z)$.

II. We suppose that $\operatorname{Im}(zl + z^2 m) = 0$. In this case the expression of p, have to $zl - \overline{zl} = 0$. How $zl + \overline{zl} = 0$ implies $zl = 0$. If $l = 0(|z| = r > 0)$ then $\bar{l} = 0$. From (6) results that $w(\zeta)$ have to verify the condition:

$$\operatorname{Re}\left\{ e^{i\psi}\left[\frac{f^2}{f - w} - f\left(\frac{w}{\zeta w'}\right)\right]\right\} \leq 0 . \qquad (24)$$

How ψ is arbitrary, real, results that $w(\zeta)$ verify next differential equation:

$$\left(\frac{w}{\zeta w'}\right)^2 = \frac{f}{f - w} . \qquad (25)$$

or $\dfrac{\sqrt{f}\,dw}{w\sqrt{f - w}} = \dfrac{d\zeta}{\zeta}$. From double integration:

$$\int_0^w \frac{\sqrt{f}\,dw}{w\sqrt{f - w}} = \int_0^\zeta \frac{d\zeta}{\zeta} . \qquad (26)$$

With denotation $\sqrt{f - w} = t$, obtained $w = f - t^2, dw = -2tdt$. The relation (26) becomes (without limits of integration):

$$\int \frac{\sqrt{f}(-2t)dt}{(f - t^2)\cdot t} = \int \frac{d\zeta}{\zeta} \text{ or } \int \frac{2\sqrt{f}}{t^2 - (\sqrt{f})^2}dt = \ln\zeta, \text{ from where}$$

$$\ln\frac{t - \sqrt{f}}{t + \sqrt{f}} = \ln\zeta \text{ or}$$

(27) $$\ln\frac{\sqrt{f-w}-\sqrt{f}}{\sqrt{f-w}+\sqrt{f}}\Big|_0^w=\ln\zeta\Big|_0^\zeta .$$

After the calculus we obtained successively:

$$\begin{cases}\ln\frac{\sqrt{f-w}-\sqrt{f}}{\sqrt{f-w}+\sqrt{f}}-\ln\frac{\sqrt{f-w}-\sqrt{f}}{\sqrt{f-w}+\sqrt{f}}\Big|_0=\ln\zeta-\ln\zeta\Big|_{\zeta=0}, \\ \text{and} \\ \ln\frac{\sqrt{f-w}-\sqrt{f}}{\sqrt{f-w}+\sqrt{f}}+\ln\left(\frac{\zeta(\sqrt{f-w}+\sqrt{f})^2}{-w}\right)\Big|_{\zeta=0}=\ln\zeta\end{cases}$$

from where:

$$\ln\left(\frac{\sqrt{f}-\sqrt{f-w}}{\sqrt{f}+\sqrt{f-w}}\cdot 4f\right)=\ln\zeta$$

and

(28) $$\frac{\sqrt{f}-\sqrt{f-w}}{\sqrt{f}+\sqrt{f-w}}=\frac{\zeta}{4f} .$$

From (28) we obtained:

(29) $$w(z)=\frac{16zf^2(z)}{(4f(z)+z)^2} .$$

How f(z) is considerate the extremal from (29) obtained after some calculus that $w(z)=\frac{z}{4}$. Observe that:

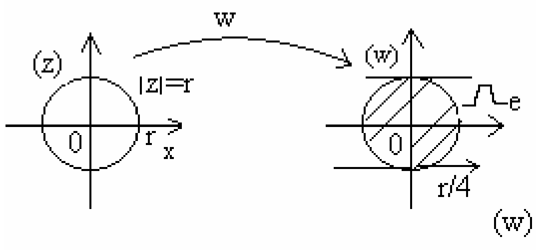

Fig 2

and the condition $\operatorname{Re} zw'(z) = 0$ implies $x = 0 (z = x + iy)$.

Then $\bar{z} = \max \operatorname{Re} w(z) = 0$, so for $z_e = 0$ and $\forall |z| > |z_e|$, any parallel to Ox intersected $f(|z| = r)$ in one point.

We exclude this ordinary case, it showing that the problem is true.

From I and II, result that the extremal function which is corresponding to the extremal region Ω_e from fig 1, has the implicitly form in equation (23).

5.Still remain to show how to determine the θ. For this we make in (11) $\zeta \to z$; and after simplifications and by multiplication of equality from $\bar{z}^{3}\bar{l}$ we obtained:

$$r^3(r^2-1)l\bar{l} \cdot p = (1-\bar{q}z)^2\left(t_0 - 2kqz + q^2 \cdot t_4 \cdot z^2\right) \cdot \bar{z}^{3} \cdot \bar{l}$$

or :

$$(30) \qquad r^3(r^2-1)l\bar{l} \cdot p = (\bar{z} - \bar{q}r^2)^2 (t_0 \bar{z}\bar{l} - 2kq\bar{l}r^2 + q^2 t_4 \bar{l} z \cdot r^2).$$

Because p is real, $r^3(r^2-1)l\bar{l}p$, is real from the relation (30) we obtain a system with two equation and determinate θ and k :

$$(31) \qquad \begin{cases} r^3(r^2-1)\bar{l}\bar{l}p = \operatorname{Re}\left[\left(\bar{z} - \bar{q}r^2\right)^2 (t_0 \bar{z}\bar{l} - 2kq\bar{l}r^2 + q^2 t_4 \bar{l} z r^2)\right] \\ \\ \operatorname{Im}\left[\left(\bar{z} - \bar{q}r^2\right)\left(t_0 \bar{z}\bar{l} - 2kq\bar{l}r^2 + q^2 t_4 \bar{l} z r^2\right)\right] = 0 \end{cases}$$

With θ and k determinated in this way the extremal function from equation (27) is well determine and with its assisting we find $z_e = \max_{f \in w} \operatorname{Re} f(z)$ with the geometrical property enounced: in domain $\Omega_e = \{z \| z| > |z_e\|\}$, any parallel to the Ox axis intersect $f(|z| = r)$ in one point (eventually in maximum one point).

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