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ON CURVE COUPLES WITH JOINT LIGHTLIKE FRENET PLANES IN MINKOWSKI 3-SPACE

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Abstract. In this study, we have investigated the possibility of whether any null Frenet plane of a given space curve in Minkowski 3-space $\mathbb{E}_1^3$ also is any null Frenet plane of another space curve in the same space. As a result, we have possible nine cases and obtain some results for given curves by matching their Frenet planes with each other one by one.

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1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problem is characterization of a regular curve. In the solution of the problem, the curvature functions k_1 (or $\varkappa$) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic or if k_1 =constant$\neq 0$ and $k_2 = 0$,then the curve is a circle with radius $(1/k_1)$, etc. Thus we can determine the shape and size of a regular curve by using its curvatures.

Another way in the solution of the problem is the relationship between the Frenet vectors of the curves (see [6]). For instance *Bertrand* curves:

In 1845, *Saint Venant*(see [6] and [8]) proposed the question whether the principal normal of a curve is the principal normal of another's on the surface generated by the principal normal of the given one. This question was answered by *Bertrand* in 1850 in a paper which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote first and second curvatures of a given curve by $k_1$and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves*.

Another interesting example is *Mannheim* curves: If there is exist a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, β is called Mannheim partner curve of α. Mannheim partner curves was studied by *Liu* and *Wang* (see [7]) in Euclidean 3-space and Minkowski 3-space.

The other way in the solution of the problem is the relationship between the Frenet planes of the curves. In ([9]), the authors asked the following question and investigated the possible answers of the question:

Is it possible that one of the Frenet planes of a given curve in $\mathbb{E}^3$ *be a Frenet plane of another space curve in the same space?* Then they give many interesting results. Also, in ([10]) and ([11]), the authors considered curve couples with joint spacelike and timelike Frenet planes in Minkowski 3-space.

In this paper, we have investigated the possibility of whether any null (lightlike) Frenet plane of a given space curve in Minkowski 3-space $\mathbb{E}^3_1$ also is any null (lightlike) Frenet plane of another space curve in the same space. We have obtained some characterizations of a given space curve by considering nine possible case.

2. Preliminaries

The Minkowski space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. Spacelike curve in $\mathbb{E}^3_1$ is called *pseudo null curve* if its principal normal vector N is null. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([12],[6],[1],[2]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a null curve, the Frenet equations are given by ([12],[1])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, g(N,N) = g(T,B) = 1$. Also, the following equations hold:

$$T \times N = -T,\ B \times T = -N,\ N \times B = -B. \tag{2}$$

Case II. if α is pseudo null curve, the Frenet formulas have the form ([12],[2])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, g(T,T) = g(N,B) = 1$. Also, the following equations hold:

$$T \times N = N,\ N \times B = T,\ B \times T = B. \tag{4}$$

3. On Curve Couples with Joint Null Frenet Planes in Minkowski 3-space

Let us consider the given two space curves C and $\overline{C}$, defined on the same open interval $I \subset \mathbb{R}$. Let us attach moving triads $\{C, T, N, B\}$ and $\{\overline{C}, \overline{T}, \overline{N}, \overline{B}\}$ to C and $\overline{C}$ at the corresponding points of C and $\overline{C}$. We denote the arcs, curvatures and torsions of C and $\overline{C}$ by s, k_1, k_2 and $\overline{s}$, $\overline{k_1}$, $\overline{k_2}$ respectively. At each point $C(s)$ of the curve C, the planes spanned by $\{T,N\}$, $\{N,B\}$, $\{T,B\}$ are known respectively as the osculating plane, the normal plane and the rectifying plane. We denote these planes by OP, NP and RP, respectively. Now, we assume that $\overline{C}$ be a arbitrary unit speed space curve with curvatures $\overline{k_1}$, $\overline{k_2}$ and Frenet vectors $\overline{T}$, $\overline{N}$, $\overline{B}$. At each point $\overline{C}(\overline{s})$ of the curve $\overline{C}$, the planes spanned by $\{\overline{T}, \overline{N}\}$, $\{\overline{N}, \overline{B}\}$, $\{\overline{T}, \overline{B}\}$ are known respectively as the osculating plane, the normal plane and the rectifying plane. We denote these planes by $\overline{OP}$, $\overline{NP}$ and $\overline{RP}$, respectively. Let

$$\frac{d\overline{s}}{ds} = f'$$

In this section we ask the following question:

"*Is it possible that one of the null(lightlike) Frenet planes of a given curve be a null(lightlike) Frenet plane of another space curve?*" and we investigate the answer of the question. For this, we consider the following possible cases:

Case	*Frenet plane of C*	*Frenet plane of $\overline{C}$*	*Condition*
1	$sp\{T,N\} = OP$	$sp\{\overline{T},\overline{N}\} = \overline{OP}$	$OP = \overline{OP}$
2	$sp\{T,N\} = OP$	$sp\{\overline{N},\overline{B}\} = \overline{NP}$	$OP = \overline{NP}$
3	$sp\{T,N\} = OP$	$sp\{\overline{T},\overline{B}\} = \overline{RP}$	$OP = \overline{RP}$
4	$sp\{N,B\} = NP$	$sp\{\overline{T},\overline{N}\} = \overline{OP}$	$NP = \overline{OP}$
5	$sp\{N,B\} = NP$	$sp\{\overline{N},\overline{B}\} = \overline{NP}$	$NP = \overline{NP}$
6	$sp\{N,B\} = NP$	$sp\{\overline{T},\overline{B}\} = \overline{RP}$	$NP = \overline{RP}$
7	$sp\{T,B\} = RP$	$sp\{\overline{T},\overline{N}\} = \overline{OP}$	$RP = \overline{OP}$
8	$sp\{T,B\} = RP$	$sp\{\overline{N},\overline{B}\} = \overline{NP}$	$RP = \overline{NP}$
9	$sp\{T,B\} = RP$	$sp\{\overline{T},\overline{B}\} = \overline{RP}$	$RP = \overline{RP}$

Now, we investigate these possible cases step by step.

If the osculating plane of C ,that is $OP = sp\{T,N\}$, is a null(lightlike) plane, we have two subcases:

Case A. T is a null(lightlike) vector and N is a spacelike vector

Case B. T is a spacelike vector and N is a null(lightlike) vector.

Case A. Since T is a null(lightlike) vector and N is a spacelike vector, C is Cartan null curve with $k_1 = 1$ satisfying the following Frenet formulae (1). Here, we have three subcases:

Case 1. $OP = \overline{OP}$

In this case, we investigate the answer of the following question: "Is it possible that the null (lightlike) osculating plane of a given space curve be the null (lightlike) osculating plane of another space curve in $\mathbb{E}_1^3$?".

Now, we investigate the answer of the question. We assume that the null (lightlike) osculating plane of the given curve C is the null (lightlike) osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a lightlike plane, we have two subcases:

(1-1) $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector,

(1-2) $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector.

Case(1-1) Since $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{T}^{\perp} =$

$sp\{\overline{T},\overline{N}\} = sp\{T,N\} = T^{\perp}$, T is parallel to $\overline{T}$. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{5}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (5) with respect to s and applying the Frenet formulas given in (1), we get

$$\overline{T}f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{6}$$

Since $\overline{T} \in Sp\{T,N\}$, we can write $\overline{T} = \lambda T + \mu N$, for some constant λ and μ. From (6) we have,

$$(\lambda T + \mu N)\, f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{7}$$

Multiplying the equation (7) by T, we obtain

$$b = 0$$

which is contradiction with our assumption.

Case(1-2) Since $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T},\overline{N}\} = sp\{T,N\} = T^{\perp}$, T is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{8}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (8) with respect to s and applying the Frenet formulas given in (1), we get

$$\overline{T}f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{9}$$

Since $\overline{T} \in Sp\{T,N\}$, we can write $\overline{T} = \lambda T + \mu N$, for some constant λ and μ. From (9) we have,

$$(\lambda T + \mu N)\, f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{10}$$

Multiplying the equation (10) by T, we obtain

$$b = 0$$

which is contradiction with our assumption. Thus, we give the following theorem:

Theorem 1. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) osculating plane, $sp\{T, N\}$ with null(lightlike) vector T and spacelike vector N, of C is null(lightlike) osculating plane of $\overline{C}$.*

Case 2. $OP = \overline{NP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) osculating plane of a given space curve be the null(lightlike) normal plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the null(lightlike) osculating plane of the given curve C is the null(lightlike) normal plane of another space curve $\overline{C}$. Since the normal plane of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{N}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{B}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{T, N\} = T^{\perp}$, $\overline{B}$ is parallel to T. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{11}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (11) with respect to s and applying the Frenet formulae given in (1), we get

$$\overline{T} f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{12}$$

First, by taking the scalar product of (12) by T, we obtain

$$b = -f'. \tag{13}$$

Next, since $N = \overline{N}$, by taking the scalar product of (12) by N, we obtain

$$a = f''. \tag{14}$$

Substituting (13) and (14) in (11), we get

$$\overline{X} = X + f''T - f'N. \tag{15}$$

Theorem 2. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) osculating plane, $sp\{T, N\}$ with null(lightlike) vector T and spacelike vector N, of the curve C is the null(lightlike) normal plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + \frac{d^2\overline{s}}{ds^2}T - \frac{d\overline{s}}{ds}N. \tag{16}$$

Case 3. $OP = \overline{RP}$

In this case we investigate the answer of the following question: "Can the null(lightlike) osculating plane of a given space curve be the null(lightlike) rectifying plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the null(lightlike) osculating plane of the given curve C is the null(lightlike) rectifying plane of another space curve $\overline{C}$. Since the rectifying of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{T, N\} = T^{\perp}$, $\overline{B}$ is parallel to T. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{17}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By differentiating (17) and using (1), we have

$$\overline{T} f' = (1 + a' + bk_2)T + (a + b')N - bB. \tag{18}$$

By taking the scalar product of (18) with T, we find

$$b = 0$$

which is a contradiction. Thus, we prove the following theorem:

Theorem 3. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) osculating plane, $sp\{T, N\}$ with null(lightlike) vector T and spacelike vector N, of C is null(lightlike) rectifying plane of $\overline{C}$.*

Case B. Since T is a spacelike vector and N is a null(lightlike) vector, C is pseudo null curve with $k_1 = 1$ satisfying the following Frenet formulae (3). Here, we have three subcases:

Case 4. $OP = \overline{OP}$

In this case, we investigate the answer of the following question: "Is it possible that the null (lightlike) osculating plane of a given space curve be the null (lightlike) osculating plane of another space curve in $\mathbb{E}_1^3$?".

Now, we investigate the answer of the question. We assume that the null (lightlike) osculating plane of the given curve C is the null (lightlike) rectifying plane of another space curve $\overline{C}$. Since the rectifying plane of $\overline{C}$ is a lightlike plane, we have two subcases:

(4-1) $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector,

(4-2) $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector.

Case(4-1) Since $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector, $\overline{C}$ is a Cartan null curve with $\overline{k} = 1$ satisfying the Frenet formulae (1). Since $\overline{T}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{T, N\} = N^{\perp}$, N is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{19}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (19) with respect to s and applying the Frenet formulas given in (3), we get

$$\overline{T} f' = (1 + a')T + (a + b' + bk_2)N. \tag{20}$$

Multiplying the equation (20) by T, we obtain

$$a = -s + c_1 \tag{21}$$

where $c_1 \in \mathbb{R}$. Substituting (21) in (20), we have

$$\overline{T} f' = (a + b' + bk_2)N. \tag{22}$$

By differentiating (22) with respect to s and using (3), we get

$$\overline{T} f'' + (f')^2 \overline{k_1} \overline{N} = (a' + ak_2 + b'' + 2b'k_2 + bk_2' + bk_2^2)N. \tag{23}$$

By taking the scalar product of (23) with itself, we get

$$(f')^4 = 0 \tag{24}$$

which is a contradiction.

Theorem 4. *There exist no a pair of space curves $(C, \overline{C})$ such that the null(lightlike) osculating plane, $sp\{T, N\}$ with spacelike vector T and null(lightlike) vector N, of C is the null(lightlike) osculating plane, $sp\{\overline{T}, \overline{N}\}$ with null(lightlike) vector $\overline{T}$ and spacelike vector $\overline{N}$, of $\overline{C}$.*

Case(4-2) Since $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector, $\overline{C}$ is a pseudo null curve with $\overline{k}_1 = 1$ satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{T, N\} = N^{\perp}$, N is parallel to $\overline{N}$. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{25}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (25) with respect to s and applying the Frenet formulas given in (3), we get

$$\overline{T} f' = (1 + a')T + (a + b' + bk_2)N. \tag{26}$$

By taking the scalar product of (26) with T, we get

$$a = \int (-1 + f')ds. \tag{27}$$

By taking the scalar product of (26) with B, we get

$$a + b' + bk_2 = 0. \tag{28}$$

Solving the differential equation (28), we obtain

$$b = e^{-\int k_2 ds} \left[c - \int e^{\int k_2 ds} (\int (-1 + f')ds)ds \right] \tag{29}$$

where $c \in \mathbb{R}$. Substituting (27) and (29) in (25), we have

$$\overline{X} = X + (\int (-1 + f')ds)T + e^{-\int k_2 ds} \left[c - \int e^{\int k_2 ds} (\int (-1 + f')ds)ds \right] N. \tag{30}$$

Thus, we give the following theorem:

Theorem 5. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) osculating plane, $sp\{T, N\}$ with spacelike vector T and null(lightlike) vector N, of C is the null(lightlike) osculating plane, $sp\{\overline{T}, \overline{N}\}$ with spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{N}$, of $\overline{C}$ with non-zero curvatures $\overline{k_1}, \overline{k_2}$ and Frenet vectors $\overline{T}, \overline{N}, \overline{B}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + (\int (-1 + f')ds)T + e^{-\int k_2 ds} \left[c - \int e^{\int k_2 ds} (\int (-1 + f')ds)ds \right] N.$$

Case 5. $OP = \overline{NP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) osculating plane of a given space curve be the null(lightlike) normal plane of another space curve?"

Now, we investigate the answer of the question. Let assume that the null(lightlike) osculating plane of the given curve C is the null(lightlike) normal plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{N}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{B}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{T, N\} = N^{\perp}$, $\overline{B}$ is parallel to N. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0, \tag{31}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (31) with respect to s and applying the Frenet formulae given in (3), we get

$$\overline{T}f' = (1 + a')T + (ak_1 + b' + bk_2)N. \tag{32}$$

By taking the scalar product of (32) with N, we get

$$f' = 0 \tag{33}$$

which is a contradiction.

Theorem 6. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) osculating plane, $sp\{T, N\}$ with spacelike vector T and null(lightlike) vector N, of C is null(lightlike) normal plane of $\overline{C}$.*

Case 6. $OP = \overline{RP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) osculating plane of a given space curve be the null(lightlike) rectifying plane of another space curve?"

Now, we investigate the answer of the question. Let assume that the null(lightlike) osculating plane of the given curve C is the null(lightlike) rectifying plane of another space curve $\overline{C}$. Since the rectifying of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{T, N\} = N^{\perp}$, $\overline{B}$ is parallel to N. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{34}$$

By differentiating (34) and using (3), we have

$$\overline{T}f' = (1 + a')T + (ak_1 + b' + bk_2)N. \tag{35}$$

By taking the scalar product of (35) with T, we find

$$a = \int (-1 + f')ds. \tag{36}$$

By taking the scalar product of (35) with B, we get

$$ak_1 + b' + bk_2 = 0. \tag{37}$$

Solving the differential equation (37), we obtain

$$b = e^{-\int k_2 ds} \left[c - \int e^{\int k_2 ds} \left(\int (-1 + f')ds \right) ds \right]. \tag{38}$$

Substituting (36) and (38) in (34), we have

$$\overline{X} = X + (\int(-1+f')ds)T + e^{-\int k_2 ds}\left[c - \int e^{\int k_2 ds}(\int(-1+f')ds)ds\right]N. \quad (39)$$

Thus, we prove the following theorem:

Theorem 7. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) osculating plane, $sp\{T,N\}$ with spacelike vector T and null(lightlike) vector N, of the curve C is the null(lightlike) rectifying plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + (\int(-1+f')ds)T + e^{-\int k_2 ds}\left[c - \int e^{\int k_2 ds}(\int(-1+f')ds)ds\right]N. \quad (40)$$

If the normal plane of C ,that is $NP = sp\{N,B\}$, is a null(lightlike) plane, then N is a spacelike vector and B is a null(lightlike) vector. Therefore, C is a Cartan null curve with $k_1 = 1$ satisfying the formulae (1). Here, we have three subcases:

Case 7. $NP = \overline{OP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) normal plane of a given space curve be the null(lightlike) osculating plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the null(lightlike) normal plane of the given curve C is the null(lightlike) osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a null(lightlike) plane, we have two cases:

(7-1). $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector

(7-2). $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector.

Case(7-1). Since $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{T}^{\perp} = sp\{\overline{T},\overline{N}\} = sp\{N,B\} = B^{\perp}$, B is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \quad (41)$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (41) with respect to s and applying the Frenet formulas given in (1), we get

$$\overline{T}f' = (1 + ak_2)T + (a' - bk_2)N + (b' - a)B. \quad (42)$$

By taking the scalar product of (42) with B, we have

$$a = -\frac{1}{k_2}. \tag{43}$$

Substituting (43) in (42), we have

$$\overline{T}f' = (a' - bk_2)N + (b' - a)B. \tag{44}$$

By taking the scalar product of (44) with itself, we have

$$b = \frac{k_2'}{k_2^3}. \tag{45}$$

Substituting (43) and (45) in (41), we have

$$\overline{X} = X - \frac{1}{k_2}N + \frac{k_2'}{k_2^3}B. \tag{46}$$

Thus, we give the following theorem:

Theorem 8. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) normal plane of the curve C is the null(lightlike) osculating plane, $sp\{\overline{T}, \overline{N}\}$ with null(lightlike) vector $\overline{T}$ and spacelike vector $\overline{N}$, of another space curve $\overline{C}$ with non-zero curvatures $\overline{k_1}$, $\overline{k_2}$ and Frenet vectors $\overline{T}$, $\overline{N}$, $\overline{B}$, then $\overline{C}$ has the following form*

$$\overline{C} = C - \frac{1}{k_2}N + \frac{k_2'}{k_2^3}B.$$

Case(7-2) Since $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{N, B\} = B^{\perp}$, B is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{47}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (47) with respect to s and applying the Frenet formulas given in (1), we get

$$\overline{T}f' = (1 + ak_2)T + (a' - bk_2)N + (b' - ak_1)B. \tag{48}$$

By taking the scalar product of (48) with B, we have

$$a = -\frac{1}{k_2}. \tag{49}$$

Substituting (49) in (48), we have

$$\overline{T}f' = (a' - bk_2)N + (b' - ak_1)B. \tag{50}$$

Differentiating (50) , we get

$$\overline{T}f''+(f')^2\overline{k_1}\overline{N} = (a'-bk_2)k_2T+(a''-2b'k_2-bk_2'+ak_1k_2)N+(-2a'k_1-ak_1'+b''+bk_1k_2)B. \tag{51}$$

By taking the scalar product of (51) with B, we have

$$a' - bk_2 = 0. \tag{52}$$

By taking the scalar product of (50) with itself, we have

$$(f')^2 = (a' - bk_2)^2. \tag{53}$$

From (52) and (53) , we find $f' = 0$ which is a contradiction. Thus, we give the following theorem:

Theorem 9. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) normal plane of C is null(lightlike) osculating plane, $sp\{\overline{T}, \overline{N}\}$ with spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{N}$, of $\overline{C}$ with non-zero curvatures $\overline{k_1}$, $\overline{k_2}$ and Frenet vectors $\overline{T}$, $\overline{N}$, $\overline{B}$.*

Case 8. $NP = \overline{NP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) normal plane of a given space curve be the null(lightlike) normal plane of another space curve?".

Now, we investigate the answer of the question. We assume that the null(lightlike) normal plane of the given curve C is the null(lightlike) normal plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{N}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{B}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{N, B\} = B^{\perp}$, $\overline{B}$ is parallel to B. Thus, we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{54}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (54) with respect to s and applying the Frenet formulae given in (1), we get

$$\overline{T}f' = (1 + ak_2)T + (a' - bk_2)N + (b' - a)B. \tag{55}$$

By taking the scalar product of (55) with B, we have

$$a = \frac{f' - 1}{k_2} \tag{56}$$

Since $N = \overline{N}$, by taking the scalar product of (55) with N, we have

$$b = \left(\frac{f' - 1}{k_2}\right)' \frac{1}{k_2} \tag{57}$$

Substituting (56) and (57) in (54), we have

$$\overline{X} = X + \left(\frac{f' - 1}{k_2}\right) N + \left(\frac{f' - 1}{k_2}\right)' \frac{1}{k_2} B. \tag{58}$$

Thus, we give the following theorem:

Theorem 10. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) normal plane of the curve C is the null(lightlike) normal plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + \left(\frac{f' - 1}{k_2}\right) N + \left(\frac{f' - 1}{k_2}\right)' \frac{1}{k_2} B. \tag{59}$$

Conclusion *Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter in theorem* 10*, then $C = \overline{C}$.*

Case 9. $NP = \overline{RP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) normal plane of a given space curve be the null(lightlike) rectifying plane of another space curve?".

We assume that the null (lightlike) normal plane of the given curve C is the null (lightlike) rectifying plane of another space curve $\overline{C}$. Since the rectifying of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{N, B\} = B^{\perp}$, $\overline{B}$ is parallel to B. Thus, we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0, \tag{60}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (60) with respect to s and applying the Frenet formulae given in (1), we get

$$\overline{T} f' = (1 + ak_2)T + (a' - bk_2)N + (b' - a)B. \tag{61}$$

Multiplying the equation (61) by B, we obtain

$$a = -\frac{1}{k_2}. \tag{62}$$

Substituting (62) in (61), we find

$$\overline{T}f' = (a' - bk_2)N + (b' - a)B. \tag{63}$$

Since $N = \overline{T}$, by taking the scalar product of (63) with T, we have

$$b' - a = 0. \tag{64}$$

From (64), we get

$$b = -\int \frac{1}{k_2} ds. \tag{65}$$

Substituting (62) and (65) in (60), we find

$$\overline{X} = X - \frac{1}{k_2}N - \left(\int \frac{1}{k_2} ds\right) B. \tag{66}$$

Thus, we give the following theorem:

Theorem 11. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) normal plane of the curve C is the null(lightlike) rectifying plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C - \frac{1}{k_2}N - \left(\int \frac{1}{k_2} ds\right) B. \tag{67}$$

If the rectifying plane of C, that is $RP = sp\{T, B\}$, is a null(lightlike) plane, then T is a spacelike vector and B is a null(lightlike) vector. Therefore, C is a pseudo null curve with $k_1 = 1$ satisfying the formulae (3). Here, we have three subcases:

Case 10. $RP = \overline{OP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) rectifying plane of a given space curve be the null(lightlike) osculating plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the null(lightlike) rectifying plane of the given curve C is the null(lightlike) osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a null(lightlike) plane, we have two cases:

(10-1). $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector, (10-2). $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector.

Case(10-1). Since $\overline{T}$ is a null(lightlike) vector and $\overline{N}$ is a spacelike vector, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{T}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{T, B\} = B^{\perp}$, B is parallel to $\overline{T}$. Thus, we have the following relation

$$\overline{X} = X + aT + bB, \quad a \neq 0, b \neq 0 \tag{68}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (68) with respect to s and applying the Frenet formulas given in (3), we get

$$\overline{T} f' = (1 + a' - b)T + aN + (b' - bk_2)B. \tag{69}$$

By taking the scalar product of (69) with B, we have

$$a = 0 \tag{70}$$

which is contradiction with our assumption.

Case(10-2) Since $\overline{T}$ is a spacelike vector and $\overline{N}$ is a null(lightlike) vector, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{T, B\} = B^{\perp}$, B is parallel to $\overline{N}$. Thus, we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{71}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (71) with respect to s and applying the Frenet formulas given in (3), we get

$$\overline{T} f' = (1 + a' - b)T + aN + (b' - bk_2)B. \tag{72}$$

By taking the scalar product of (72) with B, we have

$$a = 0 \tag{73}$$

which is contradiction with our assumption. Thus, we give the following theorem:

Theorem 12. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) rectifying plane of C is null(lightlike) osculating plane of $\overline{C}$.*

Case 11. $RP = \overline{NP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) rectifying plane of a given space curve be the null(lightlike) normal plane of another space curve?".

Now, we investigate the answer of the question. We assume that the null(lightlike) normal plane of the given curve C is the null(lightlike) normal plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{N}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a Cartan null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (1). Since $\overline{B}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{T, B\} = B^{\perp}$, $\overline{B}$ is parallel to B. Thus, we have the following relation

$$\overline{X} = X + aT + bB, \quad a \neq 0, b \neq 0, \tag{74}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (74) with respect to s and applying the Frenet formulae given in (3), we get

$$\overline{T}f' = (1 + a' - b)T + aN + (b' - bk_2)B. \tag{75}$$

By taking the scalar product of (75) with B, we have

$$a = f'. \tag{76}$$

Since $T = \overline{N}$, by taking the scalar product of (75) with T , we get

$$1 + a' - b = 0. \tag{77}$$

From (77), we find

$$b = f'' + 1. \tag{78}$$

Substituting (76) and (78) in (74), we have

$$\overline{X} = X + f'T + (f'' + 1)B. \tag{79}$$

Thus, we give the following theorem:

Theorem 13. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the null(lightlike) rectifying plane of the curve C is the null(lightlike) normal plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + f'T + (f'' + 1)B. \tag{80}$$

Case 12. $RP = \overline{RP}$

In this case, we investigate the answer of the following question: "Can the null(lightlike) rectifying plane of a given space curve be the null(lightlike) rectifying plane of another space curve?".

We assume that the null(lightlike) normal plane of the given curve C is the null(lightlike) rectifying plane of another space curve $\overline{C}$. Since the rectifying of $\overline{C}$ is a null(lightlike) plane and spanned by spacelike vector $\overline{T}$ and null(lightlike) vector $\overline{B}$, $\overline{C}$ is a pseudo null curve with $\overline{k_1} = 1$ satisfying the Frenet formulae (3). Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{T, B\} = B^{\perp}$, $\overline{B}$ is parallel to B. Thus, we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{81}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (81) with respect to s and applying the Frenet formulae given in (3), we get

$$\overline{T} f' = (1 + a' - b)T + aN + (b' - bk_2)B. \tag{82}$$

Multiplying the equation (82) by B, we obtain

$$a = 0 \tag{83}$$

which is contradiction with our assumption. Thus, we give the following theorem:

Theorem 14. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which null(lightlike) rectifying plane of C is null(lightlike) rectifying plane of $\overline{C}$.*

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