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SUBORDINATION RESULTS FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY SALAGEAN OPERATOR

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Abstract. In this paper we derive subordination results for certain subclass of analytic functions defined by using Salagean operator.

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1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (k \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g(z) \in A$ be given by:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

We also denote by K the class of functions $f(z) \in A$ that are convex in U. For $f(z) \in A$, Salagean [8] introduced the following differential operator:

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z), ..., D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N}).$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Definition 1. *(Hadamard Product or Convolution). Given two functions f and g in the class A, where $f(z)$ is given by (1.1) and $g(z)$ is given by (1.2) the Hadamard product (or convolution) of f and g is defined (as usual) by*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Definition 2. *(Subordination Principle). For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that*

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence [6, p. 4]:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 3. *[4]. Let $F_{m,n}(\alpha, \beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfy the inequality,*

$$\Re \left\{ \frac{D^m f(z)}{D^n f(z)} - \alpha \right\} > \beta \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right| \tag{1.4}$$

$$\left(-1 \leq \alpha \leq 1, \beta \geq 0; m \in \mathbb{N}; n \in \mathbb{N}_0, m > n; z \in U\right).$$

The family $F_{m,n}(\alpha, \beta)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions.

Specifying the parameters α, β, m and n, we obtain the following subclasses studied by various authors:

(*i*) $F_{n+1,n}(\alpha, \beta) = S(n, \alpha, \beta) = \left\{ f \in A : \Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - \alpha \right\} > \beta \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right|, \; 0 \leq \alpha < 1, \beta \geq 0, n \in \mathbb{N}_0, z \in U \right\}$

(see Rosy and Murugusudaramoorthy [7], see also Aouf [1]);

$$\begin{aligned}(ii) \quad F_{1,0}(\alpha,\beta) &= US(\alpha,\beta)\\ &= \left\{f \in A : \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|,\right.\\ &\quad \left. 0 \leq \alpha < 1, \beta \geq 0, z \in U\right\}\\ F_{2,1}(\alpha,\beta) &= UK(\alpha,\beta)\\ &= \left\{f \in A : \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\right.\\ &\quad \left. 0 \leq \alpha < 1, \beta \geq 0, z \in U\right\}\end{aligned}$$

(see Shams et al. [10], see also Shams and Kulkarni [9]);

Also we note that:

$$(i)\ F_{m,n}(\alpha,0) = F_{m,n}(\alpha) = \left\{f(z) \in A : \Re\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \alpha, \right.$$
$$\left. 0 \leq \alpha < 1;\ m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, z \in U\right\}.$$

Definition 4 *(Subordination Factor Sequence). A sequence $\{c_k\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z)\ (a_1 = 1; z \in U). \tag{1.5}$$

2. Main Result

Unless otherwise mentioned, we assume in the reminder of this paper that, $-1 \leq \alpha \leq 1$, $\beta \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n$ and $z \in U$.

To prove our main result we need the following lemmas.

Lemma 1. *[13]. The sequence $\{c_k\}_{k=0}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \quad (z \in U). \tag{2.1}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $F_{m,n}(\alpha,\beta)$.

Lemma 2. *A function $f(z)$ of the form* (1.1) *is in the class $F_{m,n}(\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}\left[(1+\beta)(k^m-k^n)+(1-\alpha)k^n\right]|a_k|\le 1-\alpha. \tag{2.2}$$

Proof. It suffices to show that

$$\beta\left|\frac{D^m f(z)}{D^n f(z)}-1\right|-\Re\left\{\frac{D^m f(z)}{D^n f(z)}-1\right\}<1-\alpha.$$

We have

$$\beta\left|\frac{D^m f(z)}{D^n f(z)}-1\right|-\Re\left\{\frac{D^m f(z)}{D^n f(z)}-1\right\}$$

$$\le(1+\beta)\left|\frac{D^m f(z)}{D^n f(z)}-1\right|\le\frac{(1+\beta)\sum_{k=2}^{\infty}(k^m-k^n)|a_k|}{1-\sum_{k=2}^{\infty}k^n|a_k|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\left[(1+\beta)(k^m-k^n)+(1-\alpha)k^n\right]|a_k|\le 1-\alpha,$$

and hence the proof of Lemma 2 is completed.

Remark 1. *The result obtained by Lemma 2 is giving a simplified version of the result obtained by Eker and Owa [4, Theorem 2.1].*

Taking $\beta=0$ in Lemma 2, we obtain the following coroallary:

Corollary 3. *A function $f(z)$ of the form* (1.1) *is in the class $F_{m,n}(\alpha)$ if*

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)|a_k|\le 1-\alpha. \tag{2.3}$$

Let $F^*_{m,n}(\alpha,\beta)$ and $F^*_{m,n}(\alpha)$ denote the classes offunctions $f(z)\in A$ whose coefficients satisfy the condition (2.2) and (2.3) respectively. We note that $F^*_{m,n}(\alpha,\beta)\subseteq F_{m,n}(\alpha,\beta)$ and $F^*_{m,n}(\alpha)\subseteq F_{m,n}(\alpha)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [11], we prove:

Theorem 4. *Let $f(z)\in F^*_{m,n}(\alpha,\beta)$. Then*

$$\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}(f*h)(z)\prec h(z)\ (z\in U), \tag{2.4}$$

for every function h in K, and

$$\Re\{f(z)\} > -\frac{[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}{(1+\beta)(2^m-2^n)+(1-\alpha)2^n} \ (z \in U). \tag{2.5}$$

The constant factor $\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}$ *in the subordination result* (2.4) *cannot be replaced by a larger one.*

Proof. Let $f(z) \in F^*_{m,n}(\alpha,\beta)$ and let $h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in K$. Then we have

$$\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}(f*h)(z) =$$

$$\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}\left(z+\sum_{k=2}^{\infty} a_k c_k z^k\right). \tag{2.6}$$

Thus, by Definition 4, the subordination result (2.4) will hold true if the sequence

$$\left\{\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}a_k z^k\right\} > 0 \ (z \in U). \tag{2.7}$$

Now, since

$$\Psi(k) = (1+\beta)(k^m-k^n)+(1-\alpha)k^n$$

is an increasing function of k $(k \geq 2)$, we have

$$\begin{aligned}
&\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}a_k z^k\right\} \\
= \ &\Re\left\{1+\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}z+\right. \\
&\left.\frac{1}{\left[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)\right]}\sum_{k=2}^{\infty}\left[(1+\beta)(2^m-2^n)+(1-\alpha)2^n\right]a_k z^k\right\}
\end{aligned}$$

$$\begin{aligned}
&\geq 1-\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}r- \\
&\quad \frac{1}{[(1+\beta)(2^m-2^n)+(1-\alpha)2^n+(1-\alpha)]}\sum_{k=2}^{\infty}[(1+\beta)(k^m-k^n)-(1-\alpha)k^n]\,|a_k|\,r^k \\
&> 1-\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}r-\frac{1-\alpha}{[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}r \\
&= 1-r>0\ (|z|=r<1),
\end{aligned}$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.7) holds true in U, this proves the inequality (2.4). The inequality (2.5) follows from (2.4) by taking the convex function $h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$. To prove the sharpness of the constant $\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\,[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}$, we consider the function $f_0(z)\in F^*_{m,n}(\alpha,\beta)$ given by

$$f_0(z)=z-\frac{1-\alpha}{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}z^2. \tag{2.8}$$

Thus from (2.4), we have

$$\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\,[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}f_0(z)\prec\frac{z}{1-z}\ (z\in U). \tag{2.9}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.8) that

$$\min_{|z|\leq r}\left\{\Re\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\,[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}f_0(z)\right\}=-\frac{1}{2}. \tag{2.10}$$

This shows that the constant $\frac{(1+\beta)(2^m-2^n)+(1-\alpha)2^n}{2\,[(1+\beta)(2^m-2^n)+(1-\alpha)(2^n+1)]}$ is the best possible.

Remark 2. *(i) The result obtained in Theorem 1, is giving a simplified version of the result obtained by Srivastava and Eker [12, Theorem 1];*
(ii) Taking $m=n+1(n\in\mathbb{N}_0)$ in Theorem 1, we obtain the result obtained by Aouf et al. [2, Corollary 4];
(iii) Taking $m=1$ and $n=0$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.2];
(iv) Taking $m=2$ and $n=1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.5];

Putting $\beta = 0$ in Theorem 1, we have the following corollary:

Corollary 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $F^*_{m,n}(\alpha)$ *and suppose that* $h(z) \in K$. *Then*

$$\frac{2^m - 2^n\alpha}{2\left[(2^m+1) - (2^n+1)\alpha\right]} (f * h)(z) \prec h(z) \tag{2.11}$$

and

$$\Re\{f(z)\} > -\frac{(2^m+1) - (2^n+1)\alpha}{2^m - 2^n\alpha}. \tag{2.12}$$

The constant factor $\frac{2^m-2^n\alpha}{2[(2^m+1)-(2^n+1)\alpha]}$ *in the subordination result* (2.11) *cannot be replaced by a larger one.*

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