ON A CERTAIN DIFFERENTIAL SANDWICH THEOREM ASSOCIATED WITH A NEW GENERALIZED DERIVATIVE OPERATOR

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Abstract. The purpose of this paper is to derive certain subordination and superordination results involving a new differential operator. By means of the new introduced operator, $I^m(\lambda,\beta,l)f(z)$, for certain normalized analytic functions in the open unit disc, we establish differential sandwich-type theorems. These results extend corresponding previously known results.

2000 *Mathematics Subject Classification*: 30C80, 30C45.

*Keywords:*differential subordination, differential superordination, dominant, differential operator, subordinant.

Introduction and definitions

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots .$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots\}$$

with $\mathcal{A}_1 := \mathcal{A}$.

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in U. Then we say that the function f is subordinate to g, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U$$

if there exists a Schwarz function w analytic in U such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in \mathcal{H}(U)$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$.

If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad z \in U \tag{0.1}$$

then p is a solution of the differential superordination (0.1). If f is subordinate to g, then g is superordinate to f.

An analytic function q is called a subordinant of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (0.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (0.1) is said to be the best subordinant. The best subordinant is unique up to a rotation of U. Recently Miller and Mocanu [7] obtained conditions on h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \ \Rightarrow\ q(z) \prec p(z), \quad z \in U.$$

In order to prove our subordination and superordination results, we make use of the following definition and lemmas.

Definition 1 [7] Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} - E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 1 [8] *Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)) \quad \textit{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(1) $Q(z)$ is starlike univalent in U and

(2) $\mathrm{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ *for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2 [4] *Let q be convex univalent in the unit disc U and ν and φ be analytic in a domain D containing $q(U)$. Suppose that*

(1) $\mathrm{Re}\left\{\dfrac{\nu'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$ and*

(2) $\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ *with* $p(U) \subseteq D$ *and* $\nu(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* U *and*

$$\nu(q(z)) + zq'(z)\varphi(q(z)) \prec \nu(p(z)) + zp'(z)\varphi(p(z))$$

then

$$q(z) \prec p(z)$$

and q *is the best subordinant.*

2. Main results

Definition 2 Let the function f be in the class $\mathcal{A}_n$. For $m, \beta \in \mathbb{N}_0 = \{0,1,2,\dots\}$, $\lambda \geq 0$, $l \geq 0$, we define the following differential operator

$$I^m(\lambda,\beta,l)f(z) := z + \sum_{k=n+1}^{\infty} \left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m C(\beta,k)a_k z^k \qquad (0.2)$$

where

$$C(\beta,k) := \binom{k+\beta-1}{\beta} = \frac{(\beta+1)_{k-1}}{(k-1)!}$$

and

$$(a)_n := \begin{cases} 1, & n=0 \\ a(a+1)\dots(a+n-1), & n \in \mathbb{N} = \mathbb{N}_0 - \{0\} \end{cases}$$

is Pochhamer symbol.

Using simple computation one obtains the next result.

Proposition 1 *For* $m, \beta \in \mathbb{N}_0$, $\lambda \geq 0$, $l \geq 0$

$$(l+1)I^{m+1}(\lambda,\beta,l)f(z) = (1-\lambda+l)I^m(\lambda,\beta,l)f(z) + \lambda z(I^m(\lambda,\beta,l)f(z))' \quad (0.3)$$

and

$$z(I^m(\lambda,\beta,l)f(z))' = (1+\beta)I^m(\lambda,\beta+1,l)f(z) - \beta I^m(\lambda,\beta,l)f(z). \qquad (0.4)$$

Remark 1 *Special cases of this operator includes the Ruscheweyh derivative operator* $I^0(1,\beta,0)f(z) \equiv D_\beta$ *defined in [9], the Sălăgean derivative operator* $I^m(1,0,0)f(z) \equiv D^m$*, studied in [10], the generalized Sălăgean operator* $I^m(\lambda,0,0) \equiv D^m_\lambda$ *introduced by Al-Oboudi in [1], the generalized Ruscheweyh derivative operator* $I^1(\lambda,\beta,0)f(z) \equiv D_{\lambda,\beta}$ *introduced in [6], the operator* $I^m(\lambda,\beta,0) \equiv D^m_{\lambda,\beta}$ *introduced by K. Al-Shaqsi and M. Darus in [3] and finally the operator* $I^m(\lambda,0,l) \equiv I_1(m,\lambda,l)$ *introduced in [5].*

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \prec q_2(z),$$

where $m,\beta \in \mathbb{N}_0$, $\lambda \geq 0$ and q_1, q_2 are given univalent functions in U. Also, we obtain the number of known results as their special cases.

Theorem 1 *Let* $m,\beta \in \mathbb{N}_0$*,* $\lambda > 0$ *and* q *be convex univalent in* U *with* $q(0) = 1$*. Further, assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\alpha)q(z)}{\delta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{0.5}$$

Let

$$\begin{aligned}\psi(m,\lambda,\beta,\delta,\alpha;z) = {} & \frac{\delta[1-\lambda(1+\beta)+l]}{\lambda} \cdot \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} + \\ & + \frac{\delta\lambda(\beta+1)(\beta+2)}{l+1} \cdot \frac{I^m(\lambda,\beta+2,l)f(z)}{I^m(\lambda,\beta,l)f(z)} + \\ & + \frac{\delta(1+\beta)[1-\lambda(\beta+2)+l]}{l+1} \cdot \frac{I^m(\lambda,\beta+1,l)}{I^m(\lambda,\beta,l)} + \\ & + \left[\alpha + \delta\left(1 - \frac{l+1}{\lambda}\right)\right]\left(\frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)}\right)^2. \end{aligned} \tag{0.6}$$

If $f \in \mathcal{A}_n$ satisfies

$$\psi(m, \lambda, \beta, \delta, \alpha; z) \prec \delta z q'(z) + (\delta + \alpha)(q(z))^2 \tag{0.7}$$

then

$$\frac{I^{m+1}(\lambda, \beta, l) f(z)}{I^m(\lambda, \beta, l) f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{I^{m+1}(\lambda, \beta, l) f(z)}{I^m(\lambda, \beta, l) f(z)}, \quad z \in U. \tag{0.8}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$.

Therefore, by making use of (0.3) and (0.4) we have

$$\begin{aligned} &\frac{\delta[1 - \lambda(1 + \beta) + l]}{\lambda} \cdot \frac{I^{m+1}(\lambda, \beta, l) f(z)}{I^m(\lambda, \beta, l) f(z)} + \\ &+ \frac{\delta\lambda(\beta + 1)(\beta + 2)}{l + 1} \cdot \frac{I^m(\lambda, \beta + 2, l) f(z)}{I^m(\lambda, \beta, l) f(z)} + \\ &+ \frac{\delta(1 + \beta)[1 - \lambda(\beta + 2) + l]}{l + 1} \cdot \frac{I^m(\lambda, \beta + 1, l)}{I^m(\lambda, \beta, l)} + \\ &+ \left[\alpha + \delta\left(1 - \frac{l + 1}{\lambda}\right)\right]\left(\frac{I^{m+1}(\lambda, \beta, l) f(z)}{I^m(\lambda, \beta, l) f(z)}\right)^2 = \\ &= \delta z p'(z) + (\delta + \alpha)(p(z))^2. \end{aligned} \tag{0.9}$$

By using (0.9) in (0.7) we get

$$\delta z p'(z) + (\delta + \alpha)(p(z))^2 \prec \delta z q'(z) + (\delta + \alpha)(q(z))^2.$$

By setting $\theta(w) = (\delta + \alpha) w^2$ and $\phi(w) = \delta$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$. Hence the result follows by an application of Lemma 1. $\square$

Remark 2 *Similar results were obtained earlier in [6] for the operator defined in [2].*

Let

$$q(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1$$

in Theorem 1. One obtains the following result.

Corollary 1 *Let $m, \beta \in \mathbb{N}_0$, $\lambda > 0$. Assume that (0.5) holds. If $f \in \mathcal{A}_n$, then, differential subordination*

$$\psi(m, \lambda, \beta, \delta, \alpha; z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2 \tag{0.10}$$

implies

$$\frac{I^{m+1}(\lambda, \beta, l)f(z)}{I^m(\lambda, \beta, l)f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2 *Let $m, \beta \in \mathbb{N}_0$, $\lambda > 0$. Assume that (0.5) holds. If $f \in \mathcal{A}_n$, then differential subordination*

$$\psi(m, \lambda, \beta, \delta, \alpha; z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2 \tag{0.11}$$

implies

$$\frac{I^{m+1}(\lambda, \beta, l)f(z)}{I^m(\lambda, \beta, l)f(z)} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Corollary 3 *Let $m, \beta \in \mathbb{N}_0$, $\lambda > 0$, $0 < \mu \le 1$. Assume that (0.5) holds. If $f \in \mathcal{A}_n$, then differential subordination*

$$\psi(m, \lambda, \beta, \delta, \alpha; z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\alpha+\delta)\left(\frac{1+z}{1-z}\right)^{2\mu} \tag{0.12}$$

implies

$$\frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}$$

and $\left(\dfrac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Theorem 2 *Let q be convex univalent in U with $q(0)=1$. Assume that*

$$\text{Re}\,\left\{\frac{2(\delta+\alpha)q(z)q'(z)}{\delta}\right\} > 0. \tag{0.13}$$

Let $f\in\mathcal{A}$, $\dfrac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1]\cap Q$.
If function $\psi(m,\lambda,\beta,\delta,\alpha;z)$, given by (0.6), is univalent in U and

$$(\delta+\alpha)(q(z))^2+\delta z q'(z) \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \tag{0.14}$$

then

$$q(z) \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)}$$

and q is the best subordinant.

Proof. Theorem 2 follows by using the same technique to prove Theorem 1 and by an application of Lemma 2. □

By using Theorem 2 we obtain the following corollaries.

Corollary 4 *Let $q(z)=\dfrac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $f\in\mathcal{A}$ and*

$$\frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1]\cap Q.$$

Assume that (0.13) holds. If

$$(\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\delta(A-B)z}{(1+Bz)^2} \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \tag{0.15}$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Corollary 5 *Let* $q(z)=\frac{1+z}{1-z}$, $f\in\mathcal{A}$ *and*

$$\frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1]\cap Q.$$

Assume that (0.13) holds. If

$$\frac{2\delta z}{(1-z)^2}+(\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2 \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \tag{0.16}$$

then

$$\frac{1+z}{1-z} \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)}$$

and $\frac{1+z}{1-z}$ *is the best subordinant.*

Corollary 6 *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^\mu$, $0<\mu\leq 1$, $f\in\mathcal{A}$ *and*

$$\frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1]\cap Q.$$

Assume that (0.13) holds. If

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1}+(\alpha+\delta)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \tag{0.17}$$

then

$$\left(\frac{1+z}{1-z}\right)^\mu \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)}$$

and $\left(\frac{1+z}{1-z}\right)^\mu$ *is the best subordinant.*

Combining the results of differential subordination and superordination we state the following Sandwich Theorems.

Theorem 3 *Let q_1 and q_2 be convex univalent in U and satisfy (0.13) and (0.5) respectively.*

If $f \in \mathcal{A}$, $\dfrac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi(m,\lambda,\beta,\delta,\alpha;z)$ given in (0.6) is univalent in U and

$$\delta z q_1'(z) + (\delta+\alpha)(q_1(z))^2 \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \prec \tag{0.18}$$

$$\prec \delta z q_2'(z) + (\delta+\alpha)(q_2(z))^2,$$

then

$$q_1(z) \prec \frac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \prec q_2(z)$$

and q_1 and q_2 are the best subordinant and best dominant respectively.

For $q_1(z) = \dfrac{1+A_1z}{1+B_1z}$, $q_2(z) = \dfrac{1+A_2z}{1+B_2z}$, where $-1 \le B_2 < B_1 < A_1 \le A_2 \le 1$ we have the following corollary.

Corollary 7 *If $f \in \mathcal{A}$, $\dfrac{I^{m+1}(\lambda,\beta,l)f(z)}{I^m(\lambda,\beta,l)f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2} + (\delta+\alpha)\left(\frac{1+A_1z}{1+B_1z}\right)^2 \prec \psi(m,\lambda,\beta,\delta,\alpha;z) \prec$$

$$\prec \frac{\delta(A_2-B_2)z}{(1+B_2z)^2} + (\delta+\alpha)\left(\frac{1+A_2z}{1+B_2z}\right)^2.$$

Hence $\dfrac{1+A_1z}{1+B_1z}$ and $\dfrac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant respectively.

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