Acta Universitatis Apulensis No. 39/2014
ISSN: 1582-5329 pp. 225-247
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.39.19

FURTHER TWO DIMENSIONAL SERIES EVALUATIONS VIA THE ELLIPTIC FUNCTIONS OF RAMANUJAN AND JACOBI

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Abstract. Recently, B. C. Berndt, G. Lamb and M. Rogers [4] have shown that it is possible to evaluate

$$F_{(a,b)}(x) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2}$$

for any positive rational value of x and for many values of $(a, b) \in \mathbb{N}^2$. In fact, they have [4] obtain the values of $F_{(a,b)}(x)$ for $a \in 2, 3, 4, 6$ with $b = 1$ by employing the property of Ramanujan cubic continued fraction $(G(q))$, Ramanujan Göllnitz-Gordon continued fraction $(H(q))$, Rogers-Ramanujan continued fraction $(R(q))$ and their explicit values. Furthermore they have evaluate $F_{(a,b)}(x)$ when $a > 6$ with $b = 1$ by applying elementary properties of Jacobian elliptic functions, singular moduli and class invariants. The purpose of this paper is to establish $F_{(a,b)}(x)$, for $a \in 8, 10, 12$ with $b = 1$ using the properties of continued fractions of $R(q), G(q)$ and $H(q)$.

2000 *Mathematics Subject Classification*: 11F20, 11Y65, 11Y99.

Keywords: Ramanujan continued fractions, Two dimensional series evaluations.

1. Introduction

In [11], I. J. Zucker and R. McPhedran submitted the following open problem:

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(5m)^2 + (5n+1)^2} = -\frac{\pi}{5\sqrt{5}} \log\left(\sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}}\right) + \frac{\pi}{25} \log\left(11 + 5\sqrt{5}\right). \tag{1}$$

Recently B. C. Berndt, G. Lamb and M. Rogers [4] have established the proof of (1) by employing elliptic functions and certain properties satisfied by famous Rogers-Ramanujan continued fraction. In fact they have shown that it is possible to evaluate

$$F_{(a,b)}(x) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2},$$

for any positive rational value of x and for many values of $(a,b) \in \mathbb{N}^2$, by establishing the following Theorem:

Theorem 1. *Suppose that a and b are integers with $a \geq 2, (a,b) = 1$ and assume that $Re(x) > 0$. Then*

$$F_{(a,b)}(x) = -\frac{2\pi}{ax} \sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \left(\prod_{m=0}^{\infty} \left(1 - \omega^{2j+1} q^{2m+1}\right) \left(1 - \omega^{-2j-1} q^{2m+1}\right) \right), \tag{2}$$

where $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

According to Berndt et. al. [4], (2) is not difficult to prove, but the deduction of results such as (1) from (2) is usually more difficult. In addition to (1), they have [4] also obtain the values of $F_{(a,b)}(x)$ for $a \in 2, 3, 4, 6$ with $b = 1$ by employing the property of Ramanujan cubic continued fraction ($G(q)$), Ramanujan Göllnitz-Gordon continued fraction ($H(q)$), Rogers-Ramanujan continued fraction ($R(q)$) and their explicit values. Furthermore they have evaluate $F_{(a,b)}(x)$ when $a > 6$ with $b = 1$ by applying elementary properties of Jacobian elliptic functions, singular moduli and class invariants.

The purpose of this paper is to establish $F_{(a,b)}(x)$, for $a \in 8, 10, 12$ with $b = 1$ using the properties of continued fractions of $R(q), G(q)$ and $H(q)$. We close this section by recalling certain definitions and results which are required to establish our results. For $|q| < 1$, set

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

In Chapter 16 of his second notebook [1], [3], [6] Ramanujan develops theory of theta-functions and his theta-function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \qquad |ab| < 1.$$

By Jacobi's triple product identity, we have

$$\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2} = (-a, ab)_\infty(-b, ab)_\infty(ab, ab)_\infty, \qquad |ab| < 1. \tag{3}$$

Following Ramanujan, we define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2(q^2; q^2)_\infty, \tag{4}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{5}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty \tag{6}$$

and

$$\chi(q) := (-q; q^2)_\infty. \tag{7}$$

We will often find it convenient to employ the notation

$$f_n := f(-q^n).$$

Lemma 2. *We have*

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \chi(-q) = \frac{f_1}{f_2},$$

$$\varphi(q) = \frac{f_2^5}{f_1^2\, f_4^2}, \qquad \psi(-q) = \frac{f_1\, f_4}{f_2}, \qquad \chi(q) = \frac{f_2^2}{f_1\, f_4} \quad \textit{and} \quad f(q) = \frac{f_2^3}{f_1\, f_4}.$$

The celebrated Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+\cdots}{}, \qquad |q| < 1. \tag{8}$$

The Ramanujan's cubic continued fraction $G(q)$ be defined by

$$G(q) := \frac{q^{1/3} f(-q, -q^5)}{f(-q^3, -q^3)} = \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+\cdots}{}. \tag{9}$$

The Ramanujan Göllnitz-Gordon continued fraction $H(q)$ [8] is defined by

$$H(q) := \frac{q^{1/2} f(-q, -q^7)}{f(-q^3, -q^5)} = \frac{q^{1/2}}{1+q} \underset{+}{} \frac{q^2}{1+q^3} \underset{+}{} \frac{q^4}{1+q^5} \underset{+\cdots}{}, \qquad |q| < 1. \tag{10}$$

We also define

$$K(q) := \frac{q^{1/2} f(q, q^7)}{f(q^3, q^5)}.$$

Theorem 3. *We have*

$$\varphi(q)+\varphi(q^5) = 2q^{\frac{4}{5}}f(q,q^9)R^{-1}(q^4), \tag{11}$$
$$\varphi(q)-\varphi(q^5) = 2q^{\frac{1}{5}}f(q^3,q^7)R(q^4), \tag{12}$$
$$\psi(q^2)+q\psi(q^{10}) = q^{\frac{1}{5}}f(q^2,q^8)R^{-1}(q) \tag{13}$$

and

$$\psi(q^2)-q\psi(q^{10}) = q^{-\frac{1}{5}}f(q^4,q^6)R(q). \tag{14}$$

For a proof of Theorem 3, see [2, Entry 1.7.1, p. 28], [5, Theorem 3.1].

Theorem 4. *We have*

$$\frac{1}{H(q)}-H(q) = \frac{\varphi(q^2)}{q^{\frac{1}{2}}\psi(q^4)} \tag{15}$$

and

$$\frac{1}{H(q)}+H(q) = \frac{\varphi(q)}{q^{\frac{1}{2}}\psi(q^4)}. \tag{16}$$

For a proof of Theorem 4, see [3, Entry 1, p. 221].

Theorem 5. *We have*

$$\varphi(-q)+\varphi(q^2) = 2\frac{f^2(q^3,q^5)}{\psi(q)} \tag{17}$$

and

$$\varphi(-q)-\varphi(q^2) = -2q\ \frac{f^2(q,q^7)}{\psi(q)}. \tag{18}$$

For a proof of Theorem 5, see [3, Example (iv), p. 51].

Theorem 6. *We have*

$$\frac{1}{R^5(q)}-11-R^5(q)=\frac{f^6(-q)}{q\ f^6(-q^5)}. \tag{19}$$

For a proof of Theorem 6, see [7, p. 135-177, 238-243], [9, 10], [3, p. 265-267], [2, p. 11].

2. Main Results

Lemma 7. *We have*

$$\frac{\varphi(-q^4)}{q^{1/4}\ \psi(q^2)} = \sqrt[8]{\left[\frac{1}{H(q)} - H(q)\right]^4 - 16}, \tag{20}$$

$$\frac{\psi(-q^2)\varphi(q^4)}{q^{3/4}\psi^2(q^4)}\left(\frac{\varphi(q^2)}{\varphi(q^4)}+1\right) = \frac{2}{H(q^2)\left[1-H^2(q^2)\right]}\sqrt{\frac{H(q)\left[1-K^2(q^2)\right]\left[1-H^4(q^2)\right]}{\left[1+K^2(q^2)\right]\left[1-H^2(q)\right]}} \tag{21}$$

and

$$\frac{\psi(-q^2)\varphi(q^4)}{q^{3/4}\psi^2(q^4)}\left(\frac{\varphi(q^2)}{\varphi(q^4)}-1\right) = \frac{2H(q^2)}{\left[1-H^2(q^2)\right]}\sqrt{\frac{H(q)\left[1-K^2(q^2)\right]\left[1-H^4(q^2)\right]}{\left[1+K^2(q^2)\right]\left[1-H^2(q)\right]}}. \tag{22}$$

Proof. We have from [1], [3, Entry 25, p. 40], [6],

$$\varphi(q)\psi(q^2) = \psi^2(q) \tag{23}$$

$$\varphi^2(-q^2) = \varphi(q)\varphi(-q) \tag{24}$$

and

$$\varphi^4(q) - \varphi^4(-q) = 16q\ \psi^4(q^2). \tag{25}$$

From (23) and (24), we have

$$\frac{\varphi^2(-q^2)}{q^{1/4}\ \psi^2(q)} = \frac{\varphi(q)\varphi(-q)}{q^{1/4}\ \varphi(q)\psi(q^2)} = \sqrt[4]{\frac{\varphi^4(-q)}{q\ \psi^4(q^2)}}.$$

Using (25) in the above, we get

$$\frac{\varphi^2(-q^2)}{q^{1/4}\ \psi^2(q)} = \sqrt[4]{\frac{\varphi^4(q)}{q\ \psi^4(q^2)} - 16}.$$

Now employing (15) in the above and then changing q to q^2, we obtain (20).
We have

$$\frac{\varphi(q^2)}{\varphi(q^4)} = \frac{\varphi(q^2)}{q\ \psi(q^8)} \Big/ \frac{\varphi(q^4)}{q\ \psi(q^8)}$$

Employing (15) and (16) in the above, we find that

$$\frac{\varphi(q^2)}{\varphi(q^4)} = \frac{1+H^2(q^2)}{1-H^2(q^2)}. \tag{26}$$

Which implies

$$\frac{\varphi(q^2)}{\varphi(q^4)}+1=\frac{2}{1-H^2(q^2)} \tag{27}$$

and

$$\frac{\varphi(q^2)}{\varphi(q^4)}-1=\frac{2H^2(q^2)}{1-H^2(q^2)}. \tag{28}$$

By employing (23) twice, we find that

$$\begin{aligned}\frac{\psi(-q^2)\varphi(q^4)}{q^{3/4}\ \psi^2(q^4)} &= \sqrt{\frac{\varphi(-q^2)\psi(q^4)}{q^{3/2}\ \psi^2(q^8)}}\\ &= \sqrt{\frac{\varphi(-q^2)}{\varphi(q^4)}\frac{\varphi(q^4)}{q\ \psi(q^8)}\frac{\psi(q^4)}{q^{1/2}\ \psi(q^8)}}.\end{aligned}$$

Employing (26), (15) and (16) in the above, we find that

$$\frac{\psi(-q^2)\varphi(q^4)}{q^{3/4}\ \psi^2(q^4)}=\sqrt{\left[\frac{1-K^2(q^2)}{1+K^2(q^2)}\right]\left[\frac{1}{H(q^2)}-H(q^2)\right]\left[\frac{H(q)\,[1+H^2(q^2)]}{H(q^2)\,[1-H^2(q)]}\right]}. \tag{29}$$

Now, (21) follows from (27) and (29) and (22) follows from (28) with (29).

Theorem 8. *Suppose $q=e^{-\pi/x}$, then*

$$\begin{aligned}F_{(8,1)}(x) &= -\frac{\pi}{4x}\left\{\sqrt{2+\sqrt{2}}\ \log\left(\frac{A+\sqrt{2}-\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}B-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}C}{A-\sqrt{2}+\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}B-\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}C}\right)\right.\\ &\quad\left.+\sqrt{2-\sqrt{2}}\ \log\left(\frac{A-\sqrt{2}-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}B+\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}C}{A+\sqrt{2}+\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}B-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}C}\right)\right\},\end{aligned}$$

where

$$\begin{aligned}A &= \sqrt[8]{\left[\frac{1}{H(q^{16})}-H(q^{16})\right]^4-16},\\ B &= \sqrt{\frac{2}{H(q^{32})\,[1-H^2(q^{32})]}}\sqrt[4]{\left[\frac{H(q^{16})\,[1-K^2(q^{32})]\,[1-H^4(q^{32})]}{[1+K^2(q^{32})]\,[1-H^2(q^{16})]}\right]}\end{aligned}$$

and

$$C = \sqrt{\frac{2H(q^{32})}{[1-H^2(q^{32})]}}\sqrt[4]{\left[\frac{H(q^{16})\,[1-K^2(q^{32})]\,[1-H^4(q^{32})]}{[1+K^2(q^{32})]\,[1-H^2(q^{16})]}\right]}.$$

Proof. If we set $(a, b) = (8, 1)$ in (2), we find that

$$\begin{aligned} F_{(8,1)}(x) &= -\frac{\pi}{4x}\sum_{j=0}^{7}\cos\left(\frac{\pi(2j+1)}{8}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{\pi(2j+1)}{8}\right)q^{2m+1}+q^{4m+2}\right) \\ &= -\frac{\pi}{4x}\left[\sqrt{2+\sqrt{2}}\log\prod_{m=0}^{\infty}\left(\frac{\left(1-\sqrt{2+\sqrt{2}}q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2})}{\left(1+\sqrt{2-\sqrt{2}}q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2})}\right)\right. \\ &\quad \left.+\sqrt{2-\sqrt{2}}\log\prod_{m=0}^{\infty}\left(\frac{\left(1-\sqrt{2-\sqrt{2}}q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2})}{\left(1+\sqrt{2+\sqrt{2}}q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2})}\right)\right] \quad (30) \end{aligned}$$

Letting $\omega = e^{\pi i/8}$ and then using (3), we find that the denominator of second term is equal to

$$\begin{aligned} U(q) &:= \prod_{m=0}^{\infty}\left(1+\sqrt{2+\sqrt{2}}q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2}) \\ &= (-\omega q; q^2)_\infty(-\overline{\omega}q; q^2)_\infty(q^2; q^2)_\infty \\ &= \sum_{n=-\infty}^{\infty}\omega^n q^{n^2} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n\left[q^{(8n)^2}+\omega q^{(8n+1)^2}+\omega^2 q^{(8n+2)^2}+\omega^3 q^{(8n+3)^2}\right. \\ &\qquad \left.+\omega^4 q^{(8n+4)^2}+\omega^5 q^{(8n+5)^2}+\omega^6 q^{(8n+6)^2}+\omega^7 q^{(8n+7)^2}\right]. \end{aligned}$$

From (4), we have

$$\begin{aligned} U(q) = \varphi\left(-q^{64}\right) &+ \sqrt{2+\sqrt{2}}\left[\sum_{n=0}^{\infty}(-1)^n q^{(8n+1)^2}-\sum_{n=0}^{\infty}(-1)^n q^{(8n+7)^2}\right] \\ &+ \sqrt{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(8n+2)^2}-\sum_{n=0}^{\infty}(-1)^n q^{(8n+6)^2}\right] \\ &+ \sqrt{2-\sqrt{2}}\left[\sum_{n=0}^{\infty}(-1)^n q^{(8n+3)^2}-\sum_{n=0}^{\infty}(-1)^n q^{(8n+5)^2}\right] \end{aligned}$$

$$
\begin{aligned}
= \varphi\left(-q^{64}\right) \quad &+ \quad \sqrt{2+\sqrt{2}}\; q \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{16}\right)^{4n^2+n} \\
&+ \quad \sqrt{2}\; q^4 \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{16}\right)^{4n^2+2n} \\
&+ \quad \sqrt{2-\sqrt{2}}\; q^9 \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{16}\right)^{4n^2+3n} \quad .
\end{aligned}
$$

From the definition of $f(a,b)$, we have

$$
\begin{aligned}
U(q) = \varphi(-q^{64}) \quad &+ \quad \sqrt{2+\sqrt{2}}\; q f(-q^{48}, -q^{80}) \\
&+ \quad \sqrt{2}\; q^4 f(q^{32}, q^{96}) + \sqrt{2-\sqrt{2}}\; q^9 f(-q^{16}, -q^{112}).
\end{aligned}
$$

Now employing (17) and (18) in the above, we find that

$$
\begin{aligned}
U(q) = \varphi(-q^{64}) \quad &+ \quad \sqrt{2}\; q^4 \psi(q^{32}) \\
&+ \quad \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}\; q \left[\psi(-q^{16}) \left(\varphi(q^{16}) + \varphi(q^{32})\right)\right]^{\frac{1}{2}} \\
&+ \quad \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}\; q \left[\psi(-q^{16})(\varphi(q^{16}) - \varphi(q^{32}))\right]^{\frac{1}{2}} . \qquad (31)
\end{aligned}
$$

Similarly, we obtain

$$
\prod_{m=0}^{\infty} \left(1 + \sqrt{2-\sqrt{2}} q^{2m+1} + q^{4m+2}\right) \left(1 - q^{2m+2}\right)
$$

$$
\begin{aligned}
= \varphi(-q^{64}) \quad &+ \quad \sqrt{2}\; q^4 \psi(q^{32}) \\
&+ \quad \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}\; q \left[\psi(-q^{16})(\varphi(q^{16}) + \varphi(q^{32}))\right]^{\frac{1}{2}} \\
&- \quad \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}\; q \left[\psi(-q^{16})(\varphi(q^{16}) - \varphi(q^{32}))\right]^{\frac{1}{2}} , \qquad (32)
\end{aligned}
$$

$$
\prod_{m=0}^{\infty} \left(1 - \sqrt{2-\sqrt{2}} q^{2m+1} + q^{4m+2}\right) \left(1 - q^{2m+2}\right)
$$

$$
= \varphi(-q^{64}) + \sqrt{2}\; q^4 \psi(q^{32})
$$

$$
\begin{aligned}
&- \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}\; q\left[\psi(-q^{16})(\varphi(q^{16})+\varphi(q^{32}))\right]^{\frac{1}{2}}\\
&- \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}\; q\left[\psi(-q^{16})(\varphi(q^{16})-\varphi(q^{32}))\right]^{\frac{1}{2}} \qquad (33)
\end{aligned}
$$

and

$$
\prod_{m=0}^{\infty}\left(1-\sqrt{2-\sqrt{2}}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)
$$

$$
\begin{aligned}
= \varphi(-q^{64}) \;&-\; \sqrt{2}\; q^4\psi(q^{32})\\
&- \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}\; q\left[\psi(-q^{16})(\varphi(q^{16})+\varphi(q^{32}))\right]^{\frac{1}{2}}\\
&+ \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}\; q\left[\psi(-q^{16})(\varphi(q^{16})-\varphi(q^{32}))\right]^{\frac{1}{2}}. \qquad (34)
\end{aligned}
$$

Now substituting (31), (32), (33) and (34) in (30), we obtain

$$
\begin{aligned}
F_{(8,1)}(x) = \;&-\; \frac{\pi}{4x}\left\{\sqrt{2+\sqrt{2}}\log\left(\frac{A'+\sqrt{2}-\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}B'-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}C'}{A'-\sqrt{2}+\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}B'-\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}C'}\right)\right.\\
&+\left.\sqrt{2-\sqrt{2}}\;\log\left(\frac{A'-\sqrt{2}-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}B'+\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}C'}{A'+\sqrt{2}+\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}B'-\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}C'}\right)\right\}, \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
A' &= \frac{\varphi(-q^{64})}{q^4\;\psi(q^{32})},\\
B' &= \left[\frac{\psi(-q^{16})\varphi(q^{32})}{q^6\psi^2(q^{32})}\left(\frac{\varphi(q^{16})}{\varphi(q^{32})}+1\right)\right]^{1/2}
\end{aligned}
$$

and

$$
C' = \left[\frac{\psi(-q^{16})\varphi(q^{32})}{q^6\psi^2(q^{32})}\left(\frac{\varphi(q^{16})}{\varphi(q^{32})}-1\right)\right]^{1/2}.
$$

Using (20), (21) and (22) in (35), we obtain the required result.

Lemma 9. *We have*

$$\frac{\varphi(-q)}{q^{1/4}\ \psi(q^2)} = \frac{\left[1 - H^2(q^{1/2})\right] \sqrt{H^4(q) - 6H^2(q) + 1}}{H(q^{1/2})\,[1 + H^2(q)]}.$$

Proof. From (15) and (16), we have

$$\frac{\varphi(q^2)}{\varphi(q)} = \frac{1 - H^2(q)}{1 + H^2(q)}. \tag{36}$$

Also, we have from [3, Entry 25(vi)],

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

or

$$\frac{\varphi^2(-q)}{\varphi^2(q)} = 2\ \frac{\varphi^2(q^2)}{\varphi^2(q)} - 1.$$

Employing (36) in the above, we obtain

$$\begin{aligned} \frac{\varphi^2(-q)}{\varphi^2(q)} &= 2\ \left[\frac{1 - H^2(q)}{1 + H^2(q)}\right]^2 - 1 \\ &= \frac{H^4(q) - 6H^2(q) + 1}{[1 + H^2(q)]^2}. \end{aligned} \tag{37}$$

Now consider

$$\begin{aligned} \frac{\varphi(-q)}{q^{1/4}\ \psi(q^2)} &= \sqrt{\frac{\varphi^2(-q)}{\varphi^2(q)}}\ \sqrt{\frac{\varphi^2(q)}{q^{1/2}\ \psi^2(q^2)}} \\ &= \sqrt{\frac{\varphi^2(-q)}{\varphi^2(q)}}\ \frac{\varphi(q)}{q^{1/4}\ \psi(q^2)}. \end{aligned}$$

Employing (37) and (15) in the above, we find that

$$\frac{\varphi(-q)}{q^{1/4}\ \psi(q^2)} = \frac{\left[1 - H^2(q^{1/2})\right] \sqrt{H^4(q) - 6H^2(q) + 1}}{H(q^{1/2})\,[1 + H^2(q)]}.$$

Lemma 10. *We have*

$$\frac{\psi(q)}{q^{1/2}\ \psi(q^5)} = \sqrt{\frac{1 + R(q)R^2(q^2) - R^2(q)R^4(q^2)}{R(q)R^2(q^2)}}.$$

For a proof see [2, Entry 1.8.2, p. 35], [5, Theorem 4.2].

Lemma 11. *We have*

$$\frac{\varphi(-q)}{q^{5/4}\ \psi(q^{10})} = \left(\frac{\left[1-H^2(q^{1/2})\right]\sqrt{H^4(q)-6H^2(q)+1}}{H(q^{1/2})\left[1+H^2(q)\right]}\right)\times \left(\sqrt{\frac{1+R(q^2)R^2(q^4)-R^2(q^2)R^4(q^4)}{R(q^2)R^2(q^4)}}\right).$$

Proof. We have

$$\frac{\varphi(-q)}{q^{5/4}\ \psi(q^{10})} = \frac{\varphi(-q)}{q^{1/4}\ \psi(q^2)}\ \frac{\psi(q^2)}{q\ \psi(q^{10})}. \tag{38}$$

By previous two Lemmas, we obtain the required result.

Theorem 12. *Suppose $q = e^{-\pi/x}$, then*

$$F_{(10,1)}(x) = -\frac{\pi}{5x}\left\{\frac{\sqrt{10+2\sqrt{5}}}{2}\ \log\left(\frac{E-\frac{\sqrt{10+2\sqrt{5}}}{2}F+\frac{\sqrt{5}+1}{4}G-\frac{\sqrt{10-2\sqrt{5}}}{2}H+\frac{\sqrt{5}-1}{4}I}{E+\frac{\sqrt{10+2\sqrt{5}}}{2}F+\frac{\sqrt{5}+1}{4}G+\frac{\sqrt{10-2\sqrt{5}}}{2}H+\frac{\sqrt{5}-1}{4}I}\right)\right.$$
$$\left.+\frac{\sqrt{10-2\sqrt{5}}}{2}\ \log\left(\frac{E-\frac{\sqrt{10-2\sqrt{5}}}{2}F+\frac{\sqrt{5}-1}{4}G+\frac{\sqrt{10+2\sqrt{5}}}{2}H-\frac{\sqrt{5}+1}{4}I}{E+\frac{\sqrt{10-2\sqrt{5}}}{2}F+\frac{\sqrt{5}-1}{4}G-\frac{\sqrt{10+2\sqrt{5}}}{2}H-\frac{\sqrt{5}+1}{4}I}\right)\right\},$$

where

$$E = \frac{1-H^2(q^{50})\sqrt{H^4(q^{100})-6H^2(q^{100})+1}}{H(q^{50})\left[1+H^2(q^{100})\right]},$$

$$F = \frac{1}{R(q^{20})}\left[\sqrt{\frac{1+R(q^{40})R^2(q^{80})-R^2(q^{40})R^4(q^{80})}{R(q^{40})R^2(q^{80})}}-1\right],$$

$$G = \frac{1}{R(q^{80})}\left\{\left[\left(\frac{\left[1-H^2(q^{10})\right]\sqrt{H^4(q^{20})-6H^2(q^{20})+1}}{H(q^{10})\left[1+H^2(q^{20})\right]}\right)\times \left(\sqrt{\frac{1+R(q^{40})R^2(q^{80})-R^2(q^{40})R^4(q^{80})}{R(q^{40})R^2(q^{80})}}\right)\right] - \frac{1-H^2(q^{50})\sqrt{H^4(q^{100})-6H^2(q^{100})+1}}{H(q^{50})\left[1+H^2(q^{100})\right]}\right\},$$

$$H = R(q^{20})\left[\sqrt{\frac{1+R(q^{40})R^2(q^{80})-R^2(q^{40})R^4(q^{80})}{R(q^{40})R^2(q^{80})}}+1\right]$$

and

$$I = R(q^{80})\left\{\left[\left(\frac{[1-H^2(q^{10})]\sqrt{H^4(q^{20})-6H^2(q^{20})+1}}{H(q^{10})\,[1+H^2(q^{20})]}\right)\times \left(\sqrt{\frac{1+R(q^{40})R^2(q^{80})-R^2(q^{40})R^4(q^{80})}{R(q^{40})R^2(q^{80})}}\right)\right] + \frac{1-H^2(q^{50})\sqrt{H^4(q^{100})-6H^2(q^{100})+1}}{H(q^{50})\,[1+H^2(q^{100})]}\right\}.$$

Proof. If we set $(a,b)=(10,1)$ in (2), we find that

$$F_{(10,1)}(x) = -\frac{\pi}{5x}\sum_{j=0}^{9}\cos\left(\frac{\pi(2j+1)}{10}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{\pi(2j+1)}{10}\right)q^{2m+1}+q^{4m+2}\right)$$

$$= -\frac{\pi}{5x}\left[\frac{\sqrt{10+2\sqrt{5}}}{2}\ \log\prod_{m=0}^{\infty}\left(\frac{\left(1-\frac{\sqrt{10+2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)}{\left(1+\frac{\sqrt{10+2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)}\right) + \frac{\sqrt{10-2\sqrt{5}}}{2}\ \log\prod_{m=0}^{\infty}\left(\frac{\left(1-\frac{\sqrt{10-2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)}{\left(1+\frac{\sqrt{10-2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)}\right)\right]. \quad (39)$$

Letting $\omega = e^{\pi i/10}$ and then using (3), we find that the first term of denominator is equal to

$$\begin{aligned} X(q) &:= \prod_{m=0}^{\infty}\left(1+\frac{\sqrt{10+2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right) \\ &= (-\omega q;q^2)_\infty(-\overline{\omega}q;q^2)_\infty(q^2;q^2)_\infty \\ &= \sum_{n=-\infty}^{\infty}\omega^n q^{n^2} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n\Big[q^{(10n)^2}+\omega q^{(10n+1)^2}+\omega^2q^{(10n+2)^2}+\omega^3q^{(10n+3)^2} \\ &\qquad +\omega^4q^{(10n+4)^2}+\omega^5q^{(10n+5)^2}+\omega^6q^{(10n+6)^2} \\ &\qquad +\omega^7q^{(10n+7)^2}+\omega^8q^{(10n+8)^2}+\omega^9q^{(10n+9)^2}\Big]. \end{aligned}$$

From (4), we have

$$\begin{aligned} X(q) = \varphi\left(-q^{100}\right) &+ \frac{\sqrt{10+2\sqrt{5}}}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(10n+1)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(10n+9)^2}\right] \\ &- \frac{\sqrt{5}+1}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(10n+2)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(10n+8)^2}\right] \\ &+ \frac{\sqrt{10-2\sqrt{5}}}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(10n+3)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(10n+7)^2}\right] \\ &- \frac{\sqrt{5}-1}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(10n+4)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(10n+6)^2}\right] \end{aligned}$$

$$\begin{aligned} = \varphi(-q^{100}) &+ \frac{\sqrt{10+2\sqrt{5}}}{2}\, q \sum_{n=-\infty}^{\infty}(-1)^n (q^{20})^{5n^2+n} \\ &- \frac{\sqrt{5}+1}{2}\, q^4 \sum_{n=-\infty}^{\infty}(-1)^n (q^{20})^{5n^2+2n} \\ &+ \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^9 \sum_{n=-\infty}^{\infty}(-1)^n (q^{20})^{5n^2+3n} \\ &+ \frac{\sqrt{5}-1}{2}\, q^{16} \sum_{n=-\infty}^{\infty}(-1)^n (q^{20})^{5n^2+4n}. \end{aligned}$$

Using the definition of $f(a,b)$ in the above, we obtain

$$\begin{aligned} X(q) = \varphi(-q^{100}) &+ \frac{\sqrt{10+2\sqrt{5}}}{2}\, q f(q^{80}, q^{120}) - \frac{\sqrt{5}+1}{2}\, q^4 f(-q^{60}, -q^{140}) \\ &+ \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^9 f(q^{40}, q^{160}) + \frac{\sqrt{5}-1}{2}\, q^{16} f(-q^{20}, -q^{180}). \end{aligned}$$

Now employing (11), (12), (13) and (14) in the above, we find that

$$\begin{aligned} X(q) = \varphi(-q^{100}) &+ \frac{\sqrt{10+2\sqrt{5}}}{2}\, q^5 \left[\frac{\psi(q^{40}) - q^{20}\psi(q^{200})}{R(q^{20})}\right] \\ &+ \frac{\sqrt{5}+1}{4}\left[\frac{\varphi(-q^{20}) - \varphi(-q^{100})}{R(q^{80})}\right] \\ &+ \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^5 \left[\psi(q^{40}) + q^{20}\psi(q^{200})\right] R(q^{20}) \\ &+ \frac{\sqrt{5}-1}{4}\left[\varphi(-q^{20}) + \varphi(-q^{100})\right] R(q^{80}). \end{aligned} \tag{40}$$

Similarly, we obtain

$$\begin{aligned}
&\prod_{m=0}^{\infty}\left(1+\frac{\sqrt{10-2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)\\
&= \varphi(-q^{100}) \quad + \quad \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^5 \left[\frac{\psi(q^{40})-q^{20}\psi(q^{200})}{R(q^{20})}\right]\\
&\quad + \quad \frac{\sqrt{5}-1}{4}\left[\frac{\varphi(-q^{20})-\varphi(-q^{100})}{R(q^{80})}\right]\\
&\quad - \quad \frac{\sqrt{10+2\sqrt{5}}}{2}\, q^5 \left[\psi(q^{40})+q^{20}\psi(q^{200})\right] R(q^{20})\\
&\quad - \quad \frac{\sqrt{5}+1}{4}\left[\varphi(-q^{20})+\varphi(-q^{100})\right] R(q^{80}), \qquad (41)
\end{aligned}$$

$$\begin{aligned}
&\prod_{m=0}^{\infty}\left(1-\frac{\sqrt{10+2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)\\
&= \varphi(-q^{100}) \quad - \quad \frac{\sqrt{10+2\sqrt{5}}}{2}\, q^5 \left[\frac{\psi(q^{40})-q^{20}\psi(q^{200})}{R(q^{20})}\right]\\
&\quad + \quad \frac{\sqrt{5}+1}{4}\left[\frac{\varphi(-q^{20})-\varphi(-q^{100})}{R(q^{80})}\right]\\
&\quad - \quad \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^5 \left[\psi(q^{40})+q^{20}\psi(q^{200})\right] R(q^{20})\\
&\quad + \quad \frac{\sqrt{5}-1}{4}\left[\varphi(-q^{20})+\varphi(-q^{100})\right] R(q^{80}) \qquad (42)
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{m=0}^{\infty}\left(1-\frac{\sqrt{10-2\sqrt{5}}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)\\
&= \varphi(-q^{100}) \quad - \quad \frac{\sqrt{10-2\sqrt{5}}}{2}\, q^5 \left[\frac{\psi(q^{40})-q^{20}\psi(q^{200})}{R(q^{20})}\right]\\
&\quad + \quad \frac{\sqrt{5}-1}{4}\left[\frac{\varphi(-q^{20})-\varphi(-q^{100})}{R(q^{80})}\right]\\
&\quad + \quad \frac{\sqrt{10+2\sqrt{5}}}{2}\, q^5 \left[\psi(q^{40})+q^{20}\psi(q^{200})\right] R(q^{20})\\
&\quad - \quad \frac{\sqrt{5}+1}{4}\left[\varphi(-q^{20})+\varphi(-q^{100})\right] R(q^{80}). \qquad (43)
\end{aligned}$$

Now substituting (40), (41), (42) and (43) in (39) and then employing Lemma 9, Lemma 10 and Lemma 11, we obtain the required result.

Lemma 13. *We have*

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = 1 - 8G^3(q).$$

For a proof see [3, p. 347].

Lemma 14. *We have*

$$q^{1/24}\ \frac{f(q)}{\varphi(-q^6)} = \sqrt[6]{\frac{H(q^{1/2})\left[1+H^2(q)\right]^2}{\left[1-6H^2(q)+H^4(q)\right]\left[1-H^2(q^{1/2})\right]}} \times \sqrt[4]{1-8G^3(q^2)}. \quad (44)$$

$$q^{1/24}\frac{\varphi(q^3)}{\chi(q)\varphi(-q^6)} = \sqrt[6]{\frac{\left[H^4(q)-6H^2(q)+1\right]\left[1-H^2(q^{1/2})\right]\left[1+H^2(q^2)\right]^2 H(q)}{\left[1+H^2(q)\right]^2 H(q^{1/2})\left[H^4(q^2)-6H^2(q^2)+1\right]\left[1-H^2(q)\right]}} \times \sqrt[4]{\frac{\left[1+H^2(q^3)\right]^2}{H^4(q^3)-6H^2(q^3)+1}} \quad (45)$$

$$q^{1/12}\ \frac{\varphi(-q^6)}{\chi(-q^2)\varphi(-q^3)} = \sqrt[6]{\frac{H(q)\left[1+H^2(q^2)\right]^2}{\left[1-H^2(q)\right]\left[H^4(q^2)-6H^2(q^2)+1\right]}} \times \sqrt[4]{\frac{\left[1+H^2(q^3)\right]^2}{H^4(q^3)-6H^2(q^3)+1}} \quad (46)$$

$$q^{1/8}\ \frac{\psi(-q)}{\varphi(-q^2)} = \frac{H(q^{1/2})\left[1-H^2(q^{1/4})\right]}{H(q^{1/4})\left[1+H^2(q^{1/2})\right]} \quad (47)$$

$$q^{1/3}\ \frac{\chi(q)\psi(-q^3)}{\varphi(-q^3)} = \sqrt[6]{\frac{\left[1-H^2(q)\right]\left[H^4(q^2)-6H^2(q^2)+1\right]H(q^{1/2})}{H(q)\left[1+H^2(q^2)\right]^2\left[1-H^2(q^{1/2})\right]\left[H^4(q)-6H^2(q)+1\right]}} \times$$

$$\sqrt[6]{\frac{\left[1+H^2(q)\right]^2 H(q^{3/2})\left[1+H^2(q^3)\right]^2 H(q^3)\left[1+H^2(q^6)\right]^2}{\left[1-H^2(q^{3/2})\right]\left[H^4(q^3)-6H^2(q^3)+1\right]\left[1-H^2(q^3)\right]\left[H^4(q^6)-6H^2(q^6)+1\right]}} \quad (48)$$

Proof. From Lemma 2, we have

$$\begin{aligned} q^{1/24}\ \frac{f(q)}{\varphi(-q^6)} &= q^{1/24}\ \frac{f_2^3}{f_1 f_4}\frac{1}{\varphi(-q^6)} \\ &= q^{1/24}\ \frac{1}{\chi(-q)}\frac{\varphi(-q^2)}{\varphi(-q^6)} \\ &= \sqrt[6]{\frac{1}{q^{-1/4}\ \chi^6(-q)}}\sqrt[4]{\frac{\varphi^4(-q^2)}{\varphi^4(-q^6)}}. \end{aligned} \quad (49)$$

We have

$$\chi^6(-q) = \frac{\varphi^2(-q)}{\psi^2(q)}.$$

Using (23) in the above, we have

$$\begin{aligned}\chi^6(-q) &= \frac{\varphi^2(-q)}{\varphi(q)\psi(q^2)} \\ &= \frac{\varphi^2(-q)}{\varphi^2(q)}\frac{\varphi(q)}{\psi(q^2)}.\end{aligned}$$

Employing (39) and (15) in the in the above, we obtain

$$q^{-1/4}\chi^6(-q) = \frac{\left[H^4(q) - 6H^2(q) + 1\right]\left[1 - H^2(q^{1/2})\right]}{H(q^{1/2})\left[1 + H^2(q)\right]^2}. \tag{50}$$

Now employing (50) and Lemma 13 in (49), we obtain (44).

We have

$$\chi(q) = \frac{\chi(-q^2)}{\chi(-q)}.$$

Thus,

$$q^{1/24}\frac{\varphi(q^3)}{\chi(q)\varphi(-q^6)} = q^{1/24}\frac{\chi(-q)}{\chi(-q^2)}\sqrt[4]{\frac{\varphi^4(q^3)}{\varphi^4(-q^6)}}.$$

Using (24) in the above , we find

$$\begin{aligned}q^{1/24}\frac{\varphi(q^3)}{\chi(q)\varphi(-q^6)} &= \frac{q^{1/24}\chi(-q)}{\chi(-q^2)}\sqrt[4]{\frac{\varphi^2(q^3)}{\varphi^2(-q^3)}} \\ &= \sqrt[6]{\frac{q^{-1/4}\chi^6(-q)}{q^{-1/2}\chi^6(-q^2)}}\sqrt[4]{\frac{\varphi^2(q^3)}{\varphi^2(-q^3)}}.\end{aligned}$$

Employing (37) and (50) in the above, we obtain (45).

By (24), we can see that

$$\frac{\varphi(-q^2)}{\varphi(-q)} = \sqrt[4]{\frac{\varphi^2(-q)\varphi^2(q)}{\varphi^4(-q)}} = \sqrt[4]{\frac{\varphi^2(q)}{\varphi^2(-q)}}.$$

Employing (37) in the above, we find that

$$\frac{\varphi(-q^2)}{\varphi(-q)} = \sqrt[4]{\frac{[1+H^2(q)]^2}{H^4(q)-6H^2(q)+1}}.$$

Changing q to q^3, we get

$$\frac{\varphi(-q^6)}{\varphi(-q^3)} = \sqrt[4]{\frac{[1+H^2(q^3)]^2}{H^4(q^3)-6H^2(q^3)+1}}.$$

From (50) and the above, we deduce (46).

We have, from [3, Entry 25(iii), p. 40],

$$\psi(-q) = \frac{\psi(q^2)\varphi(-q^2)}{\psi(q)}.$$

Multiplying the above equation throughout by $q^{1/8}\ \dfrac{1}{\varphi(-q^2)}$, we obtain

$$q^{1/8}\ \frac{\psi(-q)}{\varphi(-q^2)} = q^{1/8}\ \frac{\psi(q^2)\varphi(-q^2)}{\psi(q)\varphi(-q^2)}$$

i.e.

$$q^{1/8}\ \frac{\psi(-q)}{\varphi(-q^2)} = q^{1/8}\ \frac{\psi(q^2)}{\psi(q)}. \tag{51}$$

Now changing q to q^2 in (16), we obtain

$$\frac{1+H^2(q^2)}{H(q^2)} = \frac{\varphi(q^2)}{q\ \psi(q^8)}. \tag{52}$$

Dividing (15) from (52), we get

$$q^{1/2}\frac{\psi(q^8)}{\psi(q^4)} = \frac{H(q^2)\left[1-H^2(q)\right]}{H(q)\left[1+H^2(q^2)\right]}.$$

Changing q to $q^{1/4}$ in the above and substituting in (51), we obtain (47).

From Lemma 2, we have

$$\begin{aligned} q^{1/3}\frac{\chi(q)\psi(-q^3)}{\varphi(-q^3)} &= q^{1/3}\ \chi(q)\frac{f_3\ f_{12}}{f_6}\frac{f_6}{f_3^2} \\ &= q^{1/3}\ \chi(q)\frac{f_6}{f_3}\frac{f_{12}}{f_6}. \end{aligned}$$

Again using Lemma 2, we have

$$\begin{aligned}
q^{1/3}\frac{\chi(q)\psi(-q^3)}{\varphi(-q^3)} &= \frac{q^{1/3}\ \chi(q)}{\chi(-q^3)\chi(-q^6)} \\
&= q^{1/3}\ \frac{\chi(-q^2)}{\chi(-q)}\ \frac{1}{\chi(-q^3)\chi(-q^6)} \\
&= \sqrt[6]{q^2\ \frac{\chi^6(-q^2)}{\chi^6(-q)}\ \frac{1}{\chi^6(-q^3)\chi^6(-q^6)}} \\
&= \sqrt[6]{\frac{q^{-1/2}\chi^6(-q^2)}{q^{-1/4}\chi^6(-q)\ q^{-3/4}\chi^6(-q^3)\ q^{-3/2}\chi^6(-q^6)}}.
\end{aligned}$$

Employing (50) in the above equation twice, we obtain (48).

Theorem 15. *Suppose* $q = e^{-\pi/x}$, *then*

$$\begin{aligned}
F_{(12,1)}(x) &= \frac{-\pi}{6x}\left[\frac{\sqrt{6}+\sqrt{2}}{2}\log\prod_{m=0}^{\infty}\left(\frac{1-\frac{\sqrt{6}}{2}A-\frac{1}{\sqrt{2}}B+\sqrt{3}C-\sqrt{2}D+E}{1+\frac{\sqrt{6}}{2}A+\frac{1}{\sqrt{2}}B+\sqrt{3}C+\sqrt{2}D+E}\right)\right. \\
&\quad +\frac{\sqrt{6}-\sqrt{2}}{2}\log\prod_{m=0}^{\infty}\left(\frac{1-\frac{\sqrt{6}}{2}A+\frac{1}{\sqrt{2}}B-\sqrt{3}C+\sqrt{2}D+E}{1+\frac{\sqrt{6}}{2}A-\frac{1}{\sqrt{2}}B-\sqrt{3}C-\sqrt{2}D+E}\right) \\
&\quad \left.+\sqrt{2}\log\prod_{m=0}^{\infty}\left(\frac{1-\sqrt{2}B+\sqrt{2}D-2E}{1+\sqrt{2}B-\sqrt{2}D-2E}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
A &= \sqrt[6]{\frac{H(q^{12})\left[1+H^2(q^{24})\right]^2}{\left[1-6H^2(q^{24})+H^4(q^{24})\right]\left[1-H^2(q^{12})\right]}}\times\sqrt[4]{1-8G^3(q^{48})}\ , \\
B &= \sqrt[6]{\frac{H(q^{24})\left[1+H^2(q^{48})\right]^2\left[1-6H^2(q^{24})+H^4(q^{24})\right]\left[1-H^2(q^{12})\right]}{H(q^{12})\left[1+H^2(q^{24})\right]^2\left[1-6H^2(q^{48})+H^4(q^{48})\right]\left[1-H^2(q^{24})\right]}}\ \times \\
&\quad \sqrt[4]{\frac{\left[1+H^2(q^{72})\right]^2}{1-6H^2(q^{72})+H^4(q^{72})}}\ , \\
C &= \sqrt[6]{\frac{H(q^{48})\left[1+H^2(q^{96})\right]^2}{\left[1-H^2(q^{48})\right]^2\left[1+H^4(q^{96})-6H^2(q^{96})\right]}}\times\sqrt[4]{\frac{\left[1+H^2(q^{144})\right]^2}{1-6H^2(q^{144})+H^4(q^{144})}}\ ,
\end{aligned}$$

$$D = \frac{H(q^{36})\left[1-H^2(q^{18})\right]}{H(q^{18})\left[1-H^2(q^{36})\right]}$$

and

$$E = \sqrt[6]{\frac{[1-H^2(q^{48})]\,[H^4(q^{96})-6H^2(q^{96})+1]\,H(q^{24})}{H(q^{48})\,[1+H^2(q^{96})]^2\,[1-H^2(q^{24})]\,[H^4(q^{48})-6H^2(q^{48})+1]}} \times \sqrt[6]{\frac{[1+H^2(q^{48})]^2\,H(q^{72})\,[1+H^2(q^{144})]^2\,H(q^{144})\,[1+H^2(q^{288})]^2}{[1-H^2(q^{72})]\,[H^4(q^{144})-6H^2(q^{144})+1]\,[1-H^2(q^{144})]\,[H^4(q^{288})-6H^2(q^{288})+1]}}.$$

Proof. If we set $(a,b) = (12,1)$ in (2), we find that

$$\begin{aligned}
F_{(12,1)}(x) &= -\frac{\pi}{6x}\sum_{j=0}^{11}\cos\left(\frac{\pi(2j+1)}{12}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{\pi(2j+1)}{12}\right)q^{2m+1}+q^{4m+2}\right)\\
&= -\frac{\pi}{6x}\left[\frac{\sqrt{6}+\sqrt{2}}{2}\log\prod_{m=0}^{\infty}\left(\frac{\left(1-\frac{\sqrt{6}+\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}{\left(1+\frac{\sqrt{6}+\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}\right)\right.\\
&\quad+\frac{\sqrt{6}-\sqrt{2}}{2}\log\prod_{m=0}^{\infty}\left(\frac{\left(1-\frac{\sqrt{6}-\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}{\left(1+\frac{\sqrt{6}-\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}\right)\\
&\quad\left.+\sqrt{2}\log\prod_{m=0}^{\infty}\left(\frac{\left(1-\sqrt{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}{\left(1+\sqrt{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)}\right)\right]. \qquad (53)
\end{aligned}$$

Letting $\omega = e^{\pi i/12}$ and then using (3), we find that the first term of denominator is equal to

$$\begin{aligned}
V(q) &:= \prod_{m=0}^{\infty}\left(1+\frac{\sqrt{6}+\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)\\
&= (-\omega q;q^2)_\infty(-\overline{\omega}q;q^2)_\infty(q^2;q^2)_\infty\\
&= \sum_{n=-\infty}^{\infty}\omega^n q^{n^2}\\
&= \sum_{n=-\infty}^{\infty}(-1)^n\left[q^{(12n)^2}+\omega q^{(12n+1)^2}+\omega^2 q^{(12n+2)^2}+\omega^3 q^{(12n+3)^2}\right.\\
&\quad+\omega^4 q^{(12n+4)^2}+\omega^5 q^{(12n+5)^2}+\omega^6 q^{(12n+6)^2}+\omega^7 q^{(12n+7)^2}\\
&\quad\left.+\omega^8 q^{(12n+8)^2}+\omega^9 q^{(12n+9)^2}+\omega^{10} q^{(12n+10)^2}+\omega^{11} q^{(12n+11)^2}\right].
\end{aligned}$$

From (4), we have

$$\begin{aligned} V(q) = \varphi\left(-q^{144}\right) \quad &+ \quad \frac{\sqrt{6}+\sqrt{2}}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(12n+1)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(12n+11)^2}\right] \\ &+ \quad \sqrt{3}\left[\sum_{n=0}^{\infty}(-1)^n q^{(12n+2)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(12n+10)^2}\right] \\ &+ \quad \sqrt{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(12n+3)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(12n+9)^2}\right] \\ &+ \quad \left[\sum_{n=0}^{\infty}(-1)^n q^{(12n+4)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(12n+8)^2}\right] \\ &+ \quad \frac{\sqrt{6}-\sqrt{2}}{2}\left[\sum_{n=0}^{\infty}(-1)^n q^{(12n+5)^2} - \sum_{n=0}^{\infty}(-1)^n q^{(12n+7)^2}\right] \end{aligned}$$

$$\begin{aligned} V(q) = \varphi\left(-q^{144}\right) \quad &+ \quad \frac{\sqrt{6}+\sqrt{2}}{2}\, q \sum_{n=-\infty}^{\infty}(-1)^n \left(q^{24}\right)^{6n^2+n} + \sqrt{3}\, q^4 \sum_{n=-\infty}^{\infty}(-1)^n \left(q^{24}\right)^{6n^2+2n} \\ &+ \quad \sqrt{2}\, q^9 \sum_{n=-\infty}^{\infty}(-1)^n \left(q^{24}\right)^{6n^2+3n} + q^{16} \sum_{n=-\infty}^{\infty}(-1)^n \left(q^{24}\right)^{6n^2+4n} \\ &+ \quad \frac{\sqrt{6}-\sqrt{2}}{2}\, q^{25} \sum_{n=-\infty}^{\infty}(-1)^n \left(q^{24}\right)^{6n^2+5n} \end{aligned}$$

Using the definition of $f(a,b)$ in the above, we obtain

$$\begin{aligned} V(q) = \varphi\left(-q^{144}\right) \quad &+ \quad \frac{\sqrt{6}+\sqrt{2}}{2}\, q\ f\left(-q^{120},-q^{168}\right) + \sqrt{3}\, q^4\ f\left(-q^{96},-q^{192}\right) \\ &+ \quad \sqrt{2}\, q^9\ f\left(-q^{72},-q^{216}\right) + \frac{\sqrt{6}-\sqrt{2}}{2}\, q^{25}\ f\left(-q^{24},-q^{264}\right) \\ &+ \quad q^{16}\ f\left(-q^{48},-q^{240}\right) + \frac{\sqrt{6}-\sqrt{2}}{2}\, q^{25}\ f\left(-q^{24},-q^{264}\right). \end{aligned}$$

Now employing (4), (5), (6) and (7) in the above, we find that

$$\begin{aligned} V(q) = \varphi\left(-q^{144}\right) \quad &+ \quad \frac{\sqrt{6}\, q}{2}\ f\left(q^{24}\right) + \frac{q}{\sqrt{2}}\ \frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} + \sqrt{3}\, q^4\ \frac{\varphi\left(-q^{288}\right)}{\chi\left(-q^{96}\right)} \\ &+ \quad \sqrt{2}\, q^9\ \psi\left(-q^{72}\right) + q^{16}\ \chi\left(-q^{48}\right)\psi\left(-q^{144}\right). \end{aligned} \tag{54}$$

Now changing q to $-q$ in the above, we obtain

$$\prod_{m=0}^{\infty}\left(1 - \frac{\sqrt{6}+\sqrt{2}}{2} q^{2m+1} + q^{4m+2}\right)\left(1-q^{2m+2}\right)$$

$$= \varphi\left(-q^{144}\right) - \frac{\sqrt{6}\, q}{2}\, f\left(q^{24}\right) - \frac{q}{\sqrt{2}}\,\frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} + \sqrt{3}\, q^{4}\,\frac{\varphi\left(-q^{288}\right)}{\chi\left(-q^{96}\right)} - \sqrt{2}\, q^{9}\, \psi\left(-q^{72}\right) + q^{16}\,\chi\left(-q^{48}\right)\psi\left(-q^{144}\right). \quad (55)$$

Similarly, we obtain

$$\prod_{m=0}^{\infty}\left(1-\frac{\sqrt{6}-\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)$$

$$= \varphi\left(-q^{144}\right) - \frac{\sqrt{6}\, q}{2}\, f\left(q^{24}\right) + \frac{q}{\sqrt{2}}\,\frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} - \sqrt{3}\, q^{4}\,\frac{\varphi\left(-q^{288}\right)}{\chi\left(-q^{96}\right)} + \sqrt{2}\, q^{9}\, \psi\left(-q^{72}\right) + q^{16}\,\chi\left(-q^{48}\right)\psi\left(-q^{144}\right). \quad (56)$$

$$\prod_{m=0}^{\infty}\left(1+\frac{\sqrt{6}-\sqrt{2}}{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)$$

$$= \varphi\left(-q^{144}\right) + \frac{\sqrt{6}\, q}{2}\, f\left(q^{24}\right) - \frac{q}{\sqrt{2}}\,\frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} - \sqrt{3}\, q^{4}\,\frac{\varphi\left(-q^{288}\right)}{\chi\left(-q^{96}\right)} - \sqrt{2}\, q^{9}\, \psi\left(-q^{72}\right) + q^{16}\,\chi\left(-q^{48}\right)\psi\left(-q^{144}\right). \quad (57)$$

$$\prod_{m=0}^{\infty}\left(1-\sqrt{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)$$

$$= \varphi\left(-q^{144}\right) - \sqrt{2}\,\frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} + \sqrt{2}\, q^{9}\, \psi\left(-q^{72}\right) - 2q^{16}\,\chi\left(-q^{48}\right)\psi\left(-q^{144}\right) \quad (58)$$

and

$$\prod_{m=0}^{\infty}\left(1+\sqrt{2}q^{2m+1}+q^{4m+2}\right)\left(1-q^{2m+2}\right)$$

$$= \varphi\left(-q^{144}\right) + \sqrt{2}\,\frac{\varphi\left(q^{72}\right)}{\chi\left(q^{24}\right)} - \sqrt{2}\,q^{9}\,\psi(-q^{72}) - 2q^{16}\,\chi\left(-q^{48}\right)\psi(-q^{144}). \tag{59}$$

Now substituting (54), (55), (56), (57), (58) and (59) in (53) and then employing Lemma 14, we obtain the required result.

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