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SINC-GALERKIN METHOD FOR SOLUTION OF BURGERS' EQUATION

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Abstract. We develop a numerical algorithm for solving one dimensional Burgers' equation. The equation converted to the system of nonlinear ordinary differential equations by discretization first in time and subsequently in each time level we applied the Galerkin method based on sinc function in spatial direction. We proved the convergence analysis, it is shown that the approximate solution converges exponentially. The presented method is applied to two test problems, the obtained results have been compared with the exact solutions and some published numerical results in the literatures. The different measures of error in the solution verify the efficiency of sinc-Galerkin method.

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1. Introduction

We consider one-dimensional Burgers' equation:

$$u_t + uu_x = \nu u_{xx}, \quad a < x < b, \quad t > 0 \tag{1}$$

with boundary and initial conditions:

$$u(a,t) = f_1(t), \quad u(b,t) = f_2(t) \quad u(x,0) = g(x), \tag{2}$$

where $u(x,t)$ indicates the velocity for space x and time t, ν is the kinematic viscosity , f_1, f_2 and g are known functions of their arguments. This equation was first introduced by Bateman [6]. It was later treated by Burgers [7, 8] after which this equation is widely referred to as Burgers' equation. It plays an important role in the study of nonlinear waves motion since behavior of many physical systems encountered in model of traffic and fluid flow leads to Burgers' equation.

Burgers' equation has been studied by many researchers for the following reasons: first, it contain the simplest form of nonlinear advection term uu_x for simulating the physical phenomena of waves motion. Secondly, for small ν, solution can exhibit shock wave-like behavior. The exact solution of Burgers' equation has been derived by Cole [9] which is an infinite series. Many numerical methods for Burgers equation have been developed such as finite difference methods [10, 21, 28, 26, 32], spline collocation methods [11, 30], B-spline Galerkin methods [2, 4, 12, 13, 29], B-spline collocation methods [3, 14] and spectral least-squares methods [15, 16, 17, 22].

In this paper we discretize the Burgers' equation in the time direction and then the sinc-Galerkin method is applied. In this approach we do not need to use Hopf-Cole transformation to linearize the given equation. In the sinc method the test functions are translated by the sinc-function $s(x) = sin(\pi x)/(\pi x)$. The sinc method , which was developed by F. Stenger [33], is based on the Whittaker-Shannon-Kotel' nikov sampling theorem for entire functions. This method has many advantages over classical methods that use polynomials as bases. For example, in the presence of singularities, it gives a much better rate of convergence and greater accuracy than polynomial methods. In recent years, a lot of attention has been devoted to the study of the sinc method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers[5, 18, 35].

2. Preliminaries of Sinc Method

In this section, we state preliminaries of the Sinc interpolation together with some essential definitions and theorems.

The Sinc function is defined on $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$ we will denote the Sinc basis functions by

$$S(j,h)(x) = sinc(\frac{x - jh}{h}), \qquad j = 0, \pm 1, \pm 2, \dots$$

let f be a function defined on $\mathbb{R}$ then for $h > 0$ the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in Stenger [33]. These properties are derived in the infinite strip D_d of the complex plane where $d > 0$

$$D_d = \{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\}.$$

Approximations can be constructed for infinite, semi-finite, and finite intervals. But in this paper we construct approximation on the interval $(0, 1)$,we consider the conformal map

$$\phi(z) = \ln(\frac{z}{z-1}), \tag{3}$$

which maps the eye-shaped region

$$D_E = \{z = x + iy; |\arg(\frac{z}{1-z})| < d \leq \frac{\pi}{2}\},$$

onto the infinite strip D_d.

For the Sinc method, the basis functions on the interval $(0, 1)$ for $z \in D_E$ are derived from the composite translated Sinc function:

$$S_j(z) = S(j, h) \circ \phi(z) = sinc(\frac{\phi(z) - jh}{h}). \tag{4}$$

The function

$$z = \phi^{-1}(\omega) = \frac{e^\omega}{1 + e^\omega},$$

is an inverse mapping of $\omega = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\} = (0, 1).$$

The sinc grid points $z_k \in (0, 1)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ... \tag{5}$$

Definition 1. *Let $B(D_E)$ is the class of functions f which are analytic in D_E such that*

$$\int_{\psi(u+\Sigma)} |f(z)|dz \to 0, \qquad as \;\; u \to \pm\infty \tag{6}$$

where $\Sigma = \{i\eta : |\eta| < d \leq \frac{\pi}{2}\}$ and satisfy

$$\aleph(f) \equiv \int_{\partial D_E} |f(z)|dz < \infty, \tag{7}$$

where ∂D_E represents the boundary of D_E.

Theorem 1. *[27] Let $F \in B(D_E)$ and ϕ be a conformal map with constants α and C_2 so that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \leq C_2 \exp(-\alpha|\phi(x)|), \qquad x \in \Gamma,$$

by selecting $h = \sqrt{\pi d/\alpha N}$, then the Sinc trapezoidal quadrature rule is

$$\int_0^1 F(x)dx = h \sum_{j=-N}^{N} \frac{F(x_j)}{\phi'(x_j)} + o(\exp(-(\pi d \alpha N)^{1/2})).$$

The Sinc-Galerkin method requires that the derivatives of composite Sinc function be evaluated at the nodes. We need to recall the following lemma.

Lemma 2. *[27] Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (4). Then*

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{8}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j \neq k, \end{cases} \tag{9}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j \neq k, \end{cases} \tag{10}$$

in relations (8-10) h is step size and x_k is sinc grid given by (5).

3. Description of the method

We discretize the problem in time direction , therefore the solution of equation (1) with initial and boundary conditions is converted to the solution of p nonlinear ordinary differential equations with corresponding boundary conditions. Finally, we apply the sinc-Galerkin method to solve such nonlinear ordinary differential equations.

3.1. Temporal discretization

The method of discretization in time consists of the following steps:

1. The interval $[0, T]$ is divided into p subintervals of lengths $\Delta t = T/p$ where T is total time and p is chosen as a positive integer.

2. The derivative u_t is replace by the difference quotient $\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t}$ at each of points of division $t_j = j\Delta t, (j = 1, 2, \ldots, p)$, for simplicity we denote $z_j(x) = u(x, t_j)$.

3. Starting with the function $z_0(x) = u(x, 0)$, successively for $j = 1, 2, \ldots, p$, the solutions of the ordinary differential equations with boundary conditions are obtained [31].

Let us consider the Burgers' equation (1) with $a = 0$, $b = 1$ and with the boundary conditions in case of $f_1 = f_2 = 0$ at equations of (2). The method of discretization in time leads to the problem of finding, successively for $j = 1, 2, \ldots, p$, the functions $z_j(x)$ which are the solutions of the problems

$$-\nu z_j^{''}(x) + z_j(x)z_j^{'}(x) + \frac{1}{\Delta t}(z_j(x) - z_{j-1}(x)) = 0, \tag{11}$$

$$z_j(0) = 0, \qquad z_j(1) = 0, \tag{12}$$

where $z_0(x) = g(x)$. Therefore in each time level we have a nonlinear ordinary differential equation in the form of (11) with boundary conditions (12).

Now in each time level we can apply the sinc-Galerkin method to approximate the solution of nonlinear boundary value problem (11) and (12).

3.2. Sinc-Galerkin in spatial direction

The approximate solution for $z_j(x)(j = 1, 2, \ldots, p)$ is represented by

$$z_m^j(x) = \sum_{r=-N}^{N} c_r^j S_r(x), \qquad m = 2N + 1, \tag{13}$$

where $S_r(x)$ is function $S(r, h) \circ \phi(x)$ for some fixed step size h. The unknown coefficient c_r^j in relation (13) are determined by orthogonalizing the residual with respect to the basis function, i.e,

$$\langle -\nu z_j^{''}(x), S_k\rangle + \langle \frac{1}{2}(z_j^2(x))^{'}, S_k\rangle + \frac{1}{\Delta t}\langle z_j(x), S_k\rangle = \frac{1}{\Delta t}\langle z_{j-1}(x), S_k\rangle, \tag{14}$$

where $\langle ., .\rangle$ represents the inner product defined by

$$\langle f, \eta\rangle = \int_0^1 f(x).\eta(x)\omega(x)dx. \tag{15}$$

Using integrating by parts for the first two integral terms in the left hand side of (14) we have

$$\langle z_j^{'}(x)z_j(x), S_k\rangle = B_{T_1} - \frac{1}{2}\int_0^1 z_j^2(x)(S_k(x)\omega(x))^{'}dx, \tag{16}$$

$$\langle -\nu z_j^{''}(x), S_k\rangle = B_{T_2} + \int_0^1 z_j(x)\left(-\nu S_k(x)\omega(x)\right)^{''} dx, \tag{17}$$

where

$$B_{T_1} = \left[\frac{1}{2} z_j^2 S_k \omega\right]_{x=0}^1,$$

$$B_{T2} = \{z_j^{'} S_k\omega - z_j (S_k\omega)^{'}\}(x)|_0^1.$$

Suppose that $B_{Ti} = 0, i = 1,2$, then we apply the Sinc quadrature rule in Theorem 1 to the last two integrals in the right hand side of (14) and the integrals in the right hand side of (16) and (17), we can obtain the following approximations:

$$\langle -\nu z_j^{''}(x), S_k\rangle \approx h \sum_{r=-N}^{N} \sum_{i=0}^{2} \frac{z_j(x_r)}{\phi'(x_r)h^i} \delta_{kr}^{(i)} g_{2,i}(x_r), \tag{18}$$

$$\langle z_j^{'}(x) z_j(x), S_k\rangle \approx -h \sum_{r=-N}^{N} \sum_{i=0}^{1} \frac{z_j^2(x_r)}{\phi'(x_r)h^i} \delta_{kr}^{(i)} g_{1,i}(x_r), \tag{19}$$

and

$$\langle G, S_k\rangle \approx h\frac{G(x_k)\omega(x_k)}{\phi'(x_k)}, \tag{20}$$

where

$$g_{2,2} = -\nu\omega(\phi^{'})^2, \qquad g_{2,1} = -\nu\omega\phi^{''} - 2\nu\omega^{'}\phi^{'} \qquad g_{2,0} = -\nu\omega^{''}$$

$$g_{1,1} = \frac{1}{2}\omega\phi^{'}, \qquad g_{1,0} = \frac{1}{2}\omega^{'}, \qquad G = \frac{1}{\Delta t} z_j(x), \qquad or \qquad \frac{1}{\Delta t} z_{j-1}(x).$$

for $j = 1, 2, \ldots, p$.
The weight function $\omega(x)$ in the Sinc-Galerkin inner product (15) may be chosen for a variety of reasons. Although other reasons exist, a choice we make here is due to the requirement that the boundary terms $B_{Ti}, i = 1,2$ vanish. For the case of second-order problem in the sinc-Galerkin method, a convenient choice for the weight function is given by Stenger [33] as

$$\omega(x) = \frac{1}{\phi'(x)}.$$

Now replacing each term of (14) with the approximations defined in (18-20) and replacing $z_j(x_r)$ by c_r^j and dividing by h, finally we obtain the discrete sinc-Galerkin system for determination of the unknown coefficients $\{c_r^j\}_{r=-N}^{N}$ for $j = 1, 2, \ldots, p$ as

$$\sum_{r=-N}^{N} \left\{ \sum_{i=0}^{2} \frac{1}{h^i} \delta_{kr}^{(i)} \frac{g_{2,i}(x_r)}{\phi'(x_r)} c_r^j - \sum_{i=0}^{1} \frac{1}{h^i} \delta_{kr}^{(i)} \frac{g_{1,i}(x_r)}{\phi'(x_r)} (c_r^j)^2 \right\}$$
$$+ \frac{1}{\Delta t} \frac{\omega(x_k)}{\phi'(x_k)} c_k^j = \frac{1}{\Delta t} \frac{\omega(x_k)}{\phi'(x_k)} z_{j-1}(x_k) \qquad -N \leq k \leq N. \tag{21}$$

To obtain a matrix representation of the equations (21), let $I^{(i)}, 0 \leq i \leq 2$ the $m \times m$ matrixes whose jk- th entry is given by (8-10). Note that the matrix $I^{(2)}$ and $I^{(1)}$ are symmetric and skew-symmetric matrixes respectively, also $I^{(0)}$ is identity matrix. We define the $m \times m$ diagonal matrix as follow:

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_i), & i = j, \\ 0, & i \neq j, \end{cases}$$

Therefore, by using the above definitions the system (21) can be denoted by the the following matrix form:

$$\mathbf{A}^j \mathbf{C}^j + \mathbf{B}(\mathbf{C}^j)^2 = \mathbf{E}^{j-1} \qquad j = 1, 2, \ldots, p, \tag{22}$$

where
$\mathbf{C}^j$ is m-vector and $\mathbf{A}^j, \mathbf{B}^j$ are $m \times m$ matrixes and $\mathbf{E}^{j-1}$ is an m-vector as:

$$\mathbf{C}^j = (c_{-M}^j, c_{-M+1}^j, \ldots, c_N^j)^t$$

$$\mathbf{A}^j = \frac{1}{h^2} I^{(2)} \mathbf{D}(\frac{g_{2,2}}{\phi'}) + \frac{1}{h} I^{(1)} \mathbf{D}(\frac{g_{2,1}}{\phi'}) + \mathbf{D}(\frac{g_{2,0}}{\phi'} + \frac{1}{\Delta t} \frac{\omega}{\phi'})$$
$$\mathbf{B}^j = \mathbf{D}(\frac{-g_{1,0}}{\phi'}) + \frac{1}{h} I^{(1)} \mathbf{D}(\frac{-g_{1,1}}{\phi'})$$
$$\mathbf{E}^{j-1} = \mathbf{D}(\frac{\omega z_{j-1}}{\phi'}).\mathbf{1},$$

where $\mathbf{1}$ is the m-vector each of whose components are 1.

For each j, system (22) is a nonlinear system of equations which consists of m equations and m unknowns. We can obtain the coefficients in the approximate solution by solving this nonlinear system by Newton's method.

4. Convergence analysis

In this section, the convergence of sinc-galerkin method for the problem (11) with boundary conditions (12)will discuss . For this purpose, we apply the sinc- Galerkin method for the linear boundary value problem

$$(Ly)(x) \equiv -y^{''}(x)+p(x)y^{'}(x)+q(x)y(x) = \sigma(x), \quad a < x < b, y(a) = y(b) = 0. \quad (23)$$

Consider the sinc approximation by

$$y(x) \approx y_m(x) = \sum_{r=-N}^{N} c_r S_r(x), \qquad m = 2N+1 \quad (24)$$

according to concepts of section 3.2 the discrete sinc-Galerkin system for the determination of the unknown coefficient $\{c_r\}_{r=-M}^{N}$ is given by:

$$\{-\frac{1}{h^2}I^2 D(\varphi^{'}\omega) - \frac{1}{h}I^{(1)}D(\frac{\varphi^{''}\omega}{\varphi^{'}} + 2\omega^{'} + p\omega) - D(\frac{\omega^{''} + (p\omega)^{'} - q\omega}{\varphi^{'}})\}C = D(\frac{\omega\sigma}{\varphi^{'}})\mathbf{1}, \quad (25)$$

where $\mathbf{1}$ is a m-vector each of whose components are 1 and $\omega(x) = \frac{1}{\varphi^{'}}$.

The error analysis in approximating the exact solution of (23) by (24) can be proved by the following theorem in [33].

Theorem 3. *Assume that the functions of p, q and σ are analytic in D_E and that problem (23) has a unique solution $y(x)$ which is analytic in D_E, moreover we assume that $\frac{\sigma}{\varphi^{'}} \in B(D_E)$ and $yF \in B(D_E)$ where*

$$F = (\frac{1}{\varphi^{'}})^{''}, (\frac{\varphi^{''}}{\varphi^{'}}), \varphi^{'}, (\frac{p}{\varphi^{'}})^{'}, p, \frac{q}{\varphi^{'}},$$

and also there are positive constants C, α and β such that:

$$|y(x)| \leq K \begin{cases} \exp(-\alpha|\phi(x)|), & x \in \psi((-\infty, 0)), \\ \exp(-\beta|\phi(x)|), & x \in \psi((0, \infty)), \end{cases}$$

if we make the selections $h = \sqrt{\frac{\pi d}{\alpha N}}$, and the coefficient $\{c_r\}_{r=-N}^{N}$ in (24) are determined from (25),Then:

$$\|y(x) - y_m(x)\|_\infty \leq KN^2 \exp(-(\pi d\alpha N)^{1/2}),$$

where K is a constant.

By using the above theorem we can discuss convergence of sinc-Galerkin method for the equation (11). At first, we rewrite equation (11) as:

$$-\nu y^{''}(x) + y^{'}(x)y(x) + \frac{1}{\Delta t}y(x) = f(x), \tag{26}$$

where, $y(x) \equiv z_j(x), f(x) = \frac{1}{\Delta t}z_{j-1}(x)$, with boundary conditions $y(0) = y(1) = 0$.
The error bound in the solution can be obtained by using following lemma.

Lemma 4. *Let y be the exact solution of nonlinear equation*

$$Ly + G(y) = 0$$

where $(Ly)(x) = -\nu y^{''}(x) + \frac{1}{\Delta t}y(x) - f(x)$ is a linear operator and $G(y) = y^{'}(x)y(x)$ is a nonlinear operator. Let $y \in B(D_E), G^{'}(y)$ and $G^{''}(y)$ are well defined and bounded on the ball $B(y^0, r)$. Also , let $(L+G^{'}(y))^{-1}$ and $(L+G^{'}(y^0))^{-1}(Ly^0+G(y^0))$ are bounded on $B(y^0, r)$, and

$$\|Ly^0 + G(y^0)\|_\infty \le H_0, \quad \|(L + G^{'}(y))^{-1}\|_\infty \le H_1, \quad \|G^{''}(y)\|_\infty \le H_2, y \in B(y^0, r). \tag{27}$$

if $\widetilde{h} = H_1^2 H_2 H_0 < 2$ and $r > H_1 H_0 \sum_{k=0}^{\infty} (\frac{\widetilde{h}}{2})^{2^k - 1}$, then the sequence

$$y^{n+1} = y^n - \left(L + G^{'}(y^n)\right)^{-1} (Ly^n + G(y^n)) \tag{28}$$

is well defined, also $y^{n+1} \in B(D_E)$for every positive integer n and the sequence y^n converges to y^*, furthermore,

$$\|y^n - y^*\|_\infty \le H_1 H_0 \frac{(\widetilde{h}/2)^{2^n - 1}}{1 - (\widetilde{h}/2)^{2^n}}. \tag{29}$$

Proof. By applying Kantorovich's Theorem [25], we can prove the existence of the sequence $\{y^n\}_{n \ge 0}$ and the bound (29).

Theorem 5. *Let us consider all assumptions in Lemma 4 and let, the discrete equivalent of $G^{'}(y)$,$G^{''}(y)$ and $(L + G^{'}(y))^{-1}$ are well defined and bounded on the ball $\overline{B}(y^0, r)$, let the sequence v_m^n be the discrete equivalent of (28), then: (a) $\{v_m^n\}_{n \ge 0}$ converges to v_m^* and $v_m^n - v_m^*$ has a bound as defined in (29).*
(b) There exists a constant C_1 independent of m such that:

$$\|v_m^* - y^*\|_\infty \le C_1 N^2 \exp(-(\pi d \alpha N)^{1/2}). \tag{30}$$

Proof. (a) Let $\{v_m^n\}_{n\geq 0}$ be the discrete sequence by the sinc-Galerkin method that defined by the discrete equivalent of (28), similarly by using Lemma 4, the sequence $\{v_m^n\}_{n\geq 0}$ exist and converges to v_m^* and moreover we have

$$\|v_m^n - v_m^*\|_\infty \leq H_1 H_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}}, \tag{31}$$

where H_0, H_1 and $\widetilde{h}$ are defined in Lemma 4.

(b) Let the sequence y^n defined by (28), by using Lemma 4 we know that the sequence y^n exists and converges to y^* and also

$$\|y^n - y^*\|_\infty \leq H_1 H_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}}, \tag{32}$$

by considering bound $(L + G^{'}(y))^{-1}$ on the ball $\overline{B}(y^0, r)$ and theorem 2, we have:

$$\|v_m^n - y^n\|_\infty \leq C_2 N^2 \exp(-(\pi d\alpha N)^{1/2}), \tag{33}$$

Now, we consider the following inequality :

$$\|v_m^* - y^*\|_\infty \leq \|v_m^* - v_m^n\|_\infty + \|v_m^n - y^n\|_\infty + \|y^n - y^*\|_\infty, \tag{34}$$

by using theorem 5 in [37] , the following inequality can be made for n large enough,

$$H_1 H_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}} \leq C_3 N^2 \exp(-(\pi d\alpha N)^{1/2}), \tag{35}$$

By applied the relations $(31) - (35)$, we obtain :

$$\|v_m^* - y^*\|_\infty \leq C_1 N^2 \exp(-(\pi d\alpha N)^{1/2}).\square$$

Now, if we suppose that $u(x,t)$ be the exact solution and $U(x,t)$ be the numerical approximation (1) by our numerical process, then we have:

$$\|u(x,t) - U(x,t)\|_\infty \leq \rho(\exp(-(\pi d\alpha N)^{1/2}) + \Delta t),$$

where ρ is a constant.

5. Numerical results

In this section, the application of the presented method for Burgers' equation are tested on two standard problems to validate the current numerical scheme. Since the exact solution is known for these test cases, we can demonstrate the effectiveness of method and measure its accuracy. The results are also compared with other methods found in the literature. In all of the problems considered in this paper, we choose $\alpha = \beta = 1$ and $d = \frac{\pi}{2}$ which yield $h = \frac{\pi}{\sqrt{2N}}$. To measure the accuracy of our method, we compute the error under the following norms:

$$\|e\|_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{|u(y_i,t_k) - u_i^k|}{|u(y_i,t_k)|},$$

$$\|e\|_2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n} |u(y_i,t_k) - u_i^k|^2}$$

$$\|e\|_\infty = \max_{1\le i\le n} |u(y_i,t_k) - u_i^k|,$$

where y_is are uniform grids on interval $(0,1)$.

Problem 1. This problem corresponds to (1) on $0 \le x \le 1$ with boundary conditions $u(0,t) = u(1,t) = 0$, and initial condition

$$u(x,0) = g(x) = \frac{2\nu\pi \sin(\pi x)}{a + \cos(\pi x)}, \qquad a > 1.$$

Exact solution for this problem has the following nice compact closed-form, as given by Wood[34]:

$$u(x,t) = \frac{2\nu\pi e^{-\pi^2\nu t} \sin(\pi x)}{a + e^{-\pi^2\nu t}\cos(\pi x)}, \qquad a > 1.$$

This problem has been solved by using Taylor series expansion in [1]. We applying our method for the various values of $\nu = 1, 0.1, 0.01$ and $a = 2$. We compared our computed solution with [1] and exact solution, the maximum absolute errors obtained in Tables 1-3, the obtained results verified effectiveness and accuracy of our method and we observe that the error will decrease if we use the smaller values ν. Convergence curve for this problem is plotted for $\nu = 0.01, \Delta t = 0.0001$ at $t = 0.001$ in Fig. 1 (left graph). This figure indicates that the maximum pointwise errors decrease at an exponential rate with respect to N and verifies the theoretical results obtained in Section 4. Also Fig.2 depicts the exact solution with approximate results, this figure shows that treatment of the approximate solution and the exact solution is remarkably identical.

Problem 2. The initial condition for this problem is:

$$u(x,0) = g(x) = \sin \pi x, \qquad 0 < x < 1,$$

with the homogeneous boundary conditions:

$$u(0,t) = u(1,t) = 0,$$

and exact solution given by :

$$u(x,t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} a_n exp(-n^2\pi^2\nu t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n exp(-n^2\pi^2\nu t)\cos(n\pi x)},$$

with the Fourier coefficient:

$$a_0 = \int_0^1 exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}dx,$$

$$a_n = 2\int_0^1 exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)dx, \qquad n = 1,2,3,\ldots.$$

The numerical results for this problem are shown in tables 4-6 for $\nu = 1,\ 0.1,\ 0.01$. We compared our results with [1, 14, 19, 20, 23, 24, 36], the tabulated results show that our method is accurate in comparison with the other method. The convergence curve is plotted for $\nu = 0.1, \Delta t = 0.0001$ at $t = 0.1$ in Fig. 1 (right graph). This figure shows that the treatment of maximum errors is exponential with increasing N. Of course, due to the complexity of the calculations and the round of errors for $N \geq 32$ this behavior is almost near to exponential. Fig. 3 states the numerical results at different time levels $t = 0.4, 0.6, 0.8$, and $t = 1$. In order to demonstrate the stability of the this method for $\nu = 0.1$, practically exhibit the correct physical behavior of the problem.

6. Conclusions

A numerical method is developed to solve Burgers' equation. This method is based on temporal discretization and the sinc-Galerkin method in the spatial direction. The exponential convergence analysis of the method has been proved theoretically and verified numerically. The compared results show that our method is efficient with respected to the methods given in [1, 14, 19, 20, 23, 24, 36].

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Table 1: *Problem1- The errors for various values of N and $\nu = 1, a = 2, \Delta t = 0.0001$ at $t = 0.001$.*

x	Our method			
	[1]	N=8	N=16	N=64
0.1	4.4×10^{-5}	8.9×10^{-4}	8.9×10^{-5}	7.8×10^{-7}
0.2	7.7×10^{-5}	9.2×10^{-4}	2.1×10^{-5}	1.0×10^{-6}
0.3	1.2×10^{-4}	7.4×10^{-6}	5.5×10^{-6}	3.0×10^{-6}
0.4	1.7×10^{-4}	1.0×10^{-3}	2.1×10^{-5}	7.0×10^{-6}
0.5	2.5×10^{-4}	4.0×10^{-4}	1.3×10^{-5}	1.5×10^{-5}
0.6	3.5×10^{-4}	2.3×10^{-4}	7.8×10^{-5}	2.7×10^{-5}
0.7	4.8×10^{-4}	5.2×10^{-4}	9.1×10^{-6}	4.5×10^{-6}
0.8	5.6×10^{-4}	2.0×10^{-3}	9.9×10^{-5}	5.8×10^{-5}
0.9	1.0×10^{-3}	3.4×10^{-3}	1.8×10^{-5}	4.0×10^{-5}
$\|e\|_1$	1.0×10^{-4}	6.3×10^{-4}	2.3×10^{-5}	8.6×10^{-6}
$\|e\|_2$	4.4×10^{-4}	1.1×10^{-3}	5.0×10^{-5}	3.0×10^{-5}

Table 2: *Problem1- Computed solutions at grid points and different measures of error for $N = 64, a = 2$ and $\Delta t = .0001$ at $t = 1$.*

x	$\nu = 0.1$		$\nu = 0.01$	
	Computed	Exact	Computed	Exact
0.1	0.030736	0.030735	0.006147222	0.006147222
0.2	0.059809	0.059807	0.012243297	0.012243295
0.3	0.085379	0.085376	0.018185174	0.018185170
0.4	0.105300	0.105296	0.023746145	0.023746137
0.5	0.117096	0.117090	0.028463412	0.028463397
0.6	0.118171	0.118163	0.031476659	0.031476634
0.7	0.106387	0.106380	0.031384108	0.031384072
0.8	0.810476	0.810417	0.026409124	0.026409082
0.9	0.439803	0.439768	0.015453733	0.015453703
$\|e\|_1$	5.7×10^{-5}		7.6×10^{-7}	
$\|e\|_2$	5.2×10^{-6}		2.3×10^{-8}	
$\|e\|_\infty$	7.2×10^{-6}		4.2×10^{-8}	

Table 3: *Problem1- Different measures of error for* $N = 64, a = 2$ *with* $h = 0.1$ *and* $\Delta t = 0.0001$ *at* $t = 0.001$.

	$\nu = 0.1$		$\nu = 0.01$
	Our method	[1]	Our method
$\|e\|_1$	8.72×10^{-8}	4.89×10^{-6}	9×10^{-10}
$\|e\|_2$	3.16×10^{-8}	1.89×10^{-6}	3.1×10^{-11}
$\|e\|_\infty$	5.95×10^{-8}	4.35×10^{-6}	5.9×10^{-11}

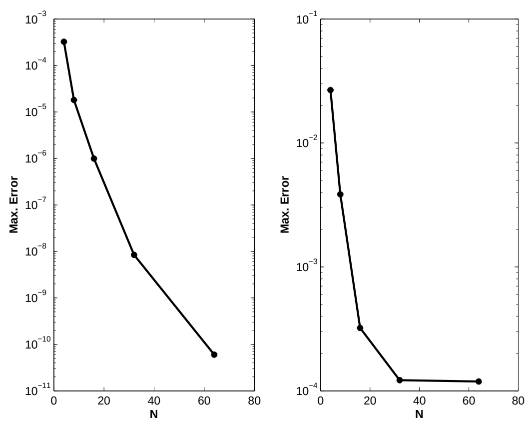

Figure 1: Convergence of the method, (left graph) Problem 1 with $\nu = 0.01$, $\Delta t = 0.0001$ at $t = 0.001$, (right graph) Problem 2 with $\nu = 0.1$, $\Delta t = 0.0001$ at $t = 0.1$.

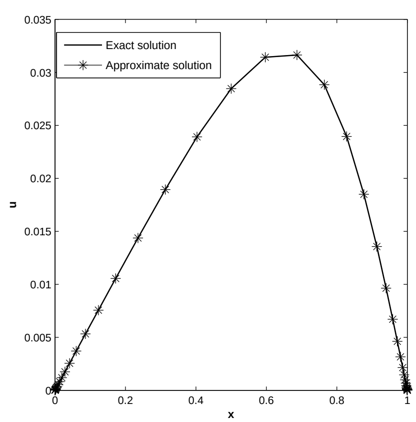

Figure 2: Exact and approximate solution problem1 for $\nu = 0.01, \Delta t = 0.0001$ at $t = 1$.

Table 4: *Problem2- Computed solution and different measures of error for* $\nu = 1, N = 64$, *and* $\Delta t = 0.00001$ *at* $t = 0.1$.

x	[14]	[23]	[36]	[24]	Our method	Exact
0.1	0.10937	0.10977	0.10920	0.10965	0.10945	0.10954
0.2	0.20945	0.21023	0.10912	0.20998	0.20980	0.20979
0.3	0.29138	0.29250	0.29088	0.29213	0.29191	0.29190
0.4	0.34726	0.34863	0.34658	0.34818	0.34794	0.34792
0.5	0.37080	0.37232	0.36997	0.37185	0.37159	0.37158
0.6	0.35823	0.35974	0.35740	0.35932	0.35906	0.35905
0.7	0.30914	0.31050	0.30847	0.31017	0.30992	0.30991
0.8	0.22722	0.22825	0.22676	0.22805	0.22783	0.22782
0.9	0.12036	0.12091	0.12012	0.12083	0.12069	0.12069
$\|e\|_1$	2.1×10^{-3}	2.0×10^{-3}	4.1×10^{-3}	8.9×10^{-4}	4.9×10^{-5}	
$\|e\|_2$	6.0×10^{-4}	5.5×10^{-3}	1.2×10^{-3}	2.2×10^{-4}	1.36×10^{-5}	

Table 5: *Comparison of absolute errors for problem 2 at different times for* $\nu = 0.1, \Delta t = 0.0001$.

x	t	[19]	[20]	[24]	Our method	
					$N = 16$	$N = 64$
0.25	0.4	8.1×10^{-3}	1.1×10^{-4}	5.2×10^{-3}	3.1×10^{-5}	3.1×10^{-5}
	0.6	7.7×10^{-3}	8.9×10^{-5}	2.8×10^{-3}	9.3×10^{-5}	5.9×10^{-5}
	0.8	7.2×10^{-3}	9.7×10^{-5}	2.4×10^{-3}	5.0×10^{-5}	1.5×10^{-5}
	1	3.4×10^{-2}	5.1×10^{-5}	2.1×10^{-3}	4.5×10^{-5}	1.4×10^{-5}
0.5	0.4	1.7×10^{-2}	1.1×10^{-4}	3.2×10^{-3}	1.8×10^{-5}	3.3×10^{-5}
	0.6	1.4×10^{-2}	8.8×10^{-5}	3.6×10^{-3}	3.3×10^{-5}	8.2×10^{-5}
	0.8	1.4×10^{-2}	7.5×10^{-5}	3.6×10^{-3}	3.8×10^{-5}	2.8×10^{-5}
	1	1.4×10^{-2}	9.2×10^{-5}	3.4×10^{-3}	4.0×10^{-5}	2.5×10^{-5}
0.75	0.4	1.1×10^{-2}	9.6×10^{-5}	4.9×10^{-3}	1.1×10^{-4}	9×10^{-6}
	0.6	1.7×10^{-2}	1.2×10^{-5}	5.4×10^{-3}	1.0×10^{-4}	8.4×10^{-5}
	0.8	1.9×10^{-2}	9.6×10^{-5}	5.2×10^{-3}	9.6×10^{-5}	2.5×10^{-5}
	1	1.8×10^{-2}	7.7×10^{-5}	4.5×10^{-3}	8.5×10^{-5}	2.9×10^{-5}

Table 6: *Comparison of absolute errors for problem 2 at different times for* $\nu = 0.01, \Delta t = 0.0001$ *and* $N = 64$.

x	t	[24]	[23]	[1]	Our method
0.25	0.4	6.2×10^{-3}	5.3×10^{-4}	2×10^{-5}	3×10^{-5}
	0.6	6.4×10^{-3}	9×10^{-5}	2×10^{-5}	6×10^{-5}
	0.8	6×10^{-3}	3×10^{-5}	2×10^{-5}	2×10^{-5}
	1	5×10^{-3}	6×10^{-5}	1×10^{-5}	2×10^{-5}
0.5	0.4	4.2×10^{-3}	8.1×10^{-4}	5×10^{-5}	3×10^{-5}
	0.6	5.8×10^{-3}	4.6×10^{-3}	5×10^{-5}	9×10^{-5}
	0.8	6.1×10^{-3}	2.3×10^{-3}	3×10^{-5}	3×10^{-5}
	1	6×10^{-3}	1×10^{-3}	2×10^{-5}	3×10^{-5}
0.75	0.4	1.7×10^{-3}	3.6×10^{-2}	2×10^{-4}	4×10^{-5}
	0.6	4×10^{-3}	1.7×10^{-2}	9×10^{-5}	9×10^{-5}
	0.8	5.1×10^{-3}	9.1×10^{-3}	4×10^{-5}	3×10^{-5}
	1	5.5×10^{-3}	5.3×10^{-3}	3×10^{-5}	4×10^{-5}

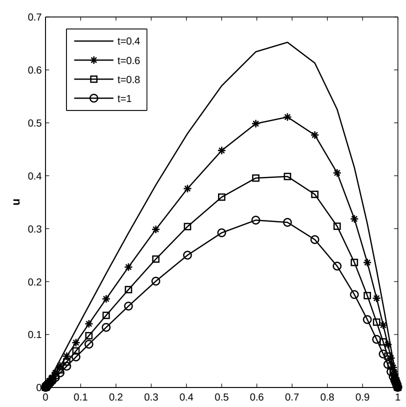

Figure 3: Solution of problem 2 at different times for $\nu = 0.1, \Delta t = 0.0001$.

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