# ON THE BICUBIC SPLINE COLLOCATION METHOD FOR POISSON'S EQUATIONS

by
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**Abstract.** In this paper is presented a bicubic spline collocation method for the numerical approximation of the solution of Dirichlet problem for the Poisson's equation .The approximating solution is efectivly determined in a bicubic Hermite spline functions space by using a suitable basis constructed as a tensorial product of univariate spline spaces.

## 1. Introduction.

Let consider the Dirichlet problem for the Poisson's equation on $\Omega \subset R^2$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $\Omega := ]0,1[ \times ]0,1[$ and $\partial\Omega$ is the boundary of $\Omega$.

In engineering applications such a problem is of the most importance .In linear elasticity , in this theory of thin plates as in the fluid mechanics , many phenomena and processes are modeling with this equation .

Various methods have been developed for solving the Dirichlet problem (1) numerically .A number of works using finite difference methods shows the efficiency of such methods , but their order of accuracy are very low .( see[3] , [7] , [10] ) .Higher order accuracy can be achieved using finite element methods (see[3] , [7] , [11] ) . There are many finite element approaches which use iterative methods , such as [3] , [4] .

In this paper we shall present a direct bicubic spline collocation method for solving numerically the Dirichlet problem for the Poisson's equations (1) .Such methods have been developed in many papers in the last decades ( see [1] ,[2] , [5] , [8] , [12] , [13] , [14] ) .

## 2. Space of Hermite cubic splines

Let N be a pozitive integer and let $\Delta : 0 = t_0 < t_1 < t_2 ... < t_N = 1$ be a uniform partition of [0,1] , such that $t_n = n \cdot h$ , n=0,1,…,N , where h=1/N is the stepsize .

Let $S_h$ be the space of Hermite cubic splines on [0,1] , defined by

$$S_h := \{ v \in C^1[0,1] : v|_{[t_n, t_{n+1}]} \in P_3 , n = 0,1,..., N-1 \} \tag{2}$$

where $P_3$ denotes the set of polinomials of degree $\leq 3$ and let define :

$$S_h^0 := \{ v \in S_h : v(0) = v(1) = 0 \} \tag{3}$$

G.Fairweather [2] constructed a useful bases for $S_h$ in the following manner . One defines the functions $v_n, s_n \in S_h, n = 0,1,..., N$ , associated with the point $t_n$ as follows :

$$v_n(t_m) = \delta_{m,n},\ v_n^{'}(t_m) = 0,\ \text{n, m} = 0\ ,1\ ,\ldots,\ \text{N}$$

$$s_n(t_m) = 0,\ s_n^{'}(t_m) = h^{-1}\delta_{m,n},\ \text{n,m} = 0,1,\ldots,\text{N}$$

where $\delta_{n,m}$ is Kronecher symbol .

To write explicit formulae for $s_n$ we define the functions :

$$a_1(t) = -2t^3 + 3t^2,\quad a_2(t) = t^3 - t^2$$

and the linear mapping $\alpha_n(t) := (t - t_n)/h$ from the interval $[t_n, t_{n+1}]$ onto [0,1] . Then we construct the following functions :

$$v_0(t) := \begin{cases} a_1(1 - \alpha_0(t)), & t \in [t_0, t_1] \\ 0, & \text{otherwise} \end{cases}$$

$$v_N(t) = \begin{cases} a_1(\alpha_{N-1}(t)), & t \in [t_{N-1}, t_N] \\ 0, & \text{otherwise} \end{cases}$$

$$v_n(t) = \begin{cases} a_1(\alpha_{n-1}(t)), & t \in [t_{n-1}, t_n], \\ a_1(1 - \alpha_n(t)), & t \in [t_n, t_{n+1}], \quad n = 1,..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_0(t) := \begin{cases} -a_2(1 - \alpha_0(t)), & t \in [t_0, t_1] \\ 0, & \text{otherwise} \end{cases}$$

$$s_N(t) = \begin{cases} a_2(\alpha_{N-1}(t)), & t \in [t_{N-1}, t_N] \\ 0, & \text{otherwise} \end{cases}$$

$$s_n(t) = \begin{cases} a_2(\alpha_{n-1}(t)), & t \in [t_{n-1}, t_n], \\ -a_2(1 - \alpha_n(t)), & t \in [t_n, t_{n+1}], \quad n = 1,2,..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

By ordering $v_n$ and $s_n$ we get two sets of basis functions $\{\Phi_n\}_{n=0}^{2N+1}$ and $\{\Psi_n\}_{n=0}^{2N+1}$ for the spline space $S_h$ :

$$\{\Phi_0, \Phi_1,..., \Phi_{2N}, \Phi_{2N+1}\} := \{v_0, v_1,..., v_{N-1}, s_0, s_1,..., s_N\}$$

$$\{\Psi_0, \Psi_1,..., \Psi_{2N}, \Psi_{2N+1}\} := \{v_0, s_0, v_1, s_1,..., v_N, s_N\}$$

and two sets of basis functions for $S_h^0$

$$\{\Phi_1,...,\Phi_{2N}\} := \{v_1,...,v_{N-1}, s_0, s_1,..., s_N\}$$

$$\{\Psi_1,...,\Psi_{2N}\} := \{s_0, v_1, s_1,..., v_{N-1}, s_{N-1}, s_N\}$$

Let $S_h \otimes S_h$ be the space of Hermite bicubic splines on $\bar{\Omega}$ , that is , the set of all functions on $\bar{\Omega}$ which are finite linear combinations of the functions of the form $u(x)v(y)$ where $u, v \in S_h$ .Identically , we define the tensorial product space $S_h^0 \otimes S_h^0$ on $\bar{\Omega}$ .Since the dimension of $S_h^0$ is 2N , the dimension of $S_h^0 \otimes S_h^0$ is $4N^2$.

Let $\{\xi_m\}_{m=1}^{2N}$ the Gauss points in $]0,1[$ given by :

$$\xi_{2n+1} := t_n + h\frac{3-\sqrt{3}}{6} ; \xi_{2n+2} := t_n + h\frac{3+\sqrt{3}}{6}, n = 0,1,..., N-1$$

and let $G := \{(x, y) : x, y \in \{\xi_m\}_{m=1}^{2N}\}$ (4)

be the set of the Gauss points in $\Omega$ .

In is known that each $v \in S_h^0$ is uniquely defined by its values at the Gauss points $\{\xi_m\}_{m=1}^{2N}$. Therefore , in what follows , $S_h^0$ is regarded as a Hilbert space with the inner product $<\cdot,\cdot>$ defined by :

$$< u, v > := \frac{h}{2}\sum_{m=1}^{2N} u(\xi_m)v(\xi_m) \quad , \quad u, v \in S_h^0 \qquad (5)$$

**3.Hermite bicubic spline collocation method .**

First we consider the homogeneons Dirichlet problem for Poisson's equation on $\Omega$ :

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad (6)$$

where $\Omega := ]0,1[ \times ]0,1[$ and $\partial\Omega$ is the boundary of $\Omega$ . Let $\{\Phi_n\}_{n=1}^{2N}$ and $\{\Psi_n\}_{n=1}^{2N}$ be the two bases for $S_h^0$ above constructed . The piecewise Hermite bicubic spline colocation approximation

$$u_h(x, y) := \sum_{i=1}^{2N}\sum_{j=1}^{2N} u_{i,j}\Phi_i(x)\Psi_j(y) \in S_h^0 \otimes S_h^0 \qquad (7)$$

to the solution u of (6) is obtained by requiring that :

$$-\Delta u_h(\xi) = f(\xi) \quad , \quad \xi \in G \qquad (8)$$

where G is defined in (4) .
The existence and uniquenessof bicubic spline $u_h$ is proved in [12] .
By introducing the vectors :

$$\mathbf{u} := \left[u_{1,1}, u_{1,2}, \ldots, u_{1,2N}, \ldots, u_{2N,1}, \ldots, u_{2N,2N}\right]^T$$

and

$$\mathbf{f} := \left[f_{1,1}, f_{1,2}, \ldots, f_{1,2N}, \ldots, f_{2N,1}, \ldots, f_{2N,2N}\right]^T \quad , \quad f_{n,m} := f\left(\xi_n, \xi_m\right)$$

the system (8) can be written as the system of linear equations

$$(A_\Phi \otimes B_\Psi + B_\Phi \otimes A_\Psi)\,\mathbf{u} = \mathbf{f} \tag{9}$$

where the matrices $A_\Phi$ and $B_\Phi$ , respectively $A_\Psi$ and $B_\Psi$ are defined by :

$$\begin{aligned} A_\Phi &= \left(a_{m,n}\right)_{m,n=1}^{2N} \ , \ a_{m,n} := -\Phi_n''\left(\xi_m\right) \ , \ B_\Phi = \left(b_{m,n}\right)_{m,n=1}^{2N} \ , \ b_{m,n} := \Phi_n\left(\xi_m\right) \\ A_\Psi &= \left(a_{m,n}\right)_{m,n=1}^{2N} \ , \ a_{m,n} := -\Psi_n''\left(\xi_m\right) \ , \ B_\Psi = \left(b_{m,n}\right)_{m,n=1}^{2N} \ , \ b_{m,n} := \Psi_n\left(\xi_m\right) \end{aligned} \tag{10}$$

and in (9) $\otimes$ denotes the matrix tensor product .

It follows from (10) and from construction of the bases of $S^0_h \otimes S^0_h$ that $A_\psi$ and $B_\psi$ are 2N x 2N almost block diagonal matrices with the first and last 2x3 matrix and the others 2x4 blocks in $A_\psi$ and $B_\psi$ given by :

$$h^{-2}\begin{bmatrix} a_1 & a_2 & -a_1 & a_3 \\ -a_1 & -a_3 & a_1 & -a_2 \end{bmatrix} \text{and} \begin{bmatrix} b_1 & b_2 & b_3 & -b_4 \\ b_3 & b_4 & b_1 & -b_2 \end{bmatrix}$$

respectively, where: $a_1 = 2\sqrt{3}$ , $a_2 = 1+\sqrt{3}$ , $a_3 = \sqrt{3}-1$ , $b_1 = \left(9+4\sqrt{3}\right)/18$

$b_2 = \left(3+\sqrt{3}\right)/36$ , $b_3 = \left(9-4\sqrt{3}\right)/18$ , $b_4 = \left(3-\sqrt{3}\right)/36$

The first 2x3 block in each matrix is obtained by removing the first column from the 2x4 block of the corresponding matrix, and the last 2x3 block is obtained by removing the third column from 2x4 block of the corresponding matrix.
Because of the special structure of the matrix $B_\Phi$ , there are at most four nonzero elements in each its column, and therefore the system (9) can be solved effectively, getting the approximating bicubic spline solution for the problem (6).

Now, we consider the nonhomogenous Dirichlet problem for the Poisson's equation on $\Omega$ :

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{11}$$

where g is a given function.
The bicubic spline approximating solution $u_h \in S_h^0 \otimes S_h^0$ for (11) is defined in the following way:

$$u_h(x,y)=\sum_{i=0}^{2N+1}\sum_{j=0}^{2N+1}u_{ij}\Phi_i(x)\Psi_j(y) \tag{12}$$

where the bases $\{\Phi_i\}^{2N+1}$ and $\{\psi_j\}^{2N+1}$ of the spline space $S^0{}_h\otimes S^0{}_h$ are defined in the previous section.
First we rewrite $u_h(x,y)$ in the form

$$u_h(x,y)=\bar{u}_h(x,y)+\tilde{u}_h(x,y) \tag{13}$$

where

$$\bar{u}_h(x,y):=\sum_{i=1}^{2N}\sum_{j=1}^{2N}u_{i,j}\Phi_i(x)\Psi_j(y) \tag{14}$$

corresponds to the bicubic spline collocation solution of the homogeneous Dirichlet problem (6) and

$$\tilde{u}_h(x,y)=\sum_{j=0}^{2N+1}u_{0,j}\Phi_0(x)\Psi_j(y)+\sum_{j=0}^{2N+1}u_{2N+1,j}\Phi_{2N+1}(x)\Psi_j(y)+\sum_{i=1}^{2N}u_{i,0}\Phi_i(x)\Psi_0(y)+\sum_{i=1}^{2N}u_{i,2N+1}\Phi_i(x)\Psi_{2N+1}(y) \tag{15}$$

corresponds to the nonhomgeneous boundary condition in (11).
The coefficients of $\tilde{u}_h$ in (15) can be determined independently therefore the existence and uniqueness of the bicubic spline approximate solution to the nonhomogeneous Dirichlet problem for Poisson's equation (11) follows directly from that of the homogeneous Dirichlet problem.
Following Bialecki and Cai [1] we present two approach to determine the coefficients $\tilde{u}_h$ in (15) which we refer to as the boundary coefficients of $u_h$.
In the first approach we approximate the boundary condition u=g using the Hermite cubic spline interpolant of g on each side of $\partial\Omega$. On the left and right hand sides of $\partial\Omega$ we require :

$$(u_h-g)(0,t_n)=0\ ,\ \frac{\partial}{\partial y}(u_h-g)(0,t_n)=0\ ,\ n=0,.1,...,N \tag{16}$$

$$(u_h-g)(1,t_n)=0\ ,\ \frac{\partial}{\partial y}(u_h-g)(1,t_n)=0\ ,\ n=0,1,...,N \tag{17}$$

Substituting (12) in (16) and (17) for the coefficients $\{u_{o,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the left hand side of $\partial\Omega$ we obtain :

$$u_{0,2n}=g(0,t_n)\ ,\ u_{0,2n+1}=h\frac{\partial g}{\partial y}(0,t_n)\ ,\ n=0,1,...,N-1$$

$$u_{0,2N}=h\frac{\partial g}{\partial y}(0,t_N)\ ,\ u_{o,2N+1}=g(0,t_N)$$

and for the coefficients $\{u_{2N+1,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the right hand side of $\partial\Omega$, we have:

$$u_{2N+1,2n} = g(1, t_n) \ , \ u_{2N+1,2n+1} = h\frac{\partial g}{\partial y}(1, t_n) \ , \ n = 0,1,..., N-1$$

$$u_{2N+1,2N} = h\frac{\partial g}{\partial y}(1, t_N) \ , \ u_{2N+1,2N+1} = g(1, t_N)$$

On the bottom an top sides of $\partial\Omega$ we require that

$$(u_h - g)(t_n, 0) = 0 \ , \ n = 1,..., N-1 \ , \ \frac{\partial}{\partial x}(u_n - g)(t_n, 0) = 0 \ , \ n = 0,..., N$$

$$(u_h - g)(t_n, 1) = 0 \ , \ n = 1,..., N-1 \ , \ \frac{\partial}{\partial x}(u_h - g)(t_n, 1) = 0 \ , \ n = 0,..., N$$

Substituting (12) into above relations we obtain explicitly the coefficients $\{u_{i,0}\}_{i=1}^{2N}$ of (15) corresponding to the bottom side of $\partial\Omega$ :

$$u_{n,0} = g(t_n, 0) \ , \ n = 1,..., N-1 \ , u_{N+n,0} = h\frac{\partial g}{\partial x}(t_n, 0) \ , \ n = 0,..., N$$

and the coefficients $\{u_{i,2N+1}\}_{i=1}^{2N}$ of (15) corresponding to the top side of $\partial\Omega$ :

$$u_{n,2N+1} = g(t_n, 1) \ , \ n = 1,..., N-1 \ ; \ u_{N+n,2N+1} = h\frac{\partial g}{\partial x}(t_n, 1) \ , n = 0,..., N$$

In the second approach we approximate u=g on $\partial\Omega$ using the cubic spline interpolant at the boundary Gauss points . Thus, on the left and right hand sides of $\partial\Omega$, we require

$$(u_h - g)(0,0) = 0 \ , \ (u_h - g)(0, \xi_m) = 0 \ , \ m = 1,...,2N \ ; \ (u_h - g)(0,1) = 0$$

and

$$(u_h - g)(1,0) = 0 \ , \ (u_h - g)(1, \xi_m) = 0 \ , \ m = 1,...,2N \ ; \ (u_h - g)(1,1) = 0$$

respectively . By substituting (12) in the first above relation we obtain the following relationships among the coefficients $\{u_{0,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the left hand side of $\partial\Omega$ :

$$u_{0,0} = g(0,0) \ , \ \sum_{j=0}^{2N+1} u_{o,j}\Psi_j(\xi_m) = g(0, \xi_m) \ , \ m = 1,...,2N \ ; \ u_{o,2N+1} = g(0,1) \quad (18)$$

If we set:

$\mathbf{u}_0 := (u_{0,0}, u_{0,1}, ..., u_{0,2N}, u_{0,2N+1})^T$; $\mathbf{g}_0 := \{g(0,0), g(0, \xi_1), ..., g(0, \xi_{2N}), g(0,1)\}^T$

then (18) can be written as the (2N+2)x(2N+2) almost block linear system, of the form :

$$\begin{pmatrix} 1 & 0 & & & & & & & & \\ x & x & x & x & & & & & & \\ x & x & x & x & & & & & & \\ & & x & x & x & x & & & & \\ & & x & x & x & x & & & & \\ & & & . & . & . & . & & & \\ & & & & & & x & x & x & x \\ & & & & & & x & x & x & x \\ & & & & & & & & 1 & 0 \end{pmatrix} \mathbf{u}_0 = \mathbf{g}_0 \qquad (19)$$

with the same 2x4 blocks. Thus the coefficients $\{u_{0,j}\}_{j=0}^{2N+1}$ can be obtained by solving the system (19). By substituting (12) into the second above boundary relation, the coefficients $\{u_{2N+1,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the right hand side of $\partial\Omega$ can be obtained in a similar way.
On the bottom and top sides of $\partial\Omega$ we require :

$$(u_h - g)(0,0) = 0 \;\; , \;\; (u_h - g)(\xi_{m,} 0) = 0 \;\; , \; m = 1,...,2N \; ; \; (u_h - g)(1,0) = 0$$

and

$$(u_h - g)(0,1) = 0, \;\; (u_h - g)(\xi_m, 1) = 0, \;\; m = 1,...,2N \;\; ; \; (u_h - g)(1,1) = 0$$

The first and last above equations give:

$$u_{0,0} = g(0,0), \;\; u_{2N+1,0} = g(1,0); \;\; u_{0,2N+1} = g(0,1); \;\; u_{2N+1,2N+1} = g(1,1)$$

Using the reordering basis functions $\{\Phi_i\}_{i=1}^{2N}$ as the basis functions $\{\Psi_i\}_{i=1}^{2N}$, the coefficients $\{u_{i,0}\}_{i=1}^{2N}$ and $\{u_{i,2N+1}\}_{i=1}^{2N}$ of (15) corresponding to the bottom and top sides of $\partial\Omega$ can also be determined by solving a almost block diagonal system of the form (19). Consequently, $\tilde{u}_h$ of (13) is determined.

Now we have only to obtain the coefficients of $\bar{u}_h$ in (14).This can be done by requiring that :

$$-\Delta\bar{u}_h(\xi) = \Delta\tilde{u}_h(\xi) + f(\xi), \quad \xi \in G \qquad (20)$$

where the right hand side is known. The functions $\bar{u}_h$ can be obtained by employing the same algorithm for solving the homogeneous Dirichlet problem (6).
Note that if g=0 , then, in both approaches, all of the coefficients in $\tilde{u}_h$ are zero.. But if $g\neq0$ , then, in general, the approximations obtained will be different.

Converting to the estimation of the error and converquence of the given bicubic spline collocation method, Bielecki and Cai [1] have proved that, on nonunifrm partitions, the $H^1$-norm error bounds for the first and second approach are

$O(h^3)$ provided that the exact solution u belongs to $H^5(\Omega)$ and $H^5(\Omega)\cap C^4(\Omega)$, respectively.

Dillery [5] improved and extended these results.In particular, it was shown that the $H^1$-norm error bound for the second approach is $O(h^2)$ under the assumption that $u\in H^6(\Omega)$, then the $L^2$-norm of the error for each approach is $O(h^6)$.

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