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GEODESIC η-INVEX AND SEMISTRICTLY GEODESIC η-PREQUASI INVEX FUNCTIONS ON RIEMANNIAN MANIFOLD

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Abstract. In this paper we consider an optimization problem on a Riemannian manifold where the objective function is geodesic η-invex, the feasible set is geodesic invex but the inequality constraints are not necessarily geodesic η-invex. We show that if the program is superconsistent and a non-degeneracy condition is satisfied then the Karush-Kuhn-Tucker (KKT) type optimality conditions are both necessary and sufficient. We also introduce the notion of semistrictly geodesic η-prequasi invex functions on Riemannian manifolds and study their properties. We construct an example of a semistrictly geodesic η-prequasi invex function on a Riemannian manifold.

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1. Introduction

Convexity plays an important role in science, engineering and optimization theory. Convexity in linear topological spaces is based upon the possibility of connecting any two points of the space by a line segment. Since convexity is often not enjoyed by real problems, various approaches to the generalizations of the usual line segment have been proposed.

Rapcsak [7] introduced a generalization of convexity called geodesic convexity and extended many results of convex analysis and optimization theory from linear spaces to Riemannian manifolds. In the year 1994, C. Udriste [8] introduced an exhaustive exposition of convex programming on Riemannian manifolds. The notion of invex function on Riemannian manifold was introduced by Pini [6]. Mititelu [5] introduced (η, θ)-invex function which is a generalization of invex function on Riemannian manifolds. Barani and Pouryayevali [3] defined geodesic invex set, geodesic η-invex function and geodesic η-preinvex function on Riemannian manifold and studied their properties. Ahmad et al. [2] extended these results by introducing geodesic η-prepseudo invex functions and geodesic η-prequasi invex functions.

In Section 2 of the paper we recall some of concepts and facts from Riemannian geometry.

In the year 2010, Lasserre [4] proved that if Slater's condition and a non-degeneracy assumption hold for a convex optimization problem on Euclidean spaces then KKT optimality conditions are necessary and sufficient even if the inequality constraints are not convex.
Motivated by the work of Lasserre [4], in Section 3 we extend KKT optimality conditions with reference to a Riemannian manifold where we take a geodesic η-invex objective function on a geodesic invex set subject to some inequality constraints which are not necessarily geodesic η-invex. We prove that if the optimization problem is superconsistent and a mild non-degeneracy condition holds then KKT optimality conditions are also necessary and sufficient.

Yang and Li [10] introduced semistrictly preinvex functions on Euclidean spaces. Agarwal et al. [1] extended the results of Yang and Li by introducing semistrictly geodesic η-preinvex functions over a Riemannian manifold.

In Section 4, we introduce the notion of semistrictly geodesic η-prequasi invex functions on Riemannian manifolds which extend semistrictly quasi invex functions introduced by Yang et al. [11]. We construct an example of a semistrictly geodesic η-prequasi invex function and study its properties.

Section 5 includes some concluding remarks.

2. Preliminaries and Definitions

In this section, we recall some definitions and basic properties about Riemannian manifolds which we use throughout the paper. One can refer [9] an easy access of the standard materials on differential geometry.
Throughout this paper M is a C^∞ smooth manifold endowed with a Riemannian metric $< .,. >_p$ on the tangent space T_pM and corresponding norm is denoted by $\|.\|_p$, which yields that M is a Riemannian manifold. The length of a piecewise C^1 curve $\gamma : [a, b] \to M$ joining p to q such that $\gamma(a) = p$ and $\gamma(b) = q$, is defined by $L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt$. We define the distance d between any two points $p, \ q \in M$ by

$$d(p, q) = \inf\{L(\gamma) : \gamma \textit{ is a piecewise } C^1 \textit{ curve joining } p \textit{ to } q\}.$$

Then d induces the original topology on M. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. We recall that a geodesic is a C^∞ smooth

path γ whose tangent is parallel along the path γ, that is γ satisfies the equation $\nabla_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p, q)$ is a geodesic, and it is called a minimal geodesic. The existence theorem for ordinary differential equations implies that for every $v \in TM$ there exist an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \to M$ with $\frac{d\gamma(0)}{dt} = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold M of TM such that for every $v \in \tilde{T}M$ the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $\exp : \tilde{T}M \to M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of exp to a fiber T_pM in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$.
Let f be a differentiable map from the manifold M to the manifold N. The linear map $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ defined by $\mathrm{d}f_p(v) = f'(p)v$ is called the differential of f at the point p.
Recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Cartan-Hadamard manifold.

Definition 1. *([3]) Let M be an n-dimensional Riemannian manifold and $\eta : M \times M \to TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_yM$. A nonempty subset S of M is said to be geodesic invex set with respect to η if for every $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0, 1] \to M$ such that*

$$\gamma_{x,y}(0) = y, \quad \gamma'_{x,y}(0) = \eta(x, y), \quad \gamma_{x,y}(t) \in S, \ \forall t \in [0, 1].$$

Definition 2. *([3]) Let M be an n-dimensional Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be geodesic η-preinvex if $\forall x, y \in S$, we have*

$$f(\gamma_{x,y}(t)) \leq tf(x) + (1 - t)f(y) \quad \forall t \in [0, 1].$$

If f be differentiable on S. We say that f is geodesic η-invex on S if the following holds

$$f(x) - f(y) \geq df_y(\eta(x, y)), \quad \forall x, y \in S.$$

Example([3]): Let M be a Cartan-Hadamard manifold and $x_0, y_0 \in M$, $x_0 \neq y_0$. Let $B(x_0, r_1) \cap B(y_0, r_2) = \emptyset$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x, r) = \{y \in M : d(x, y) < r\}$ is an open ball with center x and radius r. Let $S = B(x_0, r_1) \cup B(y_0, r_2)$. Then S is not a geodesic convex set as every geodesic curve passing x_0, y_0 is not completely lie in S. Now if we define the function $\eta : M \times M \to TM$ by

$$\eta(x, y) = \begin{cases} \exp_y^{-1} x, & x, y \in B(x_0, r_1) \text{ or } x, y \in B(y_0, r_2); \\ 0_y, & \text{otherwise.} \end{cases}$$

Then S is a geodesic invex set with respect to η.

Definition 3. *([1]) Let M be an n-dimensional Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be semistrictly geodesic η-preinvex if $\forall x, y \in S$ with $f(x) \neq f(y)$, we have*

$$f(\gamma_{x,y}(t)) < tf(x) + (1-t)f(y) \quad \forall t \in (0,1).$$

Definition 4. *([1]) Let $S \subseteq M \times \mathbb{R}$, S is said to be geodesic G-invex set if there exists $\eta : M \times M \to TM$ such that for any pair (x, α) and (y, β), we have*

$$(\gamma_{x,y}(t), t\alpha + (1-t)\beta) \in S \quad \forall t \in [0,1].$$

Definition 5. *([2]) Let M be an n-dimensional Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be geodesic η-prequasi invex on S if*

$$f(\gamma_{x,y}(t)) \leq \max\{f(x), f(y)\} \ \forall x, y \in S, \ \forall t \in [0,1].$$

3. Necessity and Sufficiency of KKT Optimality Conditions

Throughout the remaining part of the paper (M, g) denotes a complete finite-dimensional Riemannian manifold and $A \subset M$ is a geodesic invex set. T_xA is the tangent space at a point $x \in A$ and 0_x is the zero vector in T_xA.
We consider the optimization problem on the Riemannian manifold (M, g)

$$(P) \quad \min_{x \in A} f(x)$$

where $f : M \to R$ is a geodesic η-invex function which is differentiable.

$$A = \{x \in M : \psi_i(x) \leq 0; i = 1, 2, ..., m\},$$

where $\psi_i : M \to R$ is a differentiable function not necessarily geodesic η-invex for all $i = 1, 2, ..., m$.

Definition 6. *(Superconsistency) The program (P) is called superconsistent if there exists $y \in A$ such that $\psi_i(y) < 0$, for all $i = 1, 2, ..., m$.*

We assume the following non-degeneracy condition.
Assumption 1(Nondegeneracy). For every $i = 1, 2, ..., m$,

$$grad\ \psi_i(x) \neq 0_x, \quad whenever \ \ x \in A \ \ and \ \ \psi_i(x) = 0. \tag{1}$$

Lemma 1. *If the superconsistency holds for A and the constraint functions ψ_i are geodesic η-invex, then the above non-degeneracy condition holds automatically.*

Proof. By the superconsistency of A there exists a point $x_0 \in A$, such that $\psi_i(x_0) < 0$, for every $i = 1, 2, ..., m$.
Suppose that there exists $x \in A$ such that $grad\ \psi_i(x) = 0_x,\ whenever\ \psi_i(x) = 0$.
Let $\eta : M \times M \to TM$ be a function such that for every $x_0, x \in M$, $\eta(x_0, x) \in T_xM$ and $\alpha_{x_0,x}$ be a geodesic joining the point x and x_0 in A such that $\alpha_{x_0,x}(0) = x, \alpha'_{x_0,x}(0) = \eta(x_0, x)$. Since ψ_i are geodesic η-invex then we have

$$\psi_i(x_0) - \psi_i(x) \geq d\psi_{ix}(\eta(x_0, x)).$$

Now $d\psi_{ix}(\eta(x_0, x)) = g(grad\ \psi_i(x), \eta(x_0, x)) = g(0_x, \eta(x_0, x)) = 0$. i.e., $\psi_i(x_0) \geq 0$ (as $\psi_i(x) = 0$), which is a contradiction to the superconsistency of A.
Hence our assumption is false. Therefore, for every $i = 1, 2, ..., m$, $grad\ \psi_i(x) \neq 0_x,\ whenever\ x \in A\ and\ \psi_i(x) = 0$.

Lemma 2. *Let the superconsistency and the nondegeneracy assumption be hold for A. If A is geodesic invex then for every, $i = 1, 2, ..., m$,*

$$d\psi_{iy}(\eta(x, y)) \leq 0, \quad \forall x, y \in A \quad with \quad \psi_i(y) = 0. \tag{2}$$

Proof. Since A is geodesic invex with respect to η, $\forall x, y \in A$ there exists exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that $\alpha_{x,y}(0) = y,\ \alpha'_{x,y}(0) = \eta(x, y), \alpha_{x,y}(t) \in A,\ \forall t \in [0, 1]$.
If possible suppose $d\psi_{iy}(\eta(x, y)) > 0$ for some $i \in 1, 2, ..., m$ and some $x, y \in A$ with $\psi_i(y) = 0$. Then for sufficiently small $t \in [0, 1]$, we have

$$\psi_i(\alpha_{x,y}(t)) > 0, \tag{3}$$

which is a contradiction as $\alpha_{x,y}(t) \in A$ for all $0 \leq t \leq 1$ (as A is geodesic invex). Hence $d\psi_{iy}(\eta(x, y)) \leq 0,\ \forall x, y \in A\ with\ \psi_i(y) = 0$.

Let us consider the problem (P). A point $\bar{x} \in A$ is said to be a Karush-Kuhn-Tucker (KKT) point of the problem (P) if there exists scalars $\lambda_i \geq 0$, $i = 1, 2, ..., m$, such that

$$grad\ f(\bar{x}) + \sum_{i=1}^{m} \lambda_i grad\ \psi_i(\bar{x}) = 0_{\bar{x}},$$

$$\lambda_i \psi_i(\bar{x}) = 0, \forall i = 1, 2, ..., m.$$

Theorem 3. *Let the condition of superconsistency and the assumption of nondegeneracy (Assumption 1) hold for the problem (P). If f is geodesic η-invex, then every minimizer is a KKT point and conversely, every KKT point is a minimizer.*

Proof. Let $\bar{x} \in A$ be a minimizer. We first prove that $\bar{x}$ is a KKT point. Since $\bar{x}$ is a minimizer, by the Fritz-John optimality conditions we have,

$$\lambda_0 grad\ f(\bar{x}) + \sum_{i=1}^{m} \lambda_i grad\ \psi_i(\bar{x}) = 0_{\bar{x}}, \tag{4}$$

$$\lambda_i \psi_i(\bar{x}) = 0, \forall i = 1, 2, ..., m, \tag{5}$$

for some non trivial vector $\mathbf{0} \neq \lambda \in \mathbb{R}^{m+1}$.
We next prove that $\lambda_0 \neq 0$. Suppose that $\lambda_0 = 0$.
Let I $= \{i \in \{1, 2, ..., m\} : \lambda_i > 0\}$.
As $\lambda \neq \mathbf{0}$ and $\lambda_0 = 0$, the set $I \neq \emptyset$. Next by the superconsistency of A, there exists $x_0 \in A$, such that $\psi_i(x_0) < 0$, for every $i = 1, 2, ..., m$. Hence there is some $\rho > 0$, such that
$B(x_0, \rho) = \{z \in M : d(x_0, z) < \rho\} \subset A$ and $\psi_i(z) < 0, \quad \forall z \in B(x_0, \rho)$ and for all $i \in I$.
As $\lambda_0 = 0$, from (4) we have,

$$\sum_{i=1}^{m} \lambda_i grad\ \psi_i(\bar{x}) = 0_{\bar{x}}. \tag{6}$$

Since A is geodesic invex with respect to η, then for $\bar{x}, z \in A$ there exists a geodesic $\alpha_{z,\bar{x}} : [0, 1] \to M$ such that $\alpha_{z,\bar{x}}(0) = y,\ \alpha'_{z,\bar{x}}(0) = \eta(z, \bar{x}), \alpha_{z,\bar{x}}(t) \in A,\ \forall t \in [0, 1]$.
From (6) it follows that $\sum_{i \in I} \lambda_i d\psi_{i\bar{x}}(\eta(z, \bar{x})) = 0, \quad \forall z \in B(x_0, \rho)$. Hence by Lemma 2 (as $\psi_i(\bar{x}) = 0$ for $\lambda_i > 0$), we have $d\psi_{i\bar{x}}(\eta(z, \bar{x})) = 0, \quad \forall z \in B(x_0, \rho)$ *and* $i \in I$.
i.e., $d\psi_{i\bar{x}}(\eta(z, \bar{x})) = g(grad\ \psi_i(\bar{x}), \eta(z, \bar{x})) = 0, \quad \forall z \in B(x_0, \rho)$
$\Rightarrow grad\ \psi_i(\bar{x}) = 0_{\bar{x}},\ \forall i \in I$, which is a contradiction to the Assumption 1.
Hence $\lambda_0 > 0$ and we may set $\lambda_0 = 1$. So $\bar{x}$ is a KKT point.
Conversely, let $x \in A$ be an arbitrary KKT point. Hence

$$grad\ f(x) + \sum_{i=1}^{m} \lambda_i grad\ \psi_i(x) = 0_x,$$

$$\lambda_i \psi_i(x) = 0, \forall i = 1, 2, ..., m,$$

for some nonnegative $\mathbf{0} \neq \lambda \in \mathbb{R}^m$.
Since f is geodesic η-invex we have $\forall y \in A$,

$$f(y) - f(x) \geq df_x(\eta(y, x)). \tag{7}$$

As $grad\ f(x) = -\sum_{i=1}^{m} \lambda_i grad\ \psi_i(x)$, we have from (7),

$$f(y) - f(x) \geq -\sum_{i=1}^{m} \lambda_i d\psi_{ix}(\eta(y, x)). \tag{8}$$

Now, since A is geodesic invex by Lemma 2, $\forall i = 1, 2, ..., m$,
$d\psi_{ix}(\eta(y, x)) \leq 0, \ \forall x, y \in A \ with \ \psi_i(x) = 0$.
Since $\lambda_i \geq 0$, then $\sum_{i=1}^{m} \lambda_i d\psi_{ix}(\eta(y, x)) \leq 0$.
Hence from (8), $f(y) - f(x) \geq 0, \ \forall y \in A$. Hence x is a minimizer.

Hence if the feasible set is geodesic invex and both the non-degeneracy assumption and the superconsistency hold, there is a one-to-one correspondence between the KKT points and the set of minimizers of the problem (P).

4. Properties of semistrictly geodesic η-prequasi invex functions

In this section, the notion of semistrictly geodesic η-prequasi invex functions is introduced and their properties are studied.

Definition 7. *Let S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be semistrictly geodesic η-prequasi invex if $\forall x, y \in S$, $f(x) \neq f(y)$, we have*

$$f(\gamma_{x,y}(t)) < \max\{f(x), f(y)\} \quad \forall t \in (0, 1).$$

We show by an example that semistrictly geodesic η-prequasi invex function need not be geodesic η-prequasi invex function.

Example: Let $M = \{e^{i\theta} = | - \pi \leq \theta < \pi\}$ and $S = \{e^{i\theta} = | - \frac{\pi}{2} < \theta < \frac{\pi}{2}\}$. Then S is an open set in M. Let $x, y \in S$, where $x = e^{i\theta_1}, y = e^{i\theta_2}$ and $\eta(x, y) = (\theta_2 - \theta_1)(\sin\theta_2, -\cos\theta_2)$.
We define a geodesic on M as $\gamma_{x,y} : [0, 1] \to M$ such that $\gamma_{x,y}(t) = (\cos((1-t)\theta_2 + t\theta_1), \sin((1-t)\theta_2 + t\theta_1))$. Clearly

$$\gamma_{x,y}(0) = y, \quad \gamma'_{x,y}(0) = \eta(x, y), \quad \gamma_{x,y}(t) \in S, \ \forall t \in [0, 1].$$

Hence S is a geodesic invex set in M.
Now we define $f : S \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } \theta = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\forall x, y \in S$, $f(x) \neq f(y)$, we have

$$f(\gamma_{x,y}(t)) < \max\{f(x), f(y)\} \quad \forall t \in (0, 1).$$

i.e., f is semistrictly geodesic η-prequasi invex function.
Let $\theta_1 = \frac{\pi}{4}, \ \theta_2 = -\frac{\pi}{4}, \ t = \frac{1}{2}$, then

$f(\gamma_{x,y}(t)) = f(\cos(\frac{1}{2}\theta_2 + \frac{1}{2}\theta_1), \sin(\frac{1}{2}\theta_2 + \frac{1}{2}\theta_1)) = f(\cos 0, \sin 0) = 1$
$\not< \max\{f(e^{\frac{i\pi}{4}}), f(e^{-\frac{i\pi}{4}})\} = 0.$
Hence f is not geodesic η-prequasi invex function.

Remark 1. *If the function f is semistrictly geodesic η-preinvex on S, then f is semistrictly geodesic η-prequasiinvex on S.*

Proof. Let f be a semistrictly geodesic η-preinvex on S, then $\forall x, y \in S$ with $f(x) \neq f(y)$, we have

$$\begin{aligned} f(\gamma_{x,y}(t)) &< tf(x) + (1-t)f(y), \quad \forall t \in (0,1) \\ &\leq t \max\{f(x), f(y)\} + (1-t)\max\{f(x), f(y)\} \\ &= \max\{f(x), f(y)\}. \end{aligned}$$

Hence f is semistrictly geodesic η-prequasiinvex on S.

Theorem 4. *Let S be a nonempty geodesic invex subset of M with respect to $\eta : M \times M \to TM$ and $f : M \to \mathbb{R}$ be a semistrictly geodesic η-prequasi invex function. If $\bar{x} \in S$ is a local optimal solution to the optimization problem*

$$(OP) \qquad \min_{x \in S} f(x)$$

then $\bar{x}$ is a global minimum of (OP).

Proof. let $\bar{x} \in S$ be a local minimum of (OP). Then there is a neighborhood $N_\epsilon(\bar{x})$ of $\bar{x}$ such that

$$f(\bar{x}) \leq f(x) \ \ \forall x \in S \cap N_\epsilon(\bar{x}). \tag{9}$$

If possible let $\bar{x}$ is not a global minimum of f then there exists a point $x^* \in S$ such that $f(x^*) \leq f(\bar{x})$.
Since S is a geodesic invex set with respect to η, there exists exactly one geodesic $\gamma_{x^*,\bar{x}}$ joining x^*, $\bar{x}$ such that

$$\gamma_{x^*,\bar{x}}(0) = \bar{x}, \ \gamma'_{x^*,\bar{x}}(0) = \eta(x^*, \bar{x}), \ \gamma_{x^*,\bar{x}}(t) \in S \ \forall t \in [0,1].$$

Let us choose $\epsilon > 0$ small enough such that $d(\gamma_{x^*,\bar{x}}(t), \bar{x}) < \epsilon$, then $\gamma_{x^*,\bar{x}}(t) \in N_\epsilon(\bar{x})$. Since f is semistrictly geodesic η-prequasi invex function, we have

$$f(\gamma_{x^*,\bar{x}}(t)) < \max\{f(x^*), f(\bar{x})\} \quad \forall t \in (0,1).$$

i.e., for all $\gamma_{x^*,\bar{x}}(t) \in S \cap N_\epsilon(\bar{x})$, we have $f(\gamma_{x^*,\bar{x}}(t)) < f(\bar{x})$, which is a contradiction to (9).
Hence $\bar{x}$ is a global minimum of (OP).

Theorem 5. *Let S be a nonempty geodesic invex subset of M with respect to $\eta : M \times M \to TM$. Let $f : S \to \mathbb{R}$ be a semistrictly geodesic η-prequasi invex function for the same η and let $\phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Then the composite function $\phi(f)$ is a semistrictly geodesic η-prequasi invex function.*

Proof. For any $x, y \in S$, $\lambda \in (0,1)$ if $\phi(f(x)) \neq \phi(f(y))$, then $f(x) \neq f(y)$.
Since f is semistrictly geodesic η-prequasi invex function, we have

$$f(\gamma_{x,y}(t)) < \max\{f(x), f(y)\} \quad \forall t \in (0,1).$$

Since ϕ is strictly increasing, then

$$\phi[f(\gamma_{x,y}(t))] < \phi[\max\{f(x), f(y)\}] = \max\{\phi(f(x)), \phi(f(y))\}.$$

This shows that $\phi(f)$ is a semistrictly geodesic η-prequasi invex function.

5. Conclusion

In this work we prove the necessity and sufficiency of the KKT theorem for an invex optimization problem where the inequality constraints are not necessarily geodesic η-invex over a Riemannian manifold. This work generalizes the classical KKT theorem on Riemannian manifolds which had been introduced by C. Udriste [8]. We extend the notion of semistrictly quasi invex functions from Euclidean spaces to Riemannian manifolds by introducing semistrictly geodesic η-prequasi invex functions. Variational and control problems on Riemannian manifolds under geodesic η-invexity will orient the future study of the authors.

References

[1] R. P. Agarwal, I. Ahmad, A. Iqbal, S. Ali, *Geodesic G-invex sets semistrictly geodesic η-preinvex functions*, Optimization 61 (2012), 1169-1174.

[2] I. Ahmad, A. Iqbal, S. Ali, *On properties of geodesic η-preinvex functions*, Adv. Oper. Res., vol. 2009, Article ID 381831, 10 pages, doi:10.1155/2009/381831.

[3] A. Barani, M. R. Pouryayevali, *Invex sets and preinvex functions on Riemannian manifolds*, J. Math. Anal. Appl. 328 (2007), 767-779.

[4] J. B. Lasserre, *On representations of the feasible set in convex optimization*, Optim. Lett. 4 (2010), 1-5.

[5] S. Mititelu, *Generalized invexity and vector optimization on differentiable manifolds*, Differ. Geom. Dyn. Syst. 3 (2001), 21-31.

[6] R. Pini, *Convexity along curves and invexity*, Optimization 29 (1994), 301-309.

[7] T. Rapcsak, *Geodesic convexity in nonlinear optimization*, J. Optim. Theory Appl. 69 (1991), 169-183.

[8] C. Udriste, *Convex functions and optimization methods on Riemannian manifolds*, Math. Appl., vol. 297, Kluwer Academic, 1994.

[9] T. J. Willmore, *An Introduction to Differential Geometry*, Oxford University Press, 1959.

[10] X. M. Yang, D. Li, *Semistrictly preinvex functions*, J. Math. Anal. Appl. 258 (2001), 287-308.

[11] X. M. Yang, X. Q. Yang, K. L. Teo, *Characterizations and Applications of Prequasi-Invex Functios*, J. Optim. Theory Appl. 110, 3 (2001), 645-668.

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