THE LUBRICATION MODEL FOR THE FLOWS OF A THIN FILMS WITH SMALL REYNOLDS NUMBER

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Abstract. In this paper we consider the flows of a thin films for which the Reynolds number is small. This is true in many practical situations. We deduced the lubrication model for slider bearings (a slider bearing consists of a thin layer of viscous fluid confined between nearly parallel walls that are in relative tangential motion). Another application of the lubrication model is the flow of a thin film with a free boundary, with zero surface tension and respectively flow driven by surface tension gradient (Marangoni flow).

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1.Introduction

We consider the flows for which the Reynolds number Re is small. This is true in many practical situations (a pebble thrown in a pond, a bubble rising in a glass of champagne, pouring golden syrup, the formation of a tear drop, a layer of freshly applied paint, oil in an oil well, rock convecting in the earth's mantle) and we therefore consider when $Re << 1$.

The scaling used for the pressure in

$$\frac{d\overrightarrow{u}}{dt} = -\nabla p + \frac{1}{Re}\nabla^2\overrightarrow{u}, \quad \nabla \cdot \overrightarrow{u} = 0; \tag{1}$$

where $\overrightarrow{u}$ is the velocity and p is the pressure; which was chosen to make the pressure and inertia terms balance, is not now appropriate and that we need to rescale the dimensionless pressure with $\frac{1}{Re}$ so that it balances the dominant viscous terms. Replacing p by $\frac{1}{Re}p$ the equations (1) become

$$\mathrm{Re}\left(\frac{\partial\overrightarrow{u}}{\partial t} + (\overrightarrow{u}\cdot\nabla)\overrightarrow{u}\right) = -\nabla p + \nabla^2\overrightarrow{u}, \quad \nabla \cdot \overrightarrow{u} = 0. \tag{2}$$

2.Lubrication models for the flows of a thin films

We consider relatively low Reynolds number flow of a thin film. Such a film may exist between two rigid walls, as in a bearing, or in a droplet spreading under gravity on a rigid surface.

A sheet of paper can slide across a smooth floor shows that a thin layer of fluid can support a relatively large normal load while offering very little resistance to tangential motion.

More important mechanical examples occur in the lubrication of machinery and this motivates the study of slider bearings. A slider bearing consists of a thin layer of viscous fluid confined between nearly parallel walls that are in relative tangential motion.

A two-dimensional bearing is shown in figure 1, in which the plane $y = 0$ moves with constant velocity u in the x-direction and the top of the bearing is fixed [5].

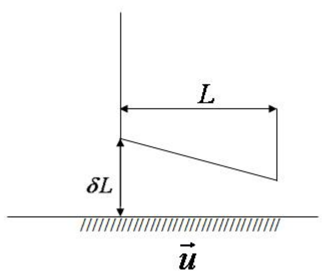

Figure 1.

The variables are nondimensionalised with respect to u and the length L of the bearing so that the position of the slider is given in the dimensionless variables used in (2) by $y = \delta H(x)$, where δL is typical gap-width of the bearing. The basic assumption of lubrication theory is that $\delta << 1$ so that we can use the ideas of boundary layer theory to simplify the Navier-Stokes equations. Starting from the steady form of (2) we rescale y, v by writing $y = \delta y'$, $v = \delta v'$ to get

$$\mathrm{Re}\ \left(u\frac{\partial u}{\partial x} + v'\frac{\partial u}{\partial y'}\right) = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{\delta^2}\cdot\frac{\partial^2 u}{\partial y'^2}; \tag{3}$$

$$\delta \cdot \mathrm{Re} \left(u\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y'} \right) = -\frac{1}{\delta}\frac{\partial p}{\partial y'} + \delta\frac{\partial^2 v'}{\partial x^2} + \frac{1}{\delta} \cdot \frac{\partial^2 v'}{\partial y'^2}; \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v'}{\partial y'} = 0 \tag{5}$$

with boundary conditions

$$\begin{aligned} u = 1 \quad v' = 0, \quad & on \quad y' = 0 \\ u = 0, \quad v' = 0, \quad & on \quad y' = H(x). \end{aligned} \tag{6}$$

The only way that (3) will not reduce to a triviality as $\delta \to 0$ is if the pressure is rescaled with $\frac{1}{\delta^2}$. Thus we write $p = \frac{1}{\delta^2}p'$ and, to lowest order the equations are (on dropping dashes) [1].

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}; \tag{7}$$

$$0 = \frac{\partial p}{\partial y}; \tag{8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{9}$$

These equations are the lubrication model and are based on the two assumptions that $\delta << 1$ and $Re\,\delta^2 << 1$. That it is not necessary for the Reynolds number based on L to be small but only that the reduced Reynolds number $Re\delta^2 = \frac{uL\delta^2}{\nu}$ be small.

The stress T_i, exerted on the upper surface is $\sigma_{ij} \cdot n_j$ where the normal $\overrightarrow{n}$ is given by $(\delta H', -1)/(1 + \delta^2 H'^2)^{1/2}$ and so, to lowest order in δ [6]

$$T = \frac{\mu u}{L}\left[\frac{1}{\delta}\left(-H'p - \frac{\partial u}{\partial y}\right), \frac{1}{\delta^2}p\right].$$

The normal stress exerted on the upper surface is therefore an order magnitude grater than the tangential stress.

Another interesting application of the lubrication model is to the flow of a thin film with a free boundary. Such a film might more under gravity as, for example, the spread of molten lava in a volcanic eruption or the motion of a raindrop down a windowpane.

We begin by considering gravity-driven flow on a horizontal surface, as in figure 2.

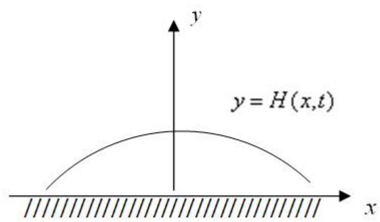

Figure 2.

Equation (8) with a gravity term added is

$$0 = -\frac{\partial p}{\partial y} - \frac{\delta^3 g L^2}{u\nu}. \tag{10}$$

Since gravity is the driving force we define u as $\frac{\delta^3 g L^2}{\nu}$, where δ, L are the initial depth and length of the film.

At the free surface we now have the kinematic boundary condition [6]

$$v = \frac{\partial H}{\partial t} + u\frac{\delta H}{\delta x}, \quad on \quad y = H(x,t) \tag{11}$$

and we need two more conditions since the position of this boundary is unknown. These conditions come from the fact that if surface tension is neglected, there is no stress applied on the free surface. This condition leads to

$$p = 0, \quad on \quad y = H(x,t); \tag{12}$$

$$\frac{\partial u}{\partial y} = 0, \quad on \quad y = H(x,t) \tag{13}$$

and

$$u = 0, v = 0 \quad on \quad y = 0. \tag{14}$$

Now, solving (10) we get

$$p = -\frac{\delta^3 g L^2}{u\nu} y + f_1(x,t),$$

and using (12)

$$f_1(x,t) = \frac{\delta^3 g L^2}{u\nu} H$$

and then

$$p = -y + H. \tag{15}$$

Then, integrating (7) and using the stress-free boundary condition (13) gives

$$\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} \cdot y + f_2(x,t)$$

$$f_2(x,t) = -\frac{\partial p}{\partial x} \cdot H$$

$$u = \frac{\partial p}{\partial x} \cdot \frac{y^2}{2} - \frac{\partial p}{\partial x} \cdot H \cdot y + f_3(x,t)$$

$$f_3(x,t) = 0$$

and

$$u = -\frac{1}{2} \cdot \frac{\partial H}{\partial x} \cdot y(2H - y). \tag{16}$$

Finally, using (9) and the kinematic boundary condition (12) leads to

$$\frac{\partial v}{\partial y} = \frac{1}{2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right) \cdot y(2H - y),$$

$$v = \frac{1}{2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right) \cdot \left(Hy^2 - \frac{y^3}{3}\right) + f_4(x,t)$$

but $f_4(x,t) = 0$ and

$$v = \frac{1}{2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right) \cdot \left(Hy^2 - \frac{y^3}{3}\right), \tag{17}$$

and we obtain the equation

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{3} \cdot H^3 \cdot \frac{\partial H}{\partial x}\right), \tag{18}$$

This a parabolic nonlinear equation for $H(x,t)$. It is possible to find solutions of (18) which have "compact support" ($H \equiv 0$ for all sufficiently large values of $\mid x \mid$).

A solution of (18) which satisfies the condition (see [5])

$$H(x,0) = \begin{cases} \left(1 - \frac{9}{10}x^2\right)^{1/3}, & \text{if } \mid x \mid < \sqrt{\frac{10}{9}} \\ 0, & \text{if } \mid x \mid > \sqrt{\frac{10}{9}} \end{cases}$$

is

$$H(x,t) = \begin{cases} (1+t)^{-\frac{1}{5}} \left(1 - \frac{9}{10}\frac{x^2}{(1+t)^{2/5}}\right)^{1/3}, & \text{if } \mid x \mid < \sqrt{\frac{10}{9}}(1+t)^{1/5} \\ 0, & \text{if } \mid x \mid > \sqrt{\frac{10}{9}}(1+t)^{1/5} \end{cases}.$$

Sketch H as a function of x for several values of t and discuss the circumstances under which this type of solution might model the evolution of a volcano.

A similar analysis is possible when the film lies on an inclined [3] or even a vertical surface [4].

Another model is the flow driven by surface tension [2]. Surface tension is a very important mechanism for small scale flows such a paint films, the motion of a contact lens on the eyeball or various wetting or coating flows. In two dimensions, the free boundary condition is [1]

$$\sigma_{ns} = 0$$
$$\sigma_{nn} = T/R$$

where s and n are tangential and normal coordinates respectively, T is the surface tension and R the radius of curvature of the surface of the film which is approximately $L/(\delta \cdot \frac{\partial^2 H}{\partial x^2})$. Thus, with a suitable scaling for the pressure, (12) is replaced by

$$p = -\frac{\partial^2 H}{\partial x^2}$$

and hence the thickness of a film on a flat base that is flowing under the influence of surface tension and viscosity satisfy a fourth-order evolution equation (Landau-Levich equation) [see 6]

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{3} \cdot H^3 \cdot \frac{\partial^3 H}{\partial x^3}\right). \tag{19}$$

This equation forms the bases of models for several of the situations mentioned above. For example, in modelling the tear film in the vicinity of a circular contact lens moving in the x-direction on a flat eyeball, we could use the two-dimensional form of (15) with the addition of a convective term $\frac{\partial H}{\partial x}$.

However, to model paint films or foams, it may be important to take surface tension gradients into account, giving rise to a different extra term in equation (15). Such gradients rise to what are called Marangoni flows [2] and they have unexpectedly been found to dominate many zero-gravity fluid dynamics experiments carried out in space, in particular those concerned with crystal growth. The ability of the surface tension to vary spatially in a crucial ingredient for the fluid to be able to form a foam (pure water has for too high a surface tension for foams to have any chance of surviving). It is also believed to be the mechanism responsible for the ripples that are often observed on solvent-based paint films.

3.Conclusions

We deduced the lubrication approximation because this model are used for investigate the flow of a thin layer. The starting point for the modeling flow of thin films are the Navier-Stokes equations. The lubrication or reduced Reynolds number approximation to the Navier-Stokes equations has been used to describe a multitude of situations: a slider bearing, a thin film flow with a free boundary, a thin film flow driven by gravity on a horizontal or inclined solid plane.

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