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ON AN INTEGRAL OPERATOR

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Abstract. In this paper we define the integral operator $J_{\alpha,\beta_i}(z)$, considered for analitic functions g_i in the open unit disk $\mathcal{U}$, and we will prove, using Becker criterion, its univalence. Also, we will present some properties of the operator.

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1. Introduction and Preliminaries

Let the open unit disk $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{A}$ the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{H}(\mathcal{U})$ the space of holomorphic functions in $\mathcal{U}$ and by $\mathcal{S} \subset \mathcal{A}$ the subclass of univalent and regular functions from $\mathcal{A}$.

The following sufficient condition for univalency of an analytic function in the unit disk was given by Becker in [1]:

Theorem 1. *Let $f \in \mathcal{A}$. If for all $z \in \mathcal{U}$ we have:*

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{2}$$

then the function f is univalent in $\mathcal{U}$.

In [3], N.N. Pascu introduced the integral operator $L_\alpha : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$ defined as:

$$L_\alpha(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} g(t) dt. \tag{3}$$

Startig from this, V. Pescar and G.L. Aldea in [4] also present the operator $J_{\alpha,\beta} : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U})$,

$$J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} (g(t))^{\beta} \, dt. \tag{4}$$

The goal of our paper is to go further with the generalization and for this we will introduce the integral operator $J_{\alpha,\beta_i}(z)$, given by:

$$J_{\alpha,\beta_i}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} (g_1(t))^{\beta_1} \ldots (g_n(t))^{\beta_n} \, dt, \tag{5}$$

with α and at least one β_i unequal with 0. We will study the univalence for it and present some properties obtained from here.

2. Main results

Theorem 2. *Let the function $g_i \in A$ of the form (1), M be a positive real number ($M \geq 1$) and α, β_i, $i = \overline{1,n}$, be complex numbers with α and at least one of β_i nonequal with 0.*

If we have:

i) $\left|\frac{g_i'(z)}{g_i(z)}\right| \leq M, \quad i = \overline{1,n};$

ii) $M - 1 \leq \frac{1 - \left|\frac{1}{\alpha} + \sum_{i=1}^{n} \beta_i - 2\right|}{\left|\sum_{i=1}^{n} \beta_i\right|},$

then the function $z^{\frac{1}{\alpha}-1} J_{\alpha,\beta_i}(z)$ is in the class $\mathcal{S}$.

Proof. We may write the operator (5) as:

$$J_{\alpha,\beta_i}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2} \left(\frac{g_1(t)}{t}\right)^{\beta_1} \ldots \left(\frac{g_n(t)}{t}\right)^{\beta_n} dt.$$

We consider now:

$$G_{\alpha,\beta_i}(z) = \frac{1}{\alpha} \int_0^z t^{\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2} \left(\frac{g_1(t)}{t}\right)^{\beta_1} \ldots \left(\frac{g_n(t)}{t}\right)^{\beta_n} dt.$$

We have:

$$G'_{\alpha,\beta_i}(z)=\frac{1}{\alpha}z^{\frac{1}{\alpha}+\sum\limits_{i=1}^{n}\beta_i-2}\left(\frac{g_1(z)}{z}\right)^{\beta_1}...\left(\frac{g_n(z)}{z}\right)^{\beta_n}.$$

and:

$$\begin{aligned}G''_{\alpha,\beta_i}(z) &= \frac{1}{\alpha}z^{\frac{1}{\alpha}+\sum\limits_{i=1}^{n}\beta_i-2}\left(\frac{g_1(z)}{z}\right)^{\beta_1}...\left(\frac{g_n(z)}{z}\right)^{\beta_n}\\ &\cdot\left[\left(\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2\right)\frac{1}{z}+\sum_{i=1}^{n}\beta_i\left(\frac{g_i'(z)}{g_i(z)}-\frac{1}{z}\right)\right].\end{aligned}$$

So we obtain:

$$\frac{G''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)}=\frac{1}{z}\left(\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2\right)+\sum_{i=1}^{n}\beta_i\left(\frac{g_i'(z)}{g_i(z)}-\frac{1}{z}\right),$$

hence:

$$\begin{aligned}\left(1-|z|^2\right)\left|\frac{zG''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)}\right| &= \left(1-|z|^2\right)\left|\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2+z\sum_{i=1}^{n}\beta_i\left(\frac{g_i'(z)}{g_i(z)}-\frac{1}{z}\right)\right|\\ &\le \left|\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2\right|+\sum_{i=1}^{n}|\beta_i|\left|\left(\frac{zg_i'(z)}{g_i(z)}-1\right)\right|.\end{aligned}$$

Using successively the properties i) and ii) for the function g_i, we have:

$$\left(1-|z|^2\right)\left|\frac{zG''_{\alpha,\beta_i}(z)}{G'_{\alpha,\beta_i}(z)}\right|\le\left|\frac{1}{\alpha}+\sum_{i=1}^{n}\beta_i-2\right|+(M-1)\sum_{i=1}^{n}|\beta_i|\le 1.$$

Hence, by Becker univalence criterion, we prove that the operator $G_{\alpha,\beta_i}(z)$ is in the class $\mathcal{S}$, so $z^{\frac{1}{\alpha}-1}J_{\alpha,\beta_i}(z)$ is in the class $\mathcal{S}$.

Remark 1. *Because $z^{\frac{1}{\alpha}-1}J_{\alpha,\beta_i}(z)\in S$, there exists $a_i, i=\overline{2,n}$ such as we may write:*

$$z^{\frac{1}{\alpha}-1}J_{\alpha,\beta_i}(z)=z+\sum_{i=2}^{\infty}a_iz^i,\quad z\in\mathcal{U},$$

so it is obviously that the operator $J_{\alpha,\beta_i}(z)$ is of the form:

$$J_{\alpha,\beta_i}(z)=z^{2-\frac{1}{\alpha}}+\sum_{i=2}^{\infty}a_iz^{i+1-\frac{1}{\alpha}},\quad z\in\mathcal{U}.$$

Remark 2. *For $\beta_i = 0, i = \overline{2,n}$, we obtain the Pescar and Aldea's operator (4) and, of course, for $\beta_1 = 1, \beta_i = 0, i = \overline{2,n}$, $J_{\alpha,\beta_i}(z)$ becames Pascu's oprator.*

Corollary 3. *Let the function $g_i \in A$ of the form (1), M be a positive real number ($M \geq 1$) and α be complex number, $\alpha \neq 0$.*

If:

i) $\left|\frac{g_i'(z)}{g_i(z)}\right| \leq M, \quad i = \overline{1,n};$

ii) $M - 1 \leq \frac{1-\left|\frac{1}{\alpha}+n-2\right|}{n}$,

then the function:

$$J_\alpha(z) = \frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-2} g_1(t)...g_n(t)dt$$

is in the class $\mathcal{S}$.

Proof. We consider $\beta_i = 1, i = \overline{1,n}$, in theorem 2.

Corollary 4. *Let the function $g_i \in A$ of the form (1). If:*

$$\left|\frac{g_i'(z)}{g_i(z)}\right| \leq \frac{2}{n}, \quad i = \overline{1,n},$$

then the function:

$$J(z) = \int_0^z t^{-1} g_1(t)...g_n(t)dt$$

is in the class $\mathcal{S}$.

Proof. We consider $\alpha = \beta_i = 1, i = \overline{1,n}$, in theorem 2.

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