

## ON THE BILATERAL TRUNCATED EXPONENTIAL DISTRIBUTIONS

by  
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**Abstract.** There is a special class of probability distributions, namely the exponential family of probability distributions, for which complete sufficient statistics with fixed dimension always exist. This class includes some, but not all, of the commonly used distributions. The objective of this paper is to give some definitions and some properties for some probability distributions which belong to such class. Also, we shall investigate some measures of the information of the unknown parameters which appear in such exponential family.

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### 1. Introduction

The notion of information plays a central role both in the life of the person and of society, as well as in all kinds of scientific research. The notion of information is so universal, it penetrates our everyday life so much that from this point of view, it can be compared only with the notion of energy. The information theory is an important branch of probability theory and it has very much application in mathematical statistics.

Let  $X$  be a random variable on the probability space  $(\Omega, \mathcal{K}, P)$ . A statistical problem arises then when the distribution of  $X$  is not known and we want to draw some inference concerning the unknown distribution of  $X$  on the basis of a limited number of observations on  $X$ . A general situation may be described as follows: the functional form of the distribution function is known and merely the values of a finite number of parameters, involved in the distribution function, are unknown; i.e., the probability density function of the random variable  $X$  is known except for the value of a finite number of parameters. In general, the parameters  $\theta_1, \theta_2, \dots, \theta_k$  will not be subject to any a priori restrictions; i.e., they may take any values. However, the parameters may in some cases be restricted to certain intervals.

Let  $X$  be a continuous random variable and its probability density function  $f(x; \theta)$  depends on an parameter  $\theta$  which is real  $k$ -dimensional parameter having values in a specified parameter space  $D_\theta$ ,  $D_\theta \subseteq \mathbf{R}^k$ ,  $k \geq 1$ . Thus we are confronted, not with one distribution of probability, but with a family of distributions. To each value of  $\theta$ ,

$\theta \in D_\theta$ , there corresponds one member of the family which will be denoted by the symbol  $\{f(x;\theta) ; \theta \in D_\theta\}$ . Any member of this family of probability density functions will be denoted by the symbol  $f(x;\theta)$ ,  $\theta \in D_\theta$ .

Let  $C$  be a statistical population and  $X$  a common property for all elements of this population. We suppose that this common property is a continuous random variable which has the probability density function  $f(x, \theta)$ ,  $\theta \in D_\theta$ ,  $D_\theta \subseteq \mathbf{R}$ ,  $\theta$ - unknown parameter and  $S_n(X)=(X_1, X_2, \dots, X_n)$  a random sample of size  $n$  from this population.

A general problem is that of defining a statistic  $\hat{\theta} = t(X_1, X_2, \dots, X_n)$ , so that if  $x_1, x_2, \dots, x_n$  are the observed experimental values of  $X_1, X_2, \dots, X_n$ , then the number  $t(x_1, x_2, \dots, x_n)$  is an estimate of  $\theta$  and it is usually written as  $\hat{\theta}_0 = t(x_1, x_2, \dots, x_n)$ , that is, we have  $g : \mathbf{R}^n \rightarrow \mathbf{R}$ , where  $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \dots \times \mathbf{R}$  is the sample space and  $\mathbf{R}$  is the one-dimensional sample space (the real line). In this case the statistic  $(X_1, X_2, \dots, X_n)$  represents an estimator for the unknown parameter  $\theta$ . Thus, we recall some very important definitions.

**Definition 1.1** An estimator  $\theta = t(X_1, X_2, \dots, X_n)$  is a function of the random sample vector (a statistic)

$$S_n(X) = (X_1, X_2, \dots, X_n)$$

that estimates  $\theta$  but is not dependent on  $\theta$ .

**Definition 1.2** Any statistic whose mathematical expectation is equal to a parameter  $\theta$  is called an unbiased estimator. Otherwise, the statistic is said to be biased.

**Definition 1.3** Any statistic that converges stochastically to a parameter  $\theta$  is called a **consistent estimator** of the parameter  $\theta$ , that is, if we have

$$\lim_{n \rightarrow \infty} P[|\hat{\theta} - \theta| \leq \varepsilon] = 1 \text{ for all } \varepsilon > 0. \quad (1.1)$$

**Definition 1.4** An estimator  $\theta = t(X_1, X_2, \dots, X_n)$  of  $\theta$  is said to be a minimum variance unbiased estimator of  $\theta$  if it has the following two properties:

- a)  $E(\theta) = \theta$ , that is,  $\theta$  is an unbiased estimator,
- b)  $\text{Var}(\theta) \leq \text{Var}(\theta^*)$  for any other estimator  $\theta^* = h(X_1, X_2, \dots, X_n)$ , which is also unbiased for  $\theta$ , that is,  $E(\theta^*) = \theta$ .

In the next we suppose that the parameter  $\theta$  is **unknown** and we estimate a specified function of  $\theta$ ,  $g(\theta)$ , with the help of statistic  $\theta = t(X_1, X_2, \dots, X_n)$  which is based on a random sample of size  $n$ ,  $S_n(X) = (X_1, X_2, \dots, X_n)$ , where  $X_i$  are independent and identically distributed random variable as the random variable  $X$ , that is, we have:  $f(x;\theta) = f(x_i;\theta)$ ,  $i=1, n$ ,  $\theta \in D_\theta$ .

A well known means of measuring the quality of the estimator

$$\theta = t(X_1, X_2, \dots, X_n) \quad (1.3)$$

is to use **the inequality of Cramér-Rao** which states that, under certain regularity conditions for  $f(x; \theta)$  (more particularly, it requires the possibility of differentiating under the integral sign) any unbiased estimator of  $g(\theta)$  has variance which satisfies the following inequality

$$\text{Var } t \geq \frac{[g'(\theta)]^2}{n \cdot I_X(\theta)} = \frac{[g'(\theta)]^2}{I_n(\theta)}, \quad (1.4)$$

where

$$I_X(\theta) = \int_{\Omega} \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 f(x; \theta) dx = \quad (1.6)$$

$$= - \int_{\Omega} \frac{1}{f(x; \theta)} \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 dx, \quad (1.7)$$

and

$$I_n(\theta) = nE \left[ \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 \right] = n \int_{\Omega} \left( \frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 f(x; \theta) dx \quad (1.8)$$

The quantity  $I_X(\theta)$  is known as Fisher's information measure and it measures the information about  $g(\theta)$  which is contained in an observation of  $X$ . Also, the quantity  $I_n(\theta) = n \cdot I_X(\theta)$  measures the information about  $g(\theta)$  contained in a random sample  $S_n(X) = (X_1, X_2, \dots, X_n)$ , then when  $X_i$ ,  $i = 1, n$ , are independent and identically distributed random variables with density  $f(x; \theta)$ ,  $\theta \in D_\theta$ .

An unbiased estimator of  $g(\theta)$  that achieves this minimum from (1.5) is known as an **efficient estimator**.

## 2. Exponential Families of Distributions

Let  $X$  be a random variable defined on a probability space  $(\Omega, K, P)$  and its probability density function (or probability function)  $f(x; \theta)$  which depends on an parameter (or random parameter)  $\theta$  with values in a specified parameter space  $D_\theta$ ,  $D_\theta \subseteq \mathbf{R}$ , that is,  $f(x; \theta)$  is an member of the family  $\{f(x; \theta); \theta \in D_\theta\}$ . There is a special family of probability distributions, namely the exponential family of probability distributions.

**Definition 2.1** [8] A family of distribution with probability density function or probability function  $f(x; \theta)$ ;  $\theta \in D_\theta$ , is said to belong to the exponential family of

distributions (the on-parameter exponential families) if  $f(x; \theta)$  can be expressed in the form

$$f(x; \theta) = c(\theta)h(x) \exp \left\{ \sum_{i=1}^k \pi_i(\theta)t_i(x) \right\}, \quad \forall \theta \in D_\theta, \quad (2.1)$$

for some measurable functions  $\pi_i, t_i, i = 1, k$ , and some integer  $k$ .

**Remark 2.1** The exponential families of distributions are frequently called the “**Koopman - Pitman – Darmois**” families of distributions from the fact that these three authors independently and almost simultaneously (1937-1938) studied their main properties.

**Remark 2.2** Examples of exponential families of distributions are: **the binomial distributions**  $\text{Bin}(n, p)$  with  $n$  known and  $0 < p < 1$ , **the Poisson distributions**  $\text{Poi}(\lambda)$ , **the negative binomial distributions**  $\text{Negbin}(a, p)$  with  $a$  known, **the geometric distributions**  $\text{Geo}(p)$ , **the normal distributions**  $N(\mu, \sigma^2)$  with  $\theta = (\mu, \sigma^2)$ , **the gamma distributions**  $\Gamma(a, b)$  with  $\theta = (a, b)$ , **the chi-squared distribution**  $\chi_n^2$ , **the exponential distribution**  $\text{Exp}(\lambda)$ , and **the beta distributions**  $\text{Beta}(\alpha, \beta)$  with  $\theta = (\alpha, \beta)$ .

**Remark 2.3** If each  $\pi_i(\theta) = \pi_i, i = 1, k$  is taken to be a parameter in (2.1), so that

$$\begin{aligned} f(x; \pi) &= c(\pi)h(x) \exp \left\{ \sum_{i=1}^k \pi_i t_i(x) \right\} = \\ &= c(\pi)h(x) \exp \{ \pi(\theta)[T(x)] \}, \end{aligned} \quad (2.2)$$

where

$$\pi = \pi(\theta) = (\pi_1, \pi_2, \dots, \pi_k) = (\pi_1(\theta), \pi_2(\theta), \dots, \pi_k(\theta)) \in \mathbf{R}^k \quad (2.3)$$

and

$$T(x) = (t_1(x), t_2(x), \dots, t_{\{k\}}(x)), \quad (2.4)$$

we say that, for the exponential family, has been given its **natural parameterization**.

**Definition 2.2** [8] In an exponential family, **the natural parameter** is the vector (2.3), and

$$\Pi = D_\pi = \left\{ \pi \in \mathbf{R}^k : \int_S h(x) \exp \left\{ \sum_{i=1}^k \pi_i t_i(x) \right\} dx < \infty \right\}, \quad (2.5)$$

is called **the natural parameter space**, where  $S$  is **the sample space**, that is, the set of possible values for the random variable  $X$ .

Obviously, for any  $\theta \in D\theta$ , the point  $(\pi_1(\theta), \pi_2(\theta), \dots, \pi_k(\theta))$  must lie in  $\Pi$ .

### 3. Definitions and properties for the bilateral truncated Gamma distributions

Let  $X$  be a continuous random variable, defined on a probability space  $(\Omega, \mathcal{K}, P)$  and  $f(x; \theta)$  its probability density function, where  $\theta = (\theta_1, \theta_2)$ ,  $\theta \in D\theta$ ,  $D\theta$ -the parameter space or the set of admissible values of  $\theta$ ,  $D_\theta \subseteq \mathbf{R}^2$ .

Then when the vector parameter covers the parameter space  $D_\theta$  we obtain the family of probability density functions  $\{f(x; \theta); \theta \in D\theta\}$ .

**Definition 3.1.** The continuous random variable  $X$  follows the **Gamma distribution** with parameters  $\theta_1 = a > 0$  and  $\theta_2 = b > 0$  if its probability density function is of the following form

$$f_X(x; a, b) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & \text{if } x > 0 \end{cases} \quad (3.1)$$

where

$$\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt \quad (3.2)$$

is **Euler's Gamma function** or the complete Euler function.

For a such random variable the mean value  $E(X)$  and the variance  $\text{Var } X$  have the following values

$$E(X) = \frac{a}{b}, \text{Var } X = \frac{a}{b^2}, \quad (3.3)$$

where  $\theta = (a, b)$  is the vector parameters,  $\theta \in D_\theta = \mathbf{R}_+^2$ .

**Definition 3.2** [5] We say that the continuous random variable  $X$  has a **Gamma distribution, truncated to the left** at  $X = \alpha$  and **to the right** at  $X = \beta$ , if its probability density function, denoted by  $f_{\alpha \leftrightarrow \beta}$ , is of the form

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \begin{cases} C(a, b) x^{a-1} e^{-bx} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{if } x < \alpha \text{ or } > \beta \end{cases}, \quad \alpha, \beta \in \mathbf{R}, \alpha \geq 0, \quad (3.4)$$

where  $C(a, b)$  is a constant with one of the following forms:

$$C(a, b) = \frac{1}{\beta \int_{\alpha} t^{a-1} e^{-bx} dt}; \alpha, \beta \in \mathbf{R}, 0 \leq \alpha \leq \beta, \quad (3.5)$$

or

$$C(a, b) = \frac{b^a}{\Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]}; \alpha, \beta \in \mathbf{R}, 0 \leq \alpha \leq \beta, \quad (3.6)$$

if  $a, b \in \mathbf{R}^+$ , or

$$C(a, b) = \frac{b^a}{\sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}]} \quad (3.7)$$

where

$$A_{a-1}^k = (a-1)(a-2) \dots [(a-1) - (k-1)], k=0, \dots, a-1, \quad (3.8)$$

if  $a \in \mathbf{N}^*$ ,  $b \in \mathbf{R}^+$ .

**Corollary 3.1** For the integral which appear at the denominator of the relation (3.5) we have one of the expressions:

$$I(\alpha, \beta) = \frac{1}{b^a} \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}] \quad (3.9)$$

if  $\alpha, \beta \in \mathbf{R}, 0 \leq \alpha \leq \beta$ , and  $a \in \mathbf{N}^*$ ,  $b \in \mathbf{R}^+$ , or

$$I(\alpha, \beta) = \frac{\Gamma(a)}{b^a} [\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)], \quad (3.10)$$

if  $\alpha, \beta \in \mathbf{R}, 0 \leq \alpha \leq \beta$ , and  $a, b \in \mathbf{R}^+$ .

**Lemma 3.1** The family of Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ ,  $\{f_{\alpha \leftrightarrow \beta}(x; a, b); a, b \in \mathbf{R}^+\}$ , is an exponential family.

**Proof.** Because, for the  $x > 0$ , we have

$$x^{a-1} e^{-bx} = e^{-bx+(a-1) \ln x}, \quad (3.11)$$

the probability density function,  $f_{\alpha \leftrightarrow \beta}(x; a, b)$ , may be rewritten in the following form

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \frac{b^a}{\Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]} \exp\left\{ \sum_{i=1}^2 \pi_i(\theta) t_i(x) \right\}. \quad (3.12)$$

If we have in view the definition 2.1, as well as, the following correspondences:

$$\theta = (a, b), \quad a, b \in \mathbf{R}^+$$

$$c(\theta) = \frac{b^a}{\Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]}, \quad \Gamma(a) = \int_0^{\infty} x^{a-1} e^{-bx} dx$$

$$\begin{aligned} h(x) &= 1, & k &= 2 \\ \pi_1(\theta) &= -b, & \pi_2(\theta) &= a-1 \\ t_1(x) &= x, & t_2(x) &= \ln x, \end{aligned} \quad (3.13)$$

we can establish that indeed the family of Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , is an exponential family.

**Theorem 3.1** If  $X_{\alpha \leftrightarrow \beta}$  is a random variable with a Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , where  $\alpha, \beta \in \mathbf{R}$ ,  $\alpha \geq 0$ , and its probability density has the form

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \begin{cases} \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} x^{a-1} e^{-bx} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{if } x < \alpha \text{ or } x > \beta \end{cases} \quad (3.14)$$

then the mean value (the expected value) has one of the following forms:

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}, \quad (3.15)$$

if  $\alpha, \beta \in \mathbf{R}$ ,  $0 \leq \alpha \leq \beta$ , and  $a, b \in \mathbf{R}^+$ , or

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}]}, \quad (3.16)$$

if  $\alpha, \beta \in \mathbf{R}$ ,  $0 \leq \alpha \leq \beta$ , and  $a \in \mathbf{N}^*$ ,  $b \in \mathbf{R}^+$ .

**Proof.** In accordance with the definition of the mean value  $E(X_{\alpha \leftrightarrow \beta})$ , we have

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \int_\alpha^\beta x^a e^{-bx} dx \quad (3.17)$$

By integrating by parts in (3.17), that is, letting

$$\begin{cases} u = x^a \\ dv = e^{-bx} dx \end{cases} \Rightarrow \begin{cases} du = ax^{a-1} dx \\ v = -\frac{1}{b} e^{-bx} \end{cases} \quad (3.17a)$$

then  $E(X_{\alpha \leftrightarrow \beta})$  can be rewritten as follows

$$\begin{aligned} E(X_{\alpha \leftrightarrow \beta}) &= \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \left\{ (1/b)[\alpha^a e^{-\alpha b} - \beta^a e^{-\beta b}] + \frac{a}{b} \int_\alpha^\beta x^{a-1} e^{-bx} dx \right\} \\ &= \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \left\{ (1/b)[\alpha^a e^{-\alpha b} - \beta^a e^{-\beta b}] + \frac{a}{b} I(\alpha, \beta) \right\}. \end{aligned} \quad (3.18)$$

Then, if we have in view the form (3.9) of the integral  $I(\alpha, \beta)$ , from (3.18) we obtain

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{1}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \left\{ \frac{a}{b} [(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}] + \frac{a}{b} \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}] \right\} \quad (3.19)$$

which is an uncomfortable form for the mean value  $E(X_{\alpha \leftrightarrow \beta})$ .

The first form of the mean value, (3.15), can be obtained using (3.10) and (3.18).

**Remark 3.1** It is easy to see that, from the relations (3.9) and (3.10), we obtain an important equality, namely

$$\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)] = \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}] \quad (3.20)$$

This last relation together with the relation (3.15) gives us the possibility to find a new form (the third form) of the mean value, namely

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}]}, \quad a > 0, b > 0. \quad (3.21)$$

**Theorem 3.2** If  $X_{\alpha \leftrightarrow \beta}$  is a random variable with a Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , where  $\alpha, \beta \in \mathbf{R}$ ,  $\alpha \geq 0$ , and its probability density has the form (3.14), then the variance  $\text{Var}(X_{\alpha \leftrightarrow \beta})$  has the form

$$\text{Var}(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b^2} - \frac{(\beta b)^{a+1} e^{-\beta b} - (\beta b)^{a+1} e^{-\alpha b}}{b^2 \Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} + \frac{a-1}{b^2} \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} - \left[ \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b \Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \right]^2 \quad (3.22)$$

**Proof.** In order to obtain the variance of the random variable  $X_{\alpha \leftrightarrow \beta}$  we shall use the formula

$$\mu_2(X_{\alpha \leftrightarrow \beta}) = \text{Var}(X_{\alpha \leftrightarrow \beta}) = E[(X_{\alpha \leftrightarrow \beta})^2] - [E(X_{\alpha \leftrightarrow \beta})]^2, \quad (3.23)$$



where  $E[(X_{\alpha \leftrightarrow \beta})^2] = \alpha_2(X_{\alpha \leftrightarrow \beta}) = \alpha_2$  is the 2-th order moment of  $X_{\alpha \leftrightarrow \beta}$  about the origin.

Using the definition for the moment  $\alpha_2$ , namely

$$E[(X_{\alpha \leftrightarrow \beta})^2] = C(a,b) \int_{\alpha}^{\beta} x^{a+1} e^{-bx} dx, \quad (3.24)$$

where

$$C(a,b) = \frac{b^a}{\Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]}, \quad (3.25)$$

and integrating by parts, that is, letting

$$\begin{cases} u = x^{a+1} & du = (a+1)x^a dx \\ dv = e^{-bx} dx & v = -\frac{1}{b}e^{-bx} \end{cases} \Rightarrow \quad (3.26)$$

we can obtain a new expression for this moment

$$\begin{aligned} E((X_{\alpha \leftrightarrow \beta})^2) &= C(a,b) \left\{ \frac{\alpha^{a+1} e^{-\alpha b} - \beta^{a+1} e^{-\beta b}}{b} + \frac{a+1}{b} \int_{\alpha}^{\beta} x^a e^{-bx} dx \right\} = \\ C(a,b) &\frac{\alpha^{a+1} e^{-\alpha b} - \beta^{a+1} e^{-\beta b}}{b} + \frac{a+1}{b} E(X_{\alpha \leftrightarrow \beta}) = \frac{(\alpha b)^{a+1} e^{-\alpha b} - (\beta b)^{a+1} e^{-\beta b}}{b^2 \Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]} + \\ &\frac{a+1}{b} \left[ \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b \Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]} \right], \end{aligned} \quad (3.27)$$

if we had in view the Theorem 3.1.

This last form, (3.27), may be express and in a form more convenient, namely

$$\begin{aligned} E((X_{\alpha \leftrightarrow \beta})^2) &= \frac{a(a+1)}{b^2} - \frac{(\alpha b)^{a+1} e^{-\alpha b} - (\beta b)^{a+1} e^{-\beta b}}{b^2 \Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]} - \\ &- \frac{a+1}{b^2} \left[ \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)[\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)]} \right]. \end{aligned} \quad (3.28)$$

Using the relations (3.28), (3.25) and (3.23) we obtain just the form (3.22) for the variance of the random variable  $X_{\alpha \leftrightarrow \beta}$  which follows the Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ ,  $\alpha, \beta \in \mathbf{R}$ ,  $\alpha \geq 0$ .

#### 4. Fisher's information measures for the truncated Gamma distribution

Let  $X_{\alpha \leftrightarrow \beta}$  be a random variable which has a Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , with probability density function of the form

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \begin{cases} C(a, b) x^{a-1} e^{-bx} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{if } x < \alpha \text{ or } > \beta \end{cases}, \quad \alpha, \beta \in \mathbf{R}, \alpha \geq 0, \quad (4.1)$$

where

$$C(a, b) = \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}; \quad \alpha, \beta \in \mathbf{R}, 0 \leq \alpha \leq \beta, \quad (4.2)$$

if  $a, b \in \mathbf{R}^+$ , where

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \quad a > 0, \quad (4.3)$$

is Euler's Gamma function, and

$$\Gamma_\beta(a) = \frac{b^a}{\Gamma(a)} \int_0^\beta x^{a-1} e^{-bx} dx, \quad \Gamma_\alpha(a) = \frac{b^a}{\Gamma(a)} \int_0^\alpha x^{a-1} e^{-bx} dx \quad (4.4)$$

**Theorem 4.1** If  $X_{\alpha \leftrightarrow \beta}$  follows a **Gamma distribution**, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , with probability density function of the form (4.1), where  $a, b \in \mathbf{R}^+$ ,  $a$ - parameter **known**,  $b$ - parameter **unknown**, then the **Fisher information measure** corresponding to  $X_{\alpha \leftrightarrow \beta}$  has the following form

$$I_{X_{\alpha \leftrightarrow \beta}}(b) = \frac{a}{b^2} - \frac{(\beta b)^{a+1} e^{-\beta b} - (\alpha b)^{a+1} e^{-\alpha b}}{b^2 \Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} + \frac{a-1}{b^2} \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} - \left( \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} \right)^2. \quad (4.5)$$

**Proof.** Because  $X_{\alpha \leftrightarrow \beta}$  is a continuous random variable and  $\theta = b$  is an unknown parameter it follows that the Fisher information measure, with respect to the unknown parameter  $b$ , has the form

$$I_{X_{\alpha \leftrightarrow \beta}}(b) = \int_\alpha^\beta \left( \frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b} \right)^2 f_{\alpha \leftrightarrow \beta}(x; a, b) dx \quad (4.6)$$

But, using the property of the probability density function  $f_{\alpha \leftrightarrow \beta}(x; a, b)$ , that is,

$$\int_a^\beta f_{\alpha \leftrightarrow \beta}(x; a, b) dx = 1, \quad (4.7)$$

and if we can differentiate twice, with respect to b, under the integral signs then, we get

$$\frac{d}{db} \int_a^\beta f_{\alpha \leftrightarrow \beta}(x; a, b) dx = \int_a^\beta \frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b} f_{\alpha \leftrightarrow \beta}(x; a, b) dx = 0, \quad (4.8)$$

and

$$\begin{aligned} \frac{d^2}{db^2} \int_a^\beta f_{\alpha \leftrightarrow \beta}(x; a, b) dx &= \int_a^\beta \frac{\partial^2 \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial^2 b} f_{\alpha \leftrightarrow \beta}(x; a, b) dx + \\ &+ \int_a^\beta \left( \frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b} \right)^2 f_{\alpha \leftrightarrow \beta}(x; a, b) dx = 0 \end{aligned} \quad (4.9)$$

From these last relations, (4.8) and (4.9), we obtain the following very important equalities, namely

$$E\left(\frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b}\right) = 0, \quad (4.10)$$

$$\begin{aligned} E\left\{\left(\frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b}\right)^2\right\} &= \int_a^\beta \left(\frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b}\right)^2 f_{\alpha \leftrightarrow \beta}(x; a, b) dx = \\ &= - \int_a^\beta \frac{\partial^2 \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial^2 b} f_{\alpha \leftrightarrow \beta}(x; a, b) dx \end{aligned} \quad (4.11)$$

Having this last equality, (4.11), the relation (4.6) can be rewritten in a new form

$$I_{X_{\alpha \leftrightarrow \beta}}(b) = - \int_a^\beta \frac{\partial^2 \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial^2 b} f_{\alpha \leftrightarrow \beta}(x; a, b) dx, \quad (4.12)$$

which represents a new relation of definition for Fisher's information measure.

Now, by means of the probability density function (4.1), we obtain

$$\ln f_{\alpha \leftrightarrow \beta}(x; a, b) = a \ln b - \ln \Gamma(a) - \ln[\Gamma_\beta(a) - \Gamma_\alpha(a)] + (a-1) \ln x - bx, \quad (4.13)$$

where  $\Gamma(a)$ ,  $\Gamma_\beta(a)$  and  $\Gamma_\alpha(a)$  have been specified in (4.3) and (4.4).

From (4.13), we find that

$$\frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b} = \frac{a}{b} - x - \frac{\frac{\partial \Gamma_{\beta}(a)}{\partial b} - \frac{\partial \Gamma_{\alpha}(a)}{\partial b}}{\Gamma_{\beta}(a) - \Gamma_{\alpha}(a)} \quad (4.14)$$

Using (3.4), we get

$$\begin{aligned} \frac{\partial \Gamma_{\beta}(a)}{\partial b} &= \frac{\partial}{\partial b} \left( \frac{b^a}{\Gamma(a)} \int_0^{\beta} x^{a-1} e^{-bx} dx \right) = \frac{ab^{a-1}}{\Gamma(a)} \int_0^{\beta} x^{a-1} e^{-bx} dx - \\ &- \frac{b^a}{\Gamma(a)} \int_0^{\beta} x^a e^{-bx} dx = \frac{a}{b} \left( \frac{b^a}{\Gamma(a)} \int_0^{\beta} x^{a-1} e^{-bx} dx \right) - \\ &\frac{\Gamma(a+1)}{b\Gamma(a)} \left( \frac{b^{a+1}}{\Gamma(a+1)} \int_0^{\beta} x^a e^{-bx} dx \right) = \frac{a}{b} [\Gamma_{\beta}(a) - \Gamma_{\beta}(a+1)]. \end{aligned} \quad (4.15)$$

But, for the integral

$$\Gamma_{\beta}(a+1) = \left( \frac{b^{a+1}}{\Gamma(a+1)} \int_0^{\beta} x^a e^{-bx} dx \right), \quad (4.16)$$

then when we apply the well known formula for integration by parts, that is, letting

$$u = x^a, \quad dv = e^{-bx} \Rightarrow du = ax^{a-1} dx, \quad v = -\frac{1}{b} e^{-bx},$$

we obtain a new form

$$\Gamma_{\beta}(a+1) = \Gamma_{\beta}(a) - \frac{(\beta b)^a}{\Gamma(a+1)} e^{-\beta b} \quad (4.17)$$

A such relation is holds and in the general case, namely

$$\Gamma_{\beta}(a+k) = \Gamma_{\beta}(a+k-1) - \frac{(\beta b)^{a+k-1}}{\Gamma(a+k)} e^{-\beta b} \quad (4.18)$$

From (4.15) and (4.17), we get the following relation

$$\frac{\partial \Gamma_{\beta}(a)}{\partial b} = \frac{(\beta b)^a}{b\Gamma(a+1)} e^{-\beta b} \quad (4.19)$$

which is holds and in general case, namely

$$\frac{\partial \Gamma_{\beta}(a+k)}{\partial b} = \frac{a+k}{b} \frac{(\beta b)^{a+k}}{\Gamma(a+k+1)} e^{-\beta b}, \quad k \in \mathbb{N}, a+k+1 > 0. \quad (4.20)$$

In a similar manner, from (4.4), we obtain

$$\frac{\partial \Gamma_{\alpha}(a)}{\partial b} = \frac{(\alpha b)^a}{b\Gamma(a+1)} e^{-\alpha b}. \quad (4.21)$$

From (4.14), (4.19) and (4.21), we conclude that

$$\frac{\partial \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b} = \frac{a}{b} - x - \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b \Gamma(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]} \quad (4.22)$$

Then, from (4.22), we obtain

$$\begin{aligned} \frac{\partial^2 \ln f_{\alpha \leftrightarrow \beta}(x; a, b)}{\partial b^2} &= -\frac{a}{b^2} - \frac{\partial}{\partial b} \left\{ \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b \Gamma(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]} \right\} = \\ &= -\frac{a}{b^2} - \frac{a-1}{b^2} \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]} + \frac{1}{b^2} \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]} \\ &\quad + \left\{ \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b \Gamma(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]} \right\}^2, \end{aligned} \quad (4.23)$$

if we have in view the following relations

$$\begin{aligned} \frac{\partial [(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}]}{\partial b} &= \frac{a}{b} [(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}] - \\ &\quad - \frac{1}{b} [(\beta b)^{a+1} e^{-\beta b} - (\alpha b)^{a+1} e^{-\alpha b}], \\ \frac{\partial \Gamma_\beta(a) [\Gamma_\beta(a) - \Gamma_\alpha(a)]}{\partial b} &= \Gamma(a) [\Gamma_\beta(a) - \Gamma_\beta(a)] + [(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}] \end{aligned} \quad (4.25)$$

Using this last relation and taking into account (4.12) we can express Fisher's information measure just in the form (4.5). Thus, the proof is complete.

**Corollary 4.1** If  $X$  follows a Gamma distribution and  $X_{\alpha \leftrightarrow \beta}$  follows a Gamma distribution, truncated to the left at  $X = \alpha$  and to the right at  $X = \beta$ , then the Fisher information  $I_{X_{\alpha \leftrightarrow \beta}}(b)$  and  $\text{Var}(X_{\alpha \leftrightarrow \beta})$  always are equal, that is, we have

$$I_{X_{\alpha \leftrightarrow \beta}}(b) = \text{Var}(X_{\alpha \leftrightarrow \beta}), \quad \alpha, \beta \in \mathbb{R}, \quad 0 \leq \alpha \leq \beta, \quad a, b \in \mathbb{R}^+.$$

**Proof.** This equality follows from the relations (3.22) and (4.5).

**Corollary 4.2** If  $\alpha$  and  $\beta \rightarrow +\infty$ , then the random variable  $X_{\alpha \leftrightarrow \beta}$  becomes an ordinary Gamma variable  $X$  and we have

$$\lim_{\alpha \rightarrow 0, \beta \rightarrow \infty} f_{\alpha \leftrightarrow \beta}(x; a, b) = f_X(x; a, b) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & \text{if } x > 0 \end{cases} \quad (4.27)$$

and

$$I_{X_{a \leftrightarrow b}}(b) = I_X(b) = \text{Var}(X) = \frac{a}{b^2}, \quad (4.28)$$

where  $a, b \in \mathbf{R}^+$ .

**Corollary 4.3** If  $\alpha, \beta \rightarrow +\infty$  and  $a = 1$ , then the random variable  $X_{\alpha \leftrightarrow \beta}$  becomes a negative exponential distribution  $X$  and we have

$$\lim_{\alpha \rightarrow 0, \beta \rightarrow \infty} f_{\alpha \leftrightarrow \beta}(x; a, b)|_{a=1} = f_X(x; b) = \begin{cases} 0 & \text{if } x \leq 0 \\ be^{-bx} & \text{if } x > 0, b > 0 \end{cases} \quad (4.29)$$

and

$$I_{X_{a \leftrightarrow b}}(b) = I_X(b) = \text{Var}(X) = \frac{1}{b^2}, \quad (4.30)$$

## REFERENCES

- [1] Lehmann, E.L., *Testing Statistical Hypotheses*, Springer-Verlag New York, Inc. 1986.
- [2] Mihoc, I., Fătu, C.I., *Fisher's Information Measures for a Truncated Distribution (I)*, Proceedings of the „Tiberiu Popoviciu” Itinerant Seminar of Functional Equations, Approximation and Convexity, Editura SRIMA, Cluj-Napoca, Romania, 2001, pp.119-131.
- [3] Mihoc, I., C.I.Fătu, *Estimation of the Multinomial Distribution*, Tiberiu Popoviciu Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, May 22 - 26, 2002, pp.165-179.
- [4] Mihoc, I., C.I.Fătu, *About some Truncated Probability Distributions*, Proceedings of the International Symposium on Numerical Analysis and Approximation Theory, Cluj-Napoca, May 9-11, 2002, pp.284-297.
- [5] Mihoc, I., C.I.Fătu., *The measure of Information and the Estimation Problem*, Perspectives in Modern Statistical Inference (II), Brno, August 14-17, 2002 (to appear).
- [6] Rao, C. R., *Linear Statistical Inference and Its Applications*, John Wiley and Sons, Inc., New York, 1965.
- [7] Rényi, A., *Probability Theory*, Akadémiai Kiado, Budapest, 1970.
- [8] Schervish, M.J., *Theory of Statistics*, Springer-Verlag New York, Inc. 1995.
- [9] Wilks, S.S., *Mathematical Statistics*, Wiley, New York, 1962.

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