GENERALIZED BIRKHOFF INTERPOLATION SCHEMES: CONDITIONS FOR ALMOST REGULARITY

by
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Abstract: Classically, Birkhoff interpolation schemes depend on a ``lower set'' S, which defines the interpolation space the solutions are required to belong to. In this paper we extend some of the notions/results to the case where S is arbitrary. Particular cases will be ``generalized Lagrange schemes'', and others (see subsection 9). After basic definitions, we present several conditions that are necessary for the almost regularity of the schemes (subsections 10-17), and also a condition that forces singularity. In the last part (subsections 18-20), we give some examples (with general S), some of which are singular, some almost regular, and also some for which the normality, the almost regularity, the regularity, or the Abel type are all equivalent.

We first introduce some basic notions:

1. If S is a subset of IN^d, define P_S as the spaceof polynomials of type

$$P(\boldsymbol{x})=P(x_1,x_2,...,x_d)=\sum_{\boldsymbol{i}\in S} a_{\boldsymbol{i}} x_1^{i_1} x_2^{i_2}...x_d^{i_d}, a_{\boldsymbol{i}} \in IR,$$

and define the spaces $P_S^{(\boldsymbol{\alpha})}=\{P^{(\boldsymbol{\alpha})} : P \in P_S\}$, where

$$P^{(\boldsymbol{\alpha})}(\boldsymbol{x})=\frac{\partial^{\alpha_1+...+\alpha_d}}{\partial x_1^{\alpha_1}...\partial x_d^{\alpha_d}} P(\boldsymbol{x})=\sum_{\boldsymbol{i}\in S} a_{\boldsymbol{i}} \frac{i_1!}{(i_1-\alpha_1)!}...\frac{i_d!}{(i_d-\alpha_d)!} x_1^{i_1-\alpha_1}...x_d^{i_d-\alpha_d},$$

for $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,...,\alpha_d)\in S$, $\boldsymbol{x}=(x_1,x_2,...,x_d)\in IR^d$, $\boldsymbol{\alpha} \le \boldsymbol{i}$ (i.e. $\alpha_k \le i_k$, $(\forall)k=\overline{1,d}$). We call it the space of derivatives of order $|\boldsymbol{\alpha}|=\alpha_1+\alpha_2+...+\alpha_d$ of P (see [8]).

2. For $\boldsymbol{\alpha}\in S \subset IN^d$, define $S_{\boldsymbol{\alpha}}$ as the set consisting of those $\boldsymbol{i}\in S$ satisfying $\boldsymbol{\alpha}\le \boldsymbol{i}$.

3. Given two subsets X and S of IN^d, we say that X is *lower* with respect to S if

$$\boldsymbol{i} \in X\,,\ \boldsymbol{j} \in S\,,\ \boldsymbol{j} \le \boldsymbol{i}\ \Rightarrow\ \boldsymbol{j} \in X$$

(see [5]).

4. *A multidimensional (polynomial) interpolation scheme (of dimension d),* denoted (Z,S,E), consists of the following

(a) A set of nodes

$$Z = \{\boldsymbol{z}_q\}_{q=1}^{m} = \left\{(z_{q,1}, z_{q,2}, \dots, z_{q,d})\right\}_{q=1}^{m} \subset IR^d\,,$$

(b) A subset S of IN^d,

(c) An incidence matrix $E = (e_{q,\boldsymbol{\alpha}})$, indexed by $1 \le q \le m$ (which label the nodes z_q) and by the elements $\boldsymbol{\alpha} \in S$.

The interpolation problem associated to the scheme (Z,S,E) consists of finding

$$P \in P_S$$

satisfying:

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}} P(z_q) = c_{q,\boldsymbol{\alpha}}$$

for all $q \in \overline{1,m}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) \in S$ with $e_{q,\boldsymbol{\alpha}} = 1$, where the $c_{q,\boldsymbol{\alpha}}$'s are arbitrary real constants.

If the incidence matrix E has a column consisting only of 1's, and the remaining elements are zero, we say that (Z,S,E) is a *generalized Lagrange scheme*. If $\boldsymbol{\alpha} \in S$ defines the non-vanishing column, we also say that (Z,S,E) is a *Lagrange scheme with respect to* $\boldsymbol{\alpha}$.

If S is lower with respect to IN^d, then (Z,S,E) is a *Birkhoff interpolation scheme* (see [12]).

5. We say that $E = (e_{q,\boldsymbol{\alpha}})$ is an Abel matrix is, for each $\boldsymbol{\alpha} \in S$, there exists precisely one

$q \in \overline{1,m}$ with the property that $e_{q,\boldsymbol{\alpha}} = 1$. The associated schemes, and the associated problems are called *Abel interpolation schemes*, and *Abel interpolation problems,* respectively.

6. For an incidence matrix E, one defines

$$|E| = \sum_{q=1}^{m}\sum_{\boldsymbol{\alpha}\in S} e_{q,\boldsymbol{\alpha}} ,$$

which is the number of nonvanishing elements of E.

An interpolation scheme (Z,S,E) is called *normal* if

$$|E| = |S|$$

(where $|S|$ is the cardinality of S).

7. Given a normal interpolation scheme (Z,S,E), we say it is:

(i) *regular* if , for all choices of the set Z of nodes, the determinant $D(Z,S,E)$ does not vanish Z,

(ii) *almost regular* if, for at least one choice of the set Z of nodes, the determinant $D(Z,S,E)$ does not vanish.

(iii) *singular* if, for any ste of nodes Z, the determinant $D(Z,S,E)$ is zero.

8. Given the scheme (Z, S, E), one defines *the support of the incidence matrix* E as the set of the order of derivatives that appear in the associated interpolation problem:

$$A = \{\boldsymbol{\alpha}\in S : e_{q,\boldsymbol{\alpha}} = 1, q = \overline{1,m}\} .$$

9. If

$$\Delta_k^d = \bigcup_{i_1=0}^{k}\bigcup_{i_2=0}^{k-i_1}\bigcup_{i_3=0}^{k-(i_1+i_2)} \dots \bigcup_{i_{d-1}=0}^{k-(i_1+\dots+i_{d-2})}\{(i_1,i_2,\dots,i_{d-1},k-(i_1+i_2+\dots+i_{d-1}))\},$$

then

$$T_n^d = \bigcup_{k=0}^{n}\Delta_k^d$$

is lower with respect to IN^d and any subset S of IN^d can be written as:

$$S = \bigcup_{t=0}^{n}\Delta'^d_{k_t}$$

where $k_n = \max_{\boldsymbol{i}\in S}|\boldsymbol{i}|$, $0 \le k_0 \le k_1 \le \dots \le k_n$, $\Delta'^d_{k_t} = S \cap \Delta^d_{k_t}$, $t = \overline{0,n}$, and $k_t \in \overline{0,k_n}$ (see [6]).

In the remaining part of this paper, unless otherwise specified, all schemes will be assumed to be normal, and S will be an arbitrary subset of

IN^d .

Of course, one would like to have criterias, hopefully simple enough so that they are usable, to decide when an interpolation scheme is regular, almost regular, or singular. In general, finding complete criterias (necessary and sufficient) is a very difficult problem, and one looks for partial criterias (which are implied by almost regularity).

Before presenting some criterias, we need the following:

10. Lemma. *Consider an interpolation scheme* (Z,S,E) *, with*

$$E_X = (e_{q,\boldsymbol{\alpha}}),\ q \in \overline{1,m},\ \boldsymbol{\alpha} \in X \subset S .$$

If A *is the support of* E *, and* $B \subseteq C \subseteq S$ *, then*

$$1)\qquad |E| = |E_A| = |A| ,$$

$$2)\qquad |E_{A\cap X}| = |E_X|,\ (\forall)X \subseteq S,$$

$$3)\qquad |E_B| \le |E_C|, |E_{C\setminus B}| = |E_C| - |E_B|.$$

Proof: 1) and 2) are immediate, and, for 3), we write

$$|E_C| = \sum_{q=1}^{m}\sum_{\boldsymbol{\alpha}\in C} e_{q,\boldsymbol{\alpha}} = \sum_{q=1}^{m}\left(\sum_{\boldsymbol{\alpha}\in B} e_{q,\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha}\in C\setminus B} e_{q,\boldsymbol{\alpha}}\right) = \sum_{q=1}^{m}\sum_{\boldsymbol{\alpha}\in B} e_{q,\boldsymbol{\alpha}} + \sum_{q=1}^{m}\sum_{\boldsymbol{\alpha}\in C\setminus B} e_{q,\boldsymbol{\alpha}} =$$

$$= |E_B| + |E_{C\setminus B}| .\square$$

In what follows we will exploit the following simple guiding principle: if a matrix has non-vanishing determinant, then the matrix cannot have ``too many zeros''. One example of this is the following simple remark:

11. Lemma. *If the matrix* $A \in M_n(IR)$ *has* a *row and* b *columns with the property that all the* ab *elements situated at the intersection of these rows and columns are zero, and* $det(A) \neq 0$ *, then*

$$a + b \le n . \qquad (1)$$

Proof: Taking the a rows from the statement, and removing the ab elements that vanish, we obtain a matrix with a rows and $n-b$ columns, call it A_1. We do the same for b columns, and we obtain a matrix A_2 with $n-a$ rows and b columns.

In the limit case, i.e. $a+b=n$, it follows that both A_1 and A_2 are suqare matrices, and the Laplace formulla tells us that

$$\det(A)=\det(A_1)\det(A_2).$$

Assume now that $a+b>n$, and we prove that $\det(A)=0$. Let b_0 so that $a+b_0=n$, and then we choose b_0 columns out of those in the statement (this is possible since $b_0<b$). We apply the first part to these b_0 columns and a rows from the statement, to conclude that $\det(A)=\det(A_1)\det(A_2)$. On the other hand, since $b_0<b$, the matrix A_1 (of type $(a,n-b_0)$) contains at least one column consisting on zero elements only. Hence $\det(A_1)=0$, and then $\det(A)=0$ as desired. □

12. Theorem. *If (Z,S,E) is almost regular, then*

$$|E_L|\geq|L| \qquad (2)$$

for all sets L which are lower with respect to S ($L\subset S$).

Proof: Let L be a subset that is lower with respect to S. Since $|S|=|L|+|S\setminus L|$, $|E|=|E_L|+|E_{S\setminus L}|$ (by Lemma 10) and $|E|=|S|$ (the normality of the scheme), it follows that

$$|E_L|+|E_{S\setminus L}|=|L|+|S\setminus L|$$

This implies that

$$|E_{S\setminus L}|\leq|S\setminus L|\Leftrightarrow|E_L|\geq|L|. \qquad (3)$$

On the other hand, the elements of the matrix $M(Z,S,E)$ situated at the intersection of the columns indexed by L become zero when we consider derivatives that come $S\setminus L$.

Applying the previous lemma, we must have

$$|L|+|E_{S\setminus L}|\leq|S|.$$

Since $|S| = |L| + |S \setminus L|$, we get $|E_{S\setminus L}| \le |S \setminus L|$ and then, by (3),

$$|E_L| \ge |L|,$$

which proves the theorem. □

13. Remark. The theorem can also be proven by the method given in [12] in the case where S is a lower set with respect to IN^d.

For the case of lower sets, the theorem above is known under the Polya condition. Accordingly, we introduce the following terminology:

14. Definition. *We say that an interpolation scheme* (Z, S, E) *(or an incidence matrix* E*) satisfies the Polya condition if* $|E_L| \ge |L|$*, for all sets* L *which are lower with respect to* S *(*$L \subset S \subset IN^d$*).*

15. Remarks. 1. In the uni-dimensional case, the Pólya condition is equivalent to the almost regularity of the scheme: a scheme is almost regular if and only if it satisfies the Polya condition (cf. [12], theorem 2.2.5). In the multidimensional case this is not longer true. Nevertheless, there are particular classes of interpolation schemes for which the Pólya condition is equivalent to the almost regularity (actually there are classes for which the normality of the scheme is equivalent to the almost regularity (see e.g. theorem 19 below).

2. A restatement of the previous theorem says that an interpolation scheme whose incidence matrix does not satisfy the Pólya condition cannot be almost regular. This is very useful in practice when we try to construct interpolation schemes that are almost regular (or regular).

Hence the theorem, and the Pólya condition, give us lower bounds for the numbers of interpolations that are necessary on the derivatives P, so that the almost regularity of the scheme is not spoiled. Next, we are looking for conditions for almost regularity which do not lower the number of nodes or the number of derivatives in the interpolation problem. Such a condition is presented in the next theorem. Although this theorem is a consequence of the Pólya condition, it does give a criteria which bounds the numbers of interpolations of a given order (for the derivatives) depending on the dimension of the space of derivatives of the given order. Of course, when we are trying to construct regular interpolation schemes, such conditions are easy to check.

16. Theorem. *If* (Z, S, E) *is almost regular, then, for all* $\boldsymbol{\alpha} \in A$ *(where* A *is* *the support of* E *, see (8))*

$$\left|E_{\{\boldsymbol{\alpha}\}}\right| \leq \dim P_S^{(\boldsymbol{\alpha})}.$$

Proof: Given $\boldsymbol{\alpha} \in A$, we consider $L = S \setminus S_{\boldsymbol{\alpha}}$, where $S_{\boldsymbol{\alpha}}$ was defined in (2). We have $L \cap A \subset A \setminus \{\boldsymbol{\alpha}\}$. Applying lemma 10,

$$\left|E_{L\cap A}\right| \leq \left|E_{A\setminus\{\boldsymbol{\alpha}\}}\right| \text{ and } \left|E_{L\cap A}\right| = \left|E_L\right|.$$

Next, we apply the Pólya condition to L (which is lower with respect S by construction), and the previous relations to conclude:

$$|S| - |S_{\boldsymbol{\alpha}}| = |S \setminus S_{\boldsymbol{\alpha}}| = |L| \leq |E_L| = |E_{L\cap A}| \leq \left|E_{A\setminus\{\boldsymbol{\alpha}\}}\right| = |E_A| - \left|E_{\{\boldsymbol{\alpha}\}}\right|.$$

Since $|S| = |E_A|$ (the normality of (Z,S,E)), it follows that

$$\left|E_{\{\boldsymbol{\alpha}\}}\right| \leq |S_{\boldsymbol{\alpha}}| = \dim P_S^{(\boldsymbol{\alpha})},$$

and the theorem is proven. □

17. Remark. Hence

$$\left|E_{\{\boldsymbol{\alpha}\}}\right| > \dim P_S^{(\boldsymbol{\alpha})},$$

with $\boldsymbol{\alpha} \in A \subset S$, is sufficient to ensure the singularity of the interpolation (Z,S,E).

We now present the two examples promised in the abstract.

18. Theorem. *If* (Z,S,E) *is a Lagrange interpolation scheme with respect to* $\boldsymbol{\alpha'} = (\alpha'_1, \alpha'_2, ..., \alpha'_d) \in S$*, then:*

- *(i) if there exists* $k \in \overline{1,d}$ *and* $\boldsymbol{i} = (i_1, i_2, ..., i_k, ..., i_d) \in S$ *such that* $i_k < \alpha'_k$*, then the scheme* (Z,S,E) *is singular,*
- *(ii) if* $\boldsymbol{\alpha'} \leq \boldsymbol{i}$ *for all* $\boldsymbol{i} = (i_1, i_2, ..., i_d) \in S$*, then the scheme* (Z,S,E) *is almost regular.*

Proof:

i) $P^{(\boldsymbol{\alpha}')}$ that appear in the interpolation problem will not contain $x_1^{i_1} x_2^{i_2} ... x_k^{i_k} ... x_d^{i_d}$, hence the corresponding column in the matrix $M(Z,S,E)$ associated to the scheme has only zero elements. Hence the determinant must be zero, and the scheme must be singular.

ii) In this case, the spaces containing the polynomial $P^{(\boldsymbol{\alpha}')}$ and P have the same dimension. Clearly, in this case there exists Z, such that the interpolation problem associated to (Z,S,E) has solution. Hence (Z,S,E) is almost regular. □

19. Theorem. *Consider the interpolation scheme* $(\Delta'^{d}_{k_t}, E_{\Delta'^{d}_{k_t}})$, $t=\overline{0,n}$, *where* $k_n = \max\limits_{\boldsymbol{i}\in S} |\boldsymbol{i}|$, $0 \le k_0 \le ... \le k_n$, $\Delta'^{d}_{k_t} = S \cap \Delta^d_{k_t}$, *and* $\Delta^d_{k_t}$, $t=\overline{0,n}$, *are those defined in (9). Then the normality of the scheme, the Pólya condition, the almost regularity, the regularity, and the Abel type condition, are all equivalent.*

Proof: Since $P \in P_S$, with $S = \Delta^d_{k_t}$, $t=\overline{0,n}$, using 1. or [6], it follows that the derivatives of order $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d) \in \Delta^d_{k_t}$, $t=\overline{0,n}$ of $P = P_{k_t}$ are

$$\frac{\partial^{\alpha_1+\alpha_2+...+\alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}} P_{k_t}(\boldsymbol{x}) = \frac{\partial^{k_t}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_{d-1}^{\alpha_{d-1}} \partial x_d^{k_t-(\alpha_1+...+\alpha_{d-1})}} P_{k_t}(\boldsymbol{x}) =$$

$$= \alpha_1! \alpha_2! ... \alpha_{d-1}! [k_t - (\alpha_1 + \alpha_2 + ... + \alpha_{d-1})]! \, a_{\alpha_1, \alpha_2, ..., \alpha_{d-1}, k_t-(\alpha_1+\alpha_2+...+\alpha_{d-1})},$$

i.e. they are constant for all $\boldsymbol{x} = (x_1, x_2, ..., x_d) \in IR^d$, and $\boldsymbol{i} = (i_1, i_2, ..., i_d) \in \Delta^d_{k_t}$, $t=\overline{0,n}$.

It follows that, if the scheme is normal, then each derivative will be interpolated exactely once, and this means that the scheme is Abel (see the definition in 5.). Conversely, if the scheme is Abel, then, from the definition, each derivative is interpolated exactely once. It follows that the number of interpolations (hence also of the equations) equals to the cardinality of the $\{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d) \in \Delta^d_{k_t}\}$, i.e. with $\dim P_S$, $S = \Delta^d_{k_t}$. This means that the scheme must be normal.

The rest is proven similarly. □

20. Corollary. *The interpolation scheme* $(\Delta'^{d}_{k_t}, E_{\Delta'^{d}_{k_t}})$, $t \in \overline{0,n}$, *cannot be Lagrange.*

Since $\boldsymbol{\alpha} \in \Delta^{d}_{k_t}$, and for $S = \Delta^{d}_{k_t}$ we have $\dim P_{\boldsymbol{S}}^{(\boldsymbol{\alpha})}=1$, it follows that each derivative can be interpolated at most once, hence the corollary.

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