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THIRD HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In the present paper we investigate the upper bounds of the Hankel determinant $H_3(1)$ for a class of analytic functions with respect to symmetric points, denoted $M_s(\alpha)$.

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1. Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Consider S the subclass of $\mathcal{A}$ consisting of univalent functions.

Recently, Selvaraj and Vasanthi [16] defined the next subclass of analytic functions with respect to symmetric points:

Definition 1. *([16]) Let $M_s(\alpha)$ denote the class of analytic functions f of the form* (1) *and satisfying the condition*

$$\operatorname{Re}\left[\frac{\alpha z^2 f''(z) + z f'(z)}{\alpha z(f(z) - f(-z))' + (1-\alpha)(f(z) - f(-z))}\right] > 0,\ 0 \le \alpha \le 1\,, z \in U. \tag{2}$$

In particular:
(i) for $\alpha = 0, M_s(0) \equiv S_s^*$,

$$S_s^* := \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{z f'(z)}{f(z) - f(-z)}\right] > 0,\ z \in U \right\}.$$

These functions are called starlike functions with respect to symmetric points and were intoduced by Sakaguchi [13].

(ii) for $\alpha = 1, M_s(1) \equiv K_s$,

$$K_s := \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{(zf'(z))'}{(f(z) - f(-z))'} \right] > 0,\ z \in U \right\}.$$

Functions in the class K_s are called convex functions with respect to symmetric points and were introduced by Das and Singh [14].

Definition 2. *([10]) Let f and g be two analytic functions in U. Then, the function f is said to be subordinate to g, written $f \prec g$, if there exists a function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, so that*

$$f(z) = g(h(z)) \text{ for all } z \in U.$$

Pommerenke [11] stated the $q-$th Hankel determinant as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{3}$$

where $n \leq 1$ and $q \leq 1$. The Hankel determinant is useful, for example, in the study of power series with integral coefficients (see [3, 4]), meromorfic functions (see [21]) and also singularities (see [4]).

It is well known that the Fekete-Szegö functional is equivalent to $H_2(1)$. In particular, sharp upper bounds on $H_2(2)$ were obtained in [6, 7, 8, 20]. Recently, the third Hankel determinant $H_3(1)$ has been considered in works [1, 12, 17].

In this paper, we determine the upper bound of $H_3(1)$ for subclasses of analytic functions with respect to symmetric and conjugate points by using Toeplitz determinants [15] and following a method devised by Libera and Zlotkiewicz (see [18, 19]).

In our proposed investigation we shall make use of the next results.

2. Preliminary Results

Let P denote the class of analytic functions p normalized by

$$p(z) = 1 + \sum_{k=1}^{\infty} t_k z^k \tag{4}$$

such that $Re\, p(z) > 0$, $z \in U$.

Lemma 1. *[5]. If $p \in P$ then the following sharp estimate holds:*

$$|t_k| \leq 2,\ k = 1, 2, \dots. \tag{5}$$

Lemma 2. *[18, 19]. Let $p \in P$. Then*

$$2t_2 = t_1^2 + x(4 - t_1^2), \tag{6}$$

$$4t_3 = t_1^3 + 2(4 - t_1^2)t_1 x - (4 - t_1^2)t_1 x^2 + 2(4 - t_1^2)(1 - |x|^2)z, \tag{7}$$

for some complex numbers x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 3. *[9]. If $p \in P$, then for λ a complex number*

$$|t_2 - \lambda t_1^2| \leq 2\max(1, |2\lambda - 1|). \tag{8}$$

This result is sharp for the functions

$$p(z) = \frac{1+z}{1-z} \text{ and } p(z) = \frac{1+z^2}{1-z^2}. \tag{9}$$

3. Main Results

Theorem 4. *Let $f \in M_s(\alpha)$. Then we have the sharp inequality*

$$|a_2a_3 - a_4| \leq \frac{1}{1+3\alpha}\max\left\{\frac{1}{2}, \frac{2(4\alpha^2+3\alpha+1)}{3(1+\alpha)(1+2\alpha)}\sqrt{\frac{4\alpha^2+3\alpha+1}{3(1+\alpha)(1+2\alpha)}}\right\}.$$

Proof. Using the definition of subordination, $f \in M_s(\alpha)$ if and only if

$$\frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z(f(z) - f(-z))' + (1-\alpha)(f(z) - f(-z))} = \frac{1+\omega(z)}{1-\omega(z)} = p(z),\ p \in P.$$

It follows that

$$z + \sum_{n=2}^{\infty} n\left[1 + \alpha(n-1)\right] a_n z^n = (1 + t_1 z + t_2 z^2 + \dots)\{z + (1+2\alpha)a_3 z^3$$
$$+ (1+4\alpha)a_5 z^5 + \dots + [1 + (2n-2)\alpha]a_{2n-1}z^{2n-1} + (1+2n\alpha)a_{2n+1}z^{2n+1} + \dots\}. \tag{10}$$

On equating the coefficients like powers of z in (10), we obtain

$$a_2 = \frac{t_1}{2(1+\alpha)},\ a_3 = \frac{t_2}{2(1+2\alpha)},\ a_4 = \frac{t_1t_2 + 2t_3}{8(1+3\alpha)}. \tag{11}$$

Assuming $t_1 = t$ and substituting for t_2 and t_3 by using Lemma 2 in (11), we have

$$a_2 = \frac{t}{2(1+\alpha)},\ a_3 = \frac{t^2 + (4-t^2)x}{4(1+2\alpha)}, \tag{12}$$

and

$$a_4 = \frac{1}{16(1+3\alpha)}(2t^3 + 3t(4-t^2)x - t(4-t^2)x^2 + 2(4-t^2)(1-|x|^2)z). \tag{13}$$

From (12) and (13) we get

$$\begin{aligned}|a_2a_3 - a_4| =& A(\alpha)| - 4\alpha^2t^3 - t(4-t^2)(6\alpha^2+3\alpha+1)x \\ &+ t(4-t^2)(1+3\alpha+2\alpha^2)x^2 - 2(4-t^2)(1+3\alpha+2\alpha^2)(1-|x|^2)z|,\end{aligned}$$

where

$$A(\alpha) = \frac{1}{16(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

Applying the triangle inequality with $t \in [0,2], |z| \leq 1$ and $\delta = |x|$, we have

$$\begin{aligned}|a_2a_3 - a_4| \leq& A(\alpha)[4t^3\alpha^2 + t(4-t^2)(6\alpha^2+3\alpha+1)\delta \\ &+ t(4-t^2)(1+3\alpha+2\alpha^2)\beta^2 + 2(4-t^2)(1+3\alpha+2\alpha^2)(1-\delta^2)] \\ &= A(\alpha)[(t-2)(4-t^2)(1+3\alpha+2\alpha^2)\delta^2 + t(4-t^2)(6\alpha^2+3\alpha+1)\delta \\ &+ 4t^3\alpha^2 + 2(4-t^2)(1+3\alpha+2\alpha^2)] = A(\alpha)F(\delta). \end{aligned} \tag{14}$$

Next, we maximize the function $F(\delta)$.
$F'(\delta) = 0$ implies $\delta = \frac{at}{2(2-t)b} \equiv d^*$ where $a = 6\alpha^2 + 3\alpha + 1$ and $b = 2(1+\alpha)(1+2\alpha)$, so we need to consider two cases.
(i) If $\delta^* > 1$, we have $\max\limits_{\delta \in [0,1]} F(\delta) = F(1)$, therefore

$$F(\delta) \leq -2t^3(2\alpha^2 + 3\alpha + 1) + 8t(4\alpha^2 + 3\alpha + 1) = G_1(t).$$

By differentiating $G_1(t)$, we get

$$G_1'(t) = -6t^2(2\alpha^2 + 3\alpha + 1) + 8(4\alpha^2 + 3\alpha + 1).$$

Setting $G_1'(t) = 0$ we obtain $t = \pm 2\sqrt{\frac{4\alpha^2+3\alpha+1}{3(2\alpha^2+3\alpha+1)}}$. Since

$$G_1''(t) = -12t(2\alpha^2 + 3\alpha + 1) \leq 0,$$

it follows that G has a maximum value at $t = 2\sqrt{\frac{4\alpha^2+3\alpha+1}{3(2\alpha^2+3\alpha+1)}} = t'$. Hence,

$$G_1(t) \leq \frac{32(4\alpha^2+3\alpha+1)}{3}\sqrt{\frac{4\alpha^2+3\alpha+1}{3(2\alpha^2+3\alpha+1)}}. \tag{15}$$

(ii) If $\delta^* \leq 1$, we find that $\max\limits_{\delta\in[0,1]} F(\delta) = F(\delta^*)$. Thus,

$$F(\delta) \leq \frac{(2+t)(a^2t^2+8b^2(2-t))}{4b} + 4\alpha^2t^3 = G_2(t).$$

It follows that G_2 has a maximum value at $t = 0$, so

$$G_2(t) \leq 16(1+\alpha)(1+2\alpha). \tag{16}$$

From the relations (14), (15) and (16) upon simplification, the theorem is proved. The result is sharp for $t_1 = t$, $t_2 = t_1^2 - 2$ and $t_3 = t_1(t_1^2 - 3)$.

Corollary 5. *[2] If $f \in S_s^*$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

Corollary 6. *[2] If $f \in K_s$, then*

$$|a_2a_3 - a_4| \leq \frac{4}{27}.$$

Theorem 7. *Let $f \in M_s(\alpha)$. Then for a complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha}\max\left(1, \left|\frac{(1+2\alpha)\mu}{(1+\alpha)^2} - 1\right|\right). \tag{17}$$

Proof. From (11), we get

$$|a_3 - \mu a_2^2| = \frac{1}{2(1+2\alpha)}\left|t_2 - \frac{(1+2\alpha)\mu}{2(1+\alpha)^2}t_1^2\right|.$$

Applying Lemma 3, the theorem is proved. This result is sharp for the functions

$$\frac{\alpha z^2 f''(z) + zf'(z)}{\alpha z(f(z)-f(-z))' + (1-\alpha)(f(z)-f(-z))} = \frac{1+z}{1-z}$$

or

$$\frac{\alpha z^2 f''(z) + zf'(z)}{\alpha z(f(z)-f(-z))' + (1-\alpha)(f(z)-f(-z))} = \frac{1+z^2}{1-z^2}.$$

For $\mu = 1$, we get $H_2(1)$.

Corollary 8. *If $f \in M_s(\alpha)$, then*

$$|a_3 - a_2^2| \leq \frac{1}{1+2\alpha}.$$

Corollary 9. *[2] If $f \in S_s^*$, then*

$$|a_3 - a_2^2| \leq 1.$$

Corollary 10. *[2] If $f \in K_s$, then*

$$|a_3 - a_2^2| \leq \frac{1}{3}.$$

Theorem 11. *Let $f \in M_s(\alpha)$. Then we have the sharp inequality*

$$|H_3(1)| \leq \frac{1}{(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)} \cdot \max\left\{52\alpha^4 + 124\alpha^3 + 88\alpha^2 + 25\alpha + 2,5; \frac{D_1 + D_2\sqrt{3(4\alpha^2+3\alpha+1)(1+\alpha)(1+2\alpha)}}{9(1+\alpha)^2}\right\}, \tag{18}$$

where

$D_1 = 18(1+\alpha)^2(1+3\alpha)^2(2\alpha^2+4\alpha+1)$ *and* $D_2 = 2(1+2\alpha)(1+4\alpha)(4\alpha^2+3\alpha+1)$.

Proof. Since $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and applying the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{19}$$

By comparing the coefficients on both sides of equation (10) and using Lemma 1 we have the sharp estimations

$$|a_3| \leq \frac{1}{1+2\alpha},\ |a_4| \leq \frac{1}{1+3\alpha} \text{ and } |a_5| \leq \frac{1}{1+4\alpha}. \tag{20}$$

Using the known inequality $|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}$ (see [20]) and (20) together with Theorem 4 and Corollary 8 in (19), the theorem is proved. The inequality (18) is sharp because each of the components functionals in (19) is sharp.

Corollary 12. *[2] If $f \in S_s^*$, then*

$$|H_3(1)| \leq \frac{5}{2}.$$

Corollary 13. *[2] If $f \in K_s$, then*

$$|H_3(1)| \leq \frac{19}{135}.$$

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