ON THE LAGRANGIAN FORMALISM AND THE STABILITY OF A DYNAMICAL SYSTEM

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Abstract. In this paper we consider a dynamical system which represents the model of the airplane flight in the vertical plane. The aim of this paper is to make some geometrical studies of the direct and inverse problems with concrete applications in the dynamical systems. It is studied the Lagrangian formalism for the inverse problem and we find a Lagrangian associated to this dynamical system. Next, we present some direct methods for the study of the stability of this dynamical system and we give the phase portraits.

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1.Introduction

We consider the mathematical model of the airplane flight in the vertical plane and we make some geometrical studies of the direct and inverse problems with concrete applications in the dynamical systems, [3, 5].

The flight regime is followed for the vertical plane for fixed direction and constant speed. In the vertical plane, will always act on the airplane the gravitational force $\vec{G}$, the bearing force $\vec{P}$, traction forces $\vec{T}$ and drag forces $\vec{R}$. The course will be on a fixed direction (horizontal, oblique) and due to small perturbations of the velocity and angle of flight occurs a pitch that destabilized the plane. We apply the classical stability criteria that will condition the parameters of this system to obtain a dynamic stability of the airplane.

The equation of motion is of the form:

$$m \dot{\vec{v}} = \vec{T} + \vec{G} + \vec{R}, \tag{1}$$

where m is the mass of the airplane, v is the speed, $\vec{T}$ is the traction force, $\vec{G} = m\vec{g}$ is the gravitational force, $\vec{R}$ is the resultant of aerodynamic forces. The resultant $\vec{R}$ has a decomposition into two forces, namely in the bearing force $\vec{P}$ and in the drag force $\vec{D}$. Here we will consider the projection of equation (1) onto tangent and normal to the trajectory in the hypothesis that the airplane has an inertial movement with $\vec{T} = \vec{0}$ and the resultant is proportional with v^2; C_D and C_L are the resistance and bearing coefficients, and the angle is very small.

$$m\,\dot{v} = -mg\sin\theta - C_D v^2 \tag{2}$$

$$mv\,\dot{\theta} = -mg\cos\theta + C_L v^2 \tag{3}$$

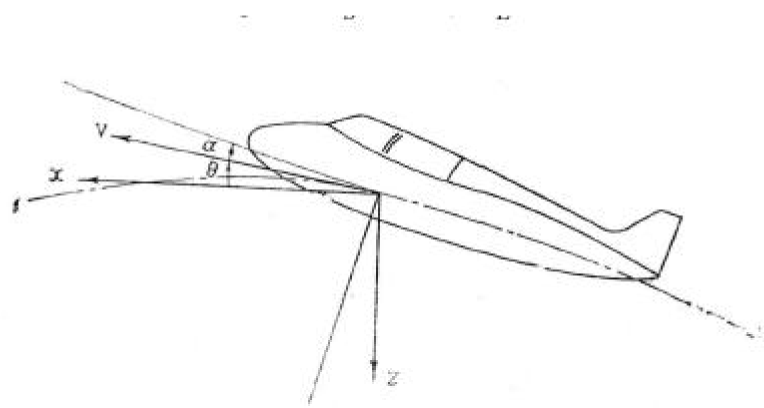

Figure 1: The flight of the airplane

Denoting as $v_0^2 = \frac{mg}{C_L}$, $\tau = gt$, $V = \frac{v}{v_0}$ and $a = \frac{C_D}{C_L}$, then from (2) and (3) we obtain, with respect to the new time variable τ, the following equivalent system:

$$\begin{cases} \dot{V} = -\sin\theta - aV^2 \\ \\ \dot{\theta} = \frac{V^2 - \cos\theta}{V} \end{cases} \quad . \tag{4}$$

Here v_0 is the flight inertial speed in the horizontal plane when the bearing force is equilibrated with the weight force in (3), with $\theta = 0$ and $a \geq 0$.

With respect to a we have the following two situations:

I) $a = 0$, horizontal flight with negligible resistance force ($C_D \approx 0$).

II) $a > 0$ and $C_D \neq 0$, dropping or raising on constant direction in the vertical plane.

In the following sections we study the Lagrangian formalism of the dynamical system from (4) in the case I) ($a = 0$), and we present some direct methods for the study of stability of this dynamical system and we give the phase portraits, in both cases.

2.The Lagrangian formalism

Let us briefly recall the Lagrangian formalism of dynamical systems for the inverse problem, (see [7, 8] and [11]).

Let

$$\dot{x} = f(t, x),\ x, f \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{5}$$

be a dynamical system. We are interested to find the conditions for which the system (5) admits the variational principle, namely, there are the functions $A_i(t, x)$, $B(t, x)$ such that the Euler-Lagrange equations system associated to the Lagrangian

$$L := \sum_i A_i(t, x)\, \dot{x}_i + B(t, x)$$

is equivalent to the system (5).

We have that $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$, $i = 1, \ldots, n$ or

$$\sum_{j=1}^{n} \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) \dot{x}_j + \frac{\partial A_i}{\partial t} - \frac{\partial B}{\partial x_i} = 0,\ i = 1, \ldots, n \tag{6}$$

must be equivalent to (5). If we denote by

$$C_{ij}=\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}\,,\; D_i=\frac{\partial A_i}{\partial t}-\frac{\partial B}{\partial x_i} \tag{7}$$

we get the system

$$\sum_{j=1}^{n}C_{ij}\;\dot{x}_j\;+D_i=0,\; i=1,\ldots,n. \tag{8}$$

We are interested to find necessary and sufficient conditions such that a system written by (8) to be equivalent with (6), namely there exist A_i and B which verify the conditions (7). For this, let us consider the differential 1-form $\varphi=\sum_i A_i dx^i+Bdt$ and its exterior derivative

$$\begin{aligned} d\varphi= &= \frac{1}{2}\sum_{i,j}\left(\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}\right)dx_i\wedge dx_j+\sum_i\left(\frac{\partial A_i}{\partial t}-\frac{\partial B}{\partial x_i}\right)dx_i\wedge dt \\ &= \frac{1}{2}\sum_{i,j}C_{ij}dx_i\wedge dx_j+\sum_i D_i dx_i\wedge dt. \end{aligned}$$

So, denoting $\Omega:=d\varphi$ it follows that for the system (8) there exist A_i and B such that the conditions (7) are fulfilled if and only if there exists an 1-form φ such that $\Omega=d\varphi$. From Poincaré Lemma is necessary and sufficient to have $d\Omega=0$, namely

$$\begin{cases} C_{ij}+C_{ji}=0 \\ \frac{\partial C_{ij}}{\partial x_k}+\frac{\partial C_{jk}}{\partial x_i}+\frac{\partial C_{ki}}{\partial x_j}=0 \\ \frac{\partial C_{ij}}{\partial t}=\frac{\partial D_i}{\partial x_j}-\frac{\partial D_j}{\partial x_i} \end{cases}, \tag{9}$$

called the *autoadjoint conditions*. If the system (8) verifies the conditions (9), then it comes from a Lagrangian.

We consider now an antisymmetric integrant factor C_{ij} for the initial system (5). Then it results that

$$\sum_j C_{ij}(\dot{x}_j\;-f_j)=0 \text{ or } \sum_j C_{ij}\;\dot{x}_j\;+D_i=0,$$

where $D_i=-\sum_j C_{ij}f_j$. If n is even and there is a matrix (C_{ij}) which verifies the autoadjoint conditions (9), then the dynamical system (5) admits the variational principle. The condition for n to be even is necessary because when n

is odd we have $\det C = 0$ and in this case the integrant factor is noninvertible and the obtained system is not equivalent to the initial system.

We apply now the method presented above for the dynamical system from (4) in the case when we have negligible resistance force ($a = 0$), and we find a Lagrangian associated to this system.

Let $C = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$ be an antisymmetric integrant factor for the dynamical system from (4), where $C = C(\tau, V, \theta)$. Multiplying the system with this factor we get

$$\begin{cases} C\left(\dot{\theta} - \frac{V^2-\cos\theta}{V}\right) = 0 \\ \\ -C\left(\dot{V} + \sin\theta\right) = 0 \quad . \end{cases} \tag{10}$$

The autoadjoint conditions in this case are given be

$$\frac{\partial C}{\partial \tau} - \sin\theta \frac{\partial C}{\partial V} + \frac{V^2 - \cos\theta}{V}\frac{\partial C}{\partial \theta} + C\frac{\sin\theta}{V} = 0. \tag{11}$$

Considering $C = C(V)$ we get $-\sin\theta \frac{\partial C}{\partial V} + C\frac{\sin\theta}{V} = 0$. The associated symmetric system of this cvasilinear partial differential equation is $\frac{dV}{V} = \frac{dC}{C}$ with the solution $C = kV$.

Taking $k = 1$, it follows that $C = V$ is an integrant factor for the our dynamical system. Now, from easy calculations we obtain

$$A_1 = \theta(V+1)\,,\; A_2 = V \text{ and } B = \frac{V^3}{3} - V\cos\theta.$$

Thus,

$$L = \theta(V+1)\,\dot{V} + V\,\dot{\theta} + \frac{V^3}{3} - V\cos\theta \tag{12}$$

is an associated Lagrangian and the dynamical system (4) (the case $a = 0$) is just the Euler-Lagrange equations system associated to this Lagrangian.

3.The study of stability

Case I) $a = 0$ **(the negligible resistance)**.

In this case the system becomes [1, 2, 5, 9]:

$$\begin{cases} \dot{V} = -\sin\theta \\ \\ \dot{\theta} = \frac{V^2 - \cos\theta}{V} \end{cases}, V = V(\tau), \theta = \theta(\tau), \tau \geq 0. \tag{13}$$

The critical points (equilibrium points) from (13) are

$$\{V^* = 1, \theta^* = 0\}. \tag{14}$$

If the flight is constant then $V^* = v_0$, if it is horizontal then $\theta^* = 0$.

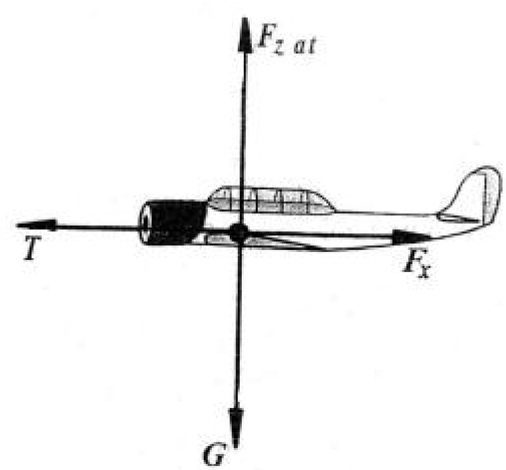

Figure 2: The horizontal flight of the airplane

From (13) we obtain

$$\frac{dV}{d\theta} = -\frac{V\sin\theta}{V^2 - \cos\theta} \tag{15}$$

with singular points $V = 0, \theta = \pm\frac{\pi}{2}$.

We'll study the stability of the point $P^*(V^* = 1, \theta^* = 0)$. The system is nonlinear and we'll determine a Liapunov function $W = W(V, \theta)$, positive defined $W(1, 0) = 0, W(V, \theta) > 0$ in the neighborhood of P^* such that

$\dot{W}(1,0) = 0$ and $\dot{W}(V,\theta) \leq 0$. The system (15) is with the total differential $W = W(V,\theta)$. So, we have

$$dW = (V^2 - \cos\theta)dV + V\sin\theta d\theta, \tag{16}$$

$$W(V,\theta) = \frac{V^3}{3} - V\cos\theta + C. \tag{17}$$

The constant is $C = -\frac{2}{3}$ from $W(1,0) = 0$ and so we have the Liapunov function

$$W(V,\theta) = \frac{V^3}{3} - V\cos\theta - \frac{2}{3}. \tag{18}$$

Taking into account of Silvester determinants is obtain $W(V,\theta) > 0$ and $\frac{dW}{d\tau} = 0$ because $W(V,\theta) = ct$ is a prime integral. So, the point $P^*(V^* = 1, \theta^* = 0)$ is a centrum - the flight regime is simple stable, this mean that at lower perturbation the trajectories are closed and oscillate around of P^* in the phases plane. Analysing the values of the constant C and $W = W(V,\theta,C)$ (the perturbed energy) we'll obtain in the physical plane (xOz) the perturbed flight trajectories compared with the horizontal flight. (Ox - the horizontal axis, Oz - the vertical axis). The components of velocity on the (xOz) system are

$$v_x = \frac{dx}{dt} = v\cos\theta, v_z = \frac{dz}{dt} = v\sin\theta. \tag{19}$$

Replacing in (2) with $C_D = 0$, $\sin\theta = \frac{\dot{z}}{v}$ we obtain the theorem of energy

$$\frac{1}{2}v^2 = -gz + c. \tag{20}$$

For the initial conditions $z = 0, v = 0, c = 0$, we obtain

$$z = -v_0^2\frac{V^2}{2g}, V = \frac{1}{v_0}\sqrt{-2gz}. \tag{21}$$

Replacing the (21) in (2), we have

$$\cos\theta = -\frac{2}{3}\frac{gz}{v_0^2} - \frac{Cv_0}{\sqrt{-2gz}} = h(z,C). \tag{22}$$

But, from $\frac{dz}{dx} = \text{tg}\theta$ taking account by (22) it is obtained $\text{tg}\theta = -\frac{1}{\sqrt{1+h^2(z,C)}}$

$$dx = \frac{dz}{tg\theta} = \frac{dz}{f(z,C)}. \tag{23}$$

By integration it is obtained the trajectory in the (xOz) plane, $F(x,z,C,k) = 0$. The function $f(z,C)$ isn't generally an elementary one and the study is made for $C \geq -\frac{2}{3}$. Thus $C = -\frac{2}{3}, \cos\theta = 1, z = -\frac{v_0^2}{2g}$, means that on the Oz axis the flight is constant, with constant speed $v_0, \theta = 0$, and at lower perturbations the trajectory is oscillating in comparison with the height $H = \frac{v_0^2}{2g}$.

For $C = 0$ integrating the relation (23) we observe that the trajectories are arcs $(x-k)^2 + z^2 = \frac{9}{4}\frac{v_0^4}{g^2}$ which represent the separation curve $-\frac{2}{3} < C < 0$. Thus for $C = -\frac{1}{3}$ we have the periodical curves with points of return. For higher speeds v_0 at perturbations $C = \frac{2}{3}$ the trajectories are periodical loopings compared with the horizontal flight.

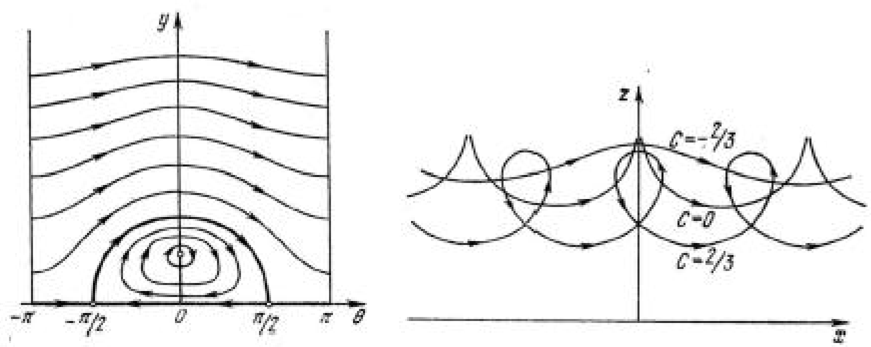

Figure 3: The trajectories of the airplane flight

Case II) $a \neq 0$.

We study the stability in first approximation for the system (4).

To find the equilibrium points we solve the system

$$\begin{cases} -\sin\theta - aV^2 = 0 \\ \\ \frac{V^2-\cos\theta}{V} = 0. \end{cases}$$

It is obtained: $P^*(\theta^*, V^*) \equiv P^*\left(-\text{arctg}a, \sqrt[4]{\frac{1}{1+a^2}}\right)$.
To have an equilibrium in the $(0,0)$ point we make the substitution: $\Theta = \theta - \theta^*$, $W = V - V^*$. It is obtained the system

$$\begin{cases} \dot{W} = -\frac{2a}{\sqrt[4]{1+a^2}}W - \frac{1}{\sqrt{1+a^2}}\Theta \\ \\ \dot{\Theta} = 2W - \frac{a}{\sqrt[4]{1+a^2}}\Theta \end{cases} \tag{24}$$

and the characteristical equation is: $\lambda^2 + \frac{3a}{\sqrt[4]{1+a^2}}\lambda + 2\sqrt{1+a^2} = 0$.

- If $a^2 - 8 < 0, a \in (0, 2\sqrt{2}) \Rightarrow \lambda_{1,2} \in C \setminus R$, with $Re(\lambda_{1,2}) < 0$ then the point is an attractive focal.
- If $a = 2\sqrt{2} \Rightarrow \lambda_1 = \lambda_2 = -\frac{-3\sqrt{2}}{\sqrt{3}}$ then the point is attractive degenerated.
- If $a > 2\sqrt{2} \Rightarrow \lambda_{1,2} \in R_-$ then the point is an attractive degenerated node.

So, with this method in first approximation we have the asymptotic stability and the airplane goes down with the oscillations at constant speed $V^* = \sqrt[4]{\frac{1}{1+a^2}}$ and it is stabilized on the inclined line $\theta^* = -\text{arctg}a$. For higer speeds, at perturbations the airplane executes some loopings and then is asymptotically stabilized.

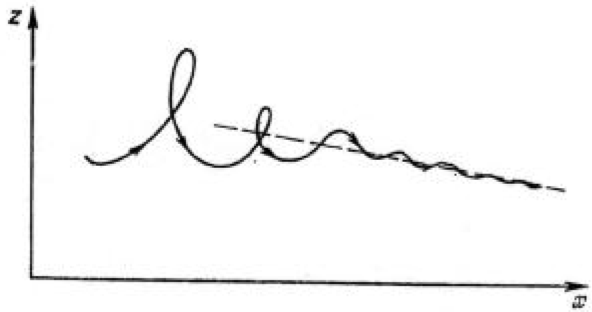

Figure 4: The loopings of the airplane

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