THE NON-CRITICALITY OF CERTAIN FAMILIES OF AFFINE VARIETIES

by
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Abstract. In this paper we show that certain closed families of affine varieties of a Euclidean space and some families of spheres of a higher dimensional sphere as well, are not critical sets for certain special real valued functions.
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1. Introduction

In this paper we first get some topological information on the complement of certain families of affine varieties of a Euclidean space, expressed in terms of homology groups. In the last section we use these information to prove that the considered families of affine varieties, are not properly critical for certain special real valued functions. The family of spheres, as a closed subset of an *m+1* dimensional sphere, produced as the closure of the inverse image by certain stereographic projections of a considered family of affine varieties, is also shown to be non-critical for the same class of special real valued functions. Let us finaly mention that the (non)-criticality problem of closed sets in the plane and in the three dimensional space has been studied before by M. Grayson, C. Pugh and A. Norton in [2,3] while the non-criticality of a family of fibers over a closed countable subset of the base space of a fibration has been studied before in [6]. In the previous papers [4,5] the non-criticality problem of finite and countable subsets of certain manifolds with respect to mappings having manifolds as target spaces is considered and studied.

2. Preliminary Results

We start this section by considering certain families of affine varieties of the *(m+1)*-dimensional Euclidean space and by showing that the homology groups of their complements are the direct sum of the homology groups of the complements of their components. A similar isomorphism is provided for the complements of some families of embedded spheres in a higher dimensional one.

1.1. Proposition *If $\{A_i\}_{i\geq 1}$ is a family of affine varieties of the Euclidean space R^{m+1} of various dimensions such that $i \neq j \Rightarrow \delta(A_i, A_j) > 0$ for some c>0, where $\delta(A_i, A_j)$ is the distance between A_i and A_j, then*

$$H_q\left(R^{m+1} \setminus \bigcup_{i\geq 1} A_i\right) \cong \bigoplus_{j\geq 1} H_q\left(R^{m+1} \setminus A_i\right) \text{ for all } q\geq 1.$$

Proof. Using Poincaré duality and the cohomology sequence, with compact supports, of the pair $(R^{m+1}, \bigcup_{i\geq 1} A_i)$, we have:

$$\begin{array}{ccccccc} 0 \cong H_{q+1}(R^{m+1}) & & & & H_q(R^{m+1}) \cong 0 & & \\ \wr\| & & & & \wr\| & & \\ \cdots \to H_c^{m-q}(R^{m+1}) & \to & H_c^{m-q}(\bigcup_{i\geq 1} A_i) & \to & H_c^{m-q+1}(R^{m+1}, \bigcup_{i\geq 1} A_i) & \to & H_c^{m-q+1}(R^{m+1}) \to \cdots \end{array}$$

which provides us the isomorphism

$$H_c^{m-q}\left(\bigcup_{i\geq 1} A_i\right) \cong H_c^{m-q+1}\left(R^{m+1}, \bigcup_{i\geq 1} A_i\right).$$

Therefore, using the duality theorem [8, Theorem 6.9.10] and the above isomorphism, we have successively

$$H_q(R^{m+1} \setminus \bigcup_{i\geq 1} A_i) \cong H_c^{m-q}(\bigcup_{i\geq 1} A_i) \cong H_c^{m-q+1}(R^{m+1}, \bigcup_{i\geq 1} A_i) \cong \bigoplus_{j\geq 1} H_c^{m-q}(A_i).$$

Further on we will show, following the same way, that $H_c^{m-q}(A_i) \cong H_c^{m-q+1}(R^{m+1}, A_i)$. Indeed using Poincaré duality and the cohomology sequence, with compact supports, of the pair (R^{m+1}, A_i), we have:

$$\begin{array}{ccccccc} 0 \cong H_{q+1}(R^{m+1}) & & & & H_q(R^{m+1}) \cong 0 & & \\ \wr\| & & & & \wr\| & & \\ \cdots \to H_c^{m-q}(R^{m+1}) & \to & H_c^{m-q}(A_i) & \to & H_c^{m-q+1}(R^{m+1}, A_i) & \to & H_c^{m-q+1}(R^{m+1}) \to \cdots \end{array}$$

which provides us the isomorphism $H_c^{m-q}(A_i) \cong H_c^{m-q+1}(R^{m+1}, A_i)$.

Therefore, using the same duality theorem [8, Theorem 6.9.10] and the above isomorphism, we have:

$$H_c^{m-q}(A_i) \cong H_c^{m-q+1}(R^{m+1}, A_i) \cong .H_q(R^{m+1} \setminus A_i),$$

and the proof is now complete.

2.2. Remark *If we replace in proposition 1.1 the Euclidean space R^{m+1} with an n-connected manifold M^{m+1} and the family $\{A_i\}_{i\geq 1}$ of affine varieties with a closed submanifold N having the connected components $\{N_i\}_{i\geq 1}$ of various dimensions, then the similar isomorphism*

$$H_q\left(M \setminus \bigcup_{i\geq 1} N_i\right) \cong \bigoplus_{j\geq 1} H_q(M \setminus N_i)$$

still holds for 1≤q≤n-1.

2.3. Corollary *Let $\{A_i\}_{i\geq 1}$ be a family of affine varieties of the Euclidean space $\mathbf{R}^{m+1}$ of various dimensions not exceeding m-1 such that $i \neq j \Rightarrow \delta(A_i, A_j) > 0$ for some c>0. In these conditions $R^{m+1} \setminus \bigcup_{i\geq 1} A_i$ is connected and for any $1 \leq q \leq m-1$ we have*

$$H_{m-q}(R^{m+1} \setminus \bigcup_{i\geq 1} A_i) \cong Z^{J_q}$$

where by J_q we denote the set $\{j \geq 1 : \dim A_j = q\}$.

Proof. Indeed, the connectedness follows easily by using a weak version of Thom theorem. Concerning the stated isomorphism we first apply proposition 1.1 to deduce that

$$H_{m-q}\left(R^{m+1} \setminus \bigcup_{i\geq 1} A_i\right) \cong \bigoplus_{j\geq 1} H_{m-q}\left(R^{m+1} \setminus A_i\right),$$

and then observe that $R^{m+1} \setminus A_i$ has the homotopy type of the sphere S^{m-q}. □

If S^{m+1} is the *(m+1)*-dimensional sphere and $p \in S^{m+1}$ a point, consider the stereographic projection $\varphi_p : S^{m+1} \setminus \{p\} \longrightarrow p^{\perp}$, $\varphi_p(x) = \dfrac{x - \langle x, p\rangle p}{1 - \langle x, p\rangle p}$, where $p^{\perp} := \{q \in S^{m+1} \mid \langle q, p\rangle = 0\}$ is the hyperplane of R^{m+2} orthogonal on *p*. It is obviously a diffeomorphism, its inverse being $\varphi_p^{-1} : p^{\perp} \longrightarrow S^{m+1} \setminus \{p\}$, $\varphi_p^{-1}(y) = \dfrac{2y + (\| y \|^2 - 1)p}{\| y \|^2 + 1}$.

Let α: $R^n \longrightarrow R^{m+2}$ be an embedding. It is easy to see that *N*=α(R^n)is a closed submanifold of R^{m+2} if $\lim_{\|x\| \to \infty} \| \alpha(x) \| = \infty$.

Consequently, for a closed submanifold *N* of $\boldsymbol{R}^{m+2}$ iffeomorphic with some Euclidean space which is contained in $p^{\perp}$ for some $p \in S^{m+1}$, we have that $\overline{\varphi_p^{-1}(N)} = \{p\} \cup \varphi_p^{-1}(N)$.

2.4. Corollary *Let $\{A_i\}_{i\geq 1}$ be a family of affine varieties of the Euclidean space* R^{m+1} *of various dimensions such that* $2 \leq \dim A_i \leq m-1$ *and* $i \neq j \Rightarrow \delta(A_i, A_j) > 0$ *for some c>0. Assume also that for any i≥1 there exists* $r(i) \in \{1,...,l\}$ *with the property that* $A_i \subset p_{r(i)}^{\perp}$ *for some* $p_1,..., p_l \in S^{m+1}$ *and such that* $\varphi_{r(i)}^{-1}(N_i) \cap \varphi_{r(j)}^{-1}(N_j) = \phi$ *for* $i \neq j$. *In these conditions the equalities*

$$\bigcup_{i\geq 1} \overline{\varphi_{r(i)}^{-1}(A_i)} = \overline{\bigcup_{i\geq 1} \varphi_{r(i)}^{-1}(A_i)} = \{p_1,...., p_l\} \cup \bigcup_{i\geq 1} \varphi_{r(i)}^{-1}(A_i)$$

hold *as well as the isomorphism*

$$H_{m-\dim A_i}\left(S^{m+1} \setminus \bigcup_{j\geq 1} \overline{\varphi_{r(j)}^{-1}(A_j)} \right) \cong \boldsymbol{Z}^{(J_{\dim A_i})}.$$

Proof. Indeed $S^{m+1} \setminus \bigcup_{j\geq 1} \overline{\varphi_{r(j)}^{-1}(Aj)} = (S^{m+1} \setminus \{p_1,...., p_l\}) \setminus \bigcup_{j\geq 1} \varphi_{r(j)}^{-1}(A_i)$ and each of $\varphi^{-1}_{r(j)}(Aj)$ is a closed submanifold of $S^{m+1}\setminus\{p_1,....,p_l\}$ diffeomorphic with $\boldsymbol{R}^{\dim A_i}$, the whole union $\bigcup_{j\geq 1}\varphi^{-1}_{r(j)}(Aj)$ being also a closed submanifold of the *(m-1)*-connected one $S^{m+1}\setminus\{p_1,....,p_l\}$. The *(m-1)*-connectedness of $S^{m+1}\setminus\{p_1,....,p_l\}$ follows from [4, Proposition 2.3] and it ensures us, according to remark 2.2, that

$$H_{m-\dim A_i}\left((S^{m+1}\setminus\{p_1,....,p_l\})\setminus\bigcup_{j\geq 1}\varphi^{-1}_{r(j)}(A_j)\right)\cong$$
$$\cong\bigoplus_{j\geq 1}H_{m-\dim A_i}\left((S^{m+1}\setminus\{p_1,....,p_l\})\setminus\varphi^{-1}_{r(j)}(A_j)\right)$$

On the other hand the inclusion of

$$(S^{m+1}\setminus\{p_1,....,p_l\})\setminus\varphi^{-1}_{r(j)}(A_j)=(S^{m+1}\setminus\overline{\varphi^{-1}_{r(j)}(A_j)})\setminus\{p_1,...,\hat{p}_{r(j)},...,p_l\}$$

in

$$(S^{m+1}\setminus\varphi^{-1}_{r(j)}(A_j))\setminus\{p_{r(j)}\}=S^{m+1}\setminus\overline{\varphi^{-1}_{r(j)}(A_j)}=(S^{m+1}\setminus\{p_{r(j)}\})\setminus\varphi^{-1}_{r(j)}(A_j)$$

induces isomorphism at the level of homotopy groups in dimension less then or equal to *m-1* for each $1\leq j\leq l$, such that, according to Whitehead theorem, it also induces isomorphism at the level of homology groups in dimension less then or equal to *m-1* for each $1\leq i\leq l$. Finaly the restriction of $\varphi_{p_{r(j)}}$ to $(S^{m+1}\setminus\{p_{r(j)}\})\setminus\varphi^{-1}_{r(j)}(A_j)$ realizes a diffeomorphism between $(S^{m+1}\setminus\{p_{r(j)}\})\setminus\varphi^{-1}_{r(j)}(A_j)$ and $p^{\perp}_{r(j)}\setminus A_j$ which in its turn has the homotopy type of $S^{m-\dim A_i}$. Consequently we have succesively

$$H_{m-\dim A_i}\left(S^{m+1}\setminus\bigcup_{j\geq 1}\overline{\varphi_{r(j)}{}^{-1}(A_j)}\right)=H_{m-\dim A_i}\left((S^{m+1}\setminus\{p_1,....,p_l\})\setminus\bigcup_{j\geq 1}\varphi^{-1}_{r(j)}(A_j)\right)\cong$$

$$\bigoplus_{j\geq 1}H_{m-\dim A_i}\left(p^{\perp}_{r(j)}\setminus A_j\right)\cong\bigoplus_{j\in J_{\dim A_i}}H_{m-\dim A_i}\left(p^{\perp}_{r(j)}\setminus A_j\right)\cong\bigoplus_{j\in J_{\dim A_i}}H_{m-\dim A_i}\left(S^{m-\dim A_j}\right)\cong$$
$$\boldsymbol{Z}^{\left(J_{\dim A_i}\right)}.$$

2.5. Remark The closed subset $\bigcup_{r(i)=j}\overline{\varphi^{-1}_{r(j)}(A_i)}$ of S^{m+1} is a union of embedded spheres any two of them having the point p_j as the only common point.

3. Application

As we have already mentioned before, in this section we use the information already obtain on the topology of the complement of a considered family *$\{A_i\}_{i\geq 1}$* of affine varieties in the Euclidean space R^{m+1} and those on the topology of the complement of a family of embedded spheres $\overline{\bigcup_{i\geq 1}\varphi_{r(i)}^{-1}(A_i)}$ in the higher dimensional one S^{m+1}, to show that the considered families are not critical sets for certain real valued functions.

Let *M,N* be two differentiable manifolds and consider $CS^{\infty}(M,N)=\{f\in C^{\infty}(M,N)\mid B(f)\cap f(C(f))=\phi\}$, where *R(f)* is the regular set of *f, B(f)=f(C(f))* is its set of critical values while *C(f)* is the critical set of *f*. Because of the empty intersection between B(f) and *f(R(f))* a mapping ***f*** from $C^{\infty}(M,N)$ obviously *separates the critical values by the regular ones*.

3.1. Proposition ([6]) $f\in CS^{\infty}(M,N)$ *iff* $C(f)=f^{-1}(B(f))$. *(ii) If M is a connected differentiable manifold and* $f\in CS^{\infty}(M,R)$ *is such that R(f)=M\C(f) is also connected, then f(R(f))=(m_f,M_f), where* $m_f=\inf_{x\in M}f(x)$, $M_f=\sup_{x\in M}f(x)$ *and* $B(f)\subseteq\{m_f,M_f\}\bigcap R$. *Moreover, if M is compact, then* $m_f,M_f\in R$ *and* $B(f)=\{m_f,M_f\}$.

3.2. Theorem *Let $\{A_i\}_{i\geq 1}$ be a family of affine varieties of the Euclidean space* R^{m+1} *of various dimensions not exceeding m-1 such that* $i\neq j\Rightarrow\delta(A_i,A_j)>0$ *for some c>0.then there is no any mapping* $f\in CS^{\infty}(M,R)$ *such that C(f)=A:=*$\bigcup_{i\geq 1}A_i$ *and the restriction* $f|_{R^{m+1}\setminus A}$ *is proper.*

Proof. Assuming that such an application exists it follows on the one hand that $B(f)\subseteq\{m_f,M_f\}$ and on the other hand its restriction $\boldsymbol{R}^{m+1}\setminus C(f)\longrightarrow$ *Im f*=(m_f,M_f), $p\mapsto f(p)$ is a proper submersion, that is a locally trivial fibration, via Ehresmann's theorem, whose compact fiber we are denoting by *F*. Its base space (m_f,M_f) being contractible, it follows that the inclusion $i_F:F\longrightarrow S^{m+1}\setminus C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms

$\pi_q(i_F):\pi_q(F)\longrightarrow\pi_q(S^{m+1}\setminus C(f))$ are all isomorphism. Consequently, using the Whitehead theorem, it follows that the induced group homomorphisms

$$H_q(i_F):H_q(F)\longrightarrow H_q(S^{m+1}\setminus C(f))$$

are also isomorphism. Because for some $i\geq 1, J_{\dim A_i}$ is infinite it follows that

$$H_{m-\dim A_i}(F)\cong H_{m-\dim A_i}(S^{m+1}\setminus C(f))\cong H_{m-\dim A_i}(S^{m+1}\setminus\overline{\bigcup_{j\geq 1}\varphi_{r(j)}^{-1}(A_j)},$$

the last one having a subgroup isomorphic with $\boldsymbol{Z}^{(J_{\dim A_i})}$, that is a contradiction with the well known fact that any compact manifold has finitely generated homology groups.

The proof of the next theorem is completely similar.

3.3. Theorem *Let $\{A_i\}_{i\geq 1}$ be a family of affine varieties of the Euclidean space R^{m+1} of various dimensions such that $2\leq\dim A_i\leq m-1$ and $i\neq j\Rightarrow\delta(A_i,A_j)>0$ for some c>0. Assume also that for any i≥1 there exists $r(i)\in\{1,\ldots,l\}$ with the property that $A_i\subset p_{r(i)}^{\perp}$ for some $p_1,\ldots,p_l\in S^{m+1}$ and such that $\varphi_{r(i)}^{-1}(N_i)\cap\varphi_{r(j)}^{-1}(N_j)=\phi$ for $i\neq j$. In these conditions there is no any mapping $f\in CS^{\infty}(S^{m+1},R)$ such that $C(f)=\overline{\bigcup_{i\geq 1}\varphi_{r(i)}^{-1}(A_i)}$.*

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