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CONCAVE MEROMORPHIC FUNCTIONS INVOLVING CONSTRUCTED OPERATORS

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Abstract. This paper involves constructed differential operators in concave meromorphic function and studied its properties. In particular, coefficient bounds, distortion theorem, and extreme points are obtained.

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1. Introduction

This paper concerns with class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ except for a simple pole at the origin. Also, this class attains certain geometrical interpretation. Explicitly, mapping $\mathbb{U}$ onto a domain whose complement is unbounded convex set.

Back to analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

it is well-known fact, that the inequality

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \qquad z \in \mathbb{U} \tag{1.1}$$

characterises *convex* functions that map the unit disk onto convex domain.

Due to the similarity, the inequality

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} < 0, \qquad z \in \mathbb{U} \tag{1.2}$$

is used sometimes as a definition of concave analytic functions (see e.g [11] and others). However, Bhowmik et al. considered another characterisation of concave analytic functions (see [6]).

In 2012, the condition (1.2) was used again and shown to be necessary and sufficient condition of *concave meromorphic mapping* in the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots \tag{1.3}$$

by Chuaqui, et al. [7].

Further in [7], the coefficients inequality

$$|a_1|^2 + 3|a_2| \leq 1$$

was deduced by applied an invariant form of Schwarz's lemma involving with Schwarzian derivative.

Later, Challab and Darus studied on concave meromorphic functions defined by Salagean Operator and Al-Oboudi operator respectively in [8, 9].

In [3], the authors estimated a_k for $k = 2, 3, ...$ for f of the form

$$f(z) = \frac{1}{z} + \sum_{k=2}^{\infty} a_k z^k. \tag{1.4}$$

Let us consider the differential operators $R^n_{\alpha,\lambda}$ and D^n_λ which introduced respectively in [10] and [4]. Then, the convoluted operator of both of them is

$$\begin{aligned}\tilde{D}^n_{\alpha,\lambda} f(z) &= D^n_\lambda f(z) * R^n_{\alpha,\lambda} f(z)\\ &= \left(\frac{1}{z} + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, a_k z^k\right) * \left(\frac{1}{z} + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, C(\alpha,k) \; a_k z^k\right)\\ &= \frac{1}{z} + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^{2n} \, C(\alpha,k) \; a_k^2 z^k\end{aligned} \tag{1.5}$$

The operator $\tilde{D}^n_{\alpha,\lambda}$ introduced in [1].

In the other hand, the authors in [2] introduced new differential operator by means of linear combination of both $R^n_{\alpha,\lambda}$ and D^n_λ as follows.

$$D^n_{\lambda,\alpha,\gamma} f(z) = (1-\gamma) R^n_\alpha f(z) + \gamma D^n_\lambda f(z), \quad z \in \mathbb{U}. \tag{1.6}$$

If $f(z)$ is an meromorphic function of the form $f(z) = \frac{1}{z} + \sum_{k=2}^{\infty} a_k z^k$, then

$$D^n_{\lambda,\alpha,\gamma} f(z) = \frac{1}{z} + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right] a_k z^k. \tag{1.7}$$

Now let us define the classes of concave meromorphic functions involving constructed differential operator $D^n_{\lambda,\alpha,\gamma}$ as follows.

Definition 1. Let $C_\Sigma(\lambda,\alpha,\gamma)$ denote the class of complex functions of the form (1.4) and satisfies

$$\Re\left\{1 + z\frac{(D^n_{\lambda,\alpha,\gamma} f(z))''}{(D^n_{\lambda,\alpha,\gamma} f(z))'}\right\} < 0, \qquad z \in \mathbb{U}^*, \tag{1.8}$$

where $\lambda,\gamma \geq 0$, and $\alpha, n \in \mathbb{N}_0$.

For constructed differential operator $\tilde{D}^n_{\alpha,\lambda}$, we define the following class of concave meromorphic functions.

Definition 2. Let $C_\Sigma(\lambda,\alpha)$ denote the class of complex functions of the form (1.4) and satisfies

$$\Re\left\{1 + z\frac{(\tilde{D}^n_{\alpha,\lambda} f(z))''}{(\tilde{D}^n_{\alpha,\lambda} f(z))'}\right\} < 0, \qquad z \in \mathbb{U}^*, \tag{1.9}$$

where $\lambda \geq 0$, and $\alpha, n \in \mathbb{N}_0$.

We begin with the coefficient bounds of the classes $C_\Sigma(\lambda,\alpha,\gamma)$ and $C_\Sigma(\lambda,\alpha)$.

2. Coefficient Bounds

First we obtain coefficient bounds of the normalised concave meromorphic functions of the form (1.4) as follow

Theorem 1. Let $f(z)$ be of the form *(1.4)* and

$$\sum_{k=2}^{\infty} k^2 |a_k| \leq 1. \tag{2.1}$$

Then $f(z)$ is concave meromorphic function.

Proof. Using the fact that $\Re w \leq 0$ if and only if $\left|\frac{w+1}{w-1}\right| < 1$, we need to show that

$$\left|\frac{1+z\frac{f''(z)}{f'(z)}+1}{1+z\frac{f''(z)}{f'(z)}-1}\right| < 1,$$

and so

$$\begin{aligned}\left|\frac{1+z\frac{f''(z)}{f'(z)}+1}{1+z\frac{f''(z)}{f'(z)}-1}\right| &= \left|\frac{2f'(z)+zf''(z)}{zf''(z)}\right| \\ &= \left|\frac{\frac{-2}{z^2}+2\sum_{k=2}^{\infty}ka_kz^{k-1}+\frac{2}{z^2}+\sum_{k=2}^{\infty}k(k-1)a_kz^{k-1}}{\frac{2}{z^2}+\sum_{k=2}^{\infty}k(k-1)a_kz^{k-1}}\right| \\ &= \left|\frac{\sum_{k=2}^{\infty}\left(2k+k(k-1)\right)a_kz^{k-1}}{\frac{2}{z^2}+\sum_{k=2}^{\infty}k(k-1)a_kz^{k-1}}\right| \\ &\leq \frac{\sum_{k=2}^{\infty}k(k+1)|a_k|}{2-\sum_{k=2}^{\infty}k(k-1)|a_k|}.\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{k=2}^{\infty}k(k+1)|a_k| < 2-\sum_{k=2}^{\infty}k(k-1)|a_k|,$$

which equivalent to (2.1). The other side of the assertion is trivial.
Therefore, $f(z)$ is concave meromorphic function.

This result was obtained by the authors in [3]. For classes $C_\Sigma(\lambda,\alpha,\gamma)$ and $C_\Sigma(\lambda,\alpha)$ we provide the following theorems.

Theorem 2. Let $f(z)$ be of the form *(1.4)*, $\lambda,\gamma \geq 0, \alpha, n \in \mathbb{N}_0$ and

$$\sum_{k=2}^{\infty}k^2[1+\lambda(k-1)]^n\left[\gamma+(1-\gamma)\,C(\alpha,k)\right]|a_k| \leq 1. \tag{2.2}$$

Then $f(z) \in C_\Sigma(\lambda,\alpha,\gamma)$.

Proof. Using the fact that $\Re w \leq 0$ if and only if $\left|\frac{w+1}{w-1}\right| < 1$, we need to show that

$$\left|\frac{1+z\frac{(D^n_{\lambda,\alpha,\gamma}f)''(z)}{(D^n_{\lambda,\alpha,\gamma}f)'(z)}+1}{1+z\frac{(D^n_{\lambda,\alpha,\gamma}f)''(z)}{(D^n_{\lambda,\alpha,\gamma}f)(z)}-1}\right| < 1.$$

Following the steps of proof Theorem 1, the result is straightforward.

Theorem 3. Let $f(z)$ be of the form *(1.4)*, $\lambda \geq 0$, $\alpha, n \in \mathbb{N}_0$ and

$$\sum_{k=2}^{\infty} k^2 \left[1 + \lambda(k-1)\right]^{2n} C(\alpha, k)|a_k|^2 \leq 1. \tag{2.3}$$

Then $f(z) \in C_\Sigma(\lambda, \alpha)$.

Proof. Using the fact that $\Re w \leq 0$ if and only if $\left|\frac{w+1}{w-1}\right| < 1$, we need to show that

$$\left| \frac{1 + z\frac{(\tilde{D}^n_{\alpha,\lambda} f(z))''(z)}{(\tilde{D}^n_{\alpha,\lambda} f(z))'(z)} + 1}{1 + z\frac{(\tilde{D}^n_{\alpha,\lambda} f(z))''(z)}{(\tilde{D}^n_{\alpha,\lambda} f(z))(z)} - 1} \right| < 1.$$

Following the steps of proof Theorem 1, the result is straightforward.

The following two sections are concerting on the class $C_\Sigma(\lambda, \alpha, \gamma)$.

3. Distortion Theorem

The forgoing theorem obtain the the bound of $|f(z)|$ for the class $C_\Sigma(\lambda, \alpha, \gamma)$.

Theorem 4. Let $f(z)$ be of the form *(1.4)*and in the class $C_\Sigma(\lambda, \alpha, \gamma)$. Then for $z \in \mathbb{U}^*$

$$|f(z)| \leq \frac{1}{|z|} + \sum_{k=2}^{\infty} \frac{1}{4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha, 2)\right]} |z|^2$$

and

$$|f(z)| \geq \frac{1}{|z|} + \sum_{k=2}^{\infty} \frac{1}{4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha, 2)\right]} |z|^2.$$

Proof. Using Theorem 2 we have,

$$4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha, 2)\right] \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} k^2 [1 + \lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha, k)\right] |a_k| \leq 1.$$

That is,

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{1}{4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha, 2)\right]}$$

$$
\begin{aligned}
|f(z)| &= \left| \frac{1}{z} + \sum_{k=2}^{\infty} a_k z^k \right| \\
&\leq \left| \frac{1}{z} \right| + \sum_{k=2}^{\infty} |a_k||z|^k \\
&\leq \frac{1}{|z|} + \sum_{k=2}^{\infty} |a_k||z|^2 \\
&\leq \frac{1}{|z|} + \sum_{k=2}^{\infty} \frac{1}{4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha,2)\right]} |z|^2.
\end{aligned}
$$

The other assertion can be proved as follows

$$
\begin{aligned}
|f(z)| &= \left| \frac{1}{z} + \sum_{k=2}^{\infty} a_k z^k \right| \\
&\geq \left| \frac{1}{z} \right| + \sum_{k=2}^{\infty} |a_k||z|^k \\
&\geq \frac{1}{|z|} + \sum_{k=2}^{\infty} |a_k||z|^2 \\
&\geq \frac{1}{|z|} + \sum_{k=2}^{\infty} \frac{1}{4[1+\lambda]^n \left[\gamma + (1-\gamma)\, C(\alpha,2)\right]} |z|^2.
\end{aligned}
$$

This completes the proof.

3.1. Extreme Points

In this subsection, extreme points of the normalised concave meromorphic functions of the form (1.4) are obtained.

Theorem 5. Let $f_1(z) = \frac{1}{z}$ and $f_k(z) = \frac{1}{z} + \frac{1}{k^2[1+\lambda(k-1)]^n[\gamma+(1-\gamma)C(\alpha,k)]} z^k$. Then $f(z)$ concave meromorphic function of the form *(1.4)* if and only if it can be expressed in the form

$$
f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z),
$$

where $\delta_k \geq 0$ and $\sum_{k=1}^{\infty} \delta_k = 1$.

Proof. Assume that

$$
f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z).
$$

Then

$$
\begin{aligned}
f(z) &= \sum_{k=1}^{\infty} \delta_k f_k(z) \\
&= \delta_1 \frac{1}{z} + \sum_{k=2}^{\infty} \delta_k \left(\frac{1}{z} + \frac{1}{k^2[1+\lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right]} z^k \right) \\
&= \left(\sum_{k=1}^{\infty} \delta_k \right) \frac{1}{z} + \sum_{k=2}^{\infty} \delta_k \frac{1}{k^2[1+\lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right]} z^k \\
&= \frac{1}{z} + \sum_{k=2}^{\infty} \delta_k \frac{1}{k^2[1+\lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right]} z^k .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{k=2}^{\infty} \delta_k \frac{1}{k^2} k^2 \\
&= \sum_{k=2}^{\infty} \delta_k = 1 - \delta_1 < 1.
\end{aligned}
$$

Therefore, $f(z)$ is a concave meromorphic function of the form (1.4).

Conversely, suppose that $f(z)$ is concave meromorphic function of the form (1.4). So

$$
a_k \leq \frac{1}{k^2}, \qquad (k = 2, 3, ...).
$$

We can set

$$
\begin{aligned}
\delta_k &:= k^2 \\
\delta_1 &:= 1 - \sum_{k=2}^{\infty} \delta_k = 1.
\end{aligned}
$$

Then,

$$
\begin{aligned}
f(z) &= \frac{1}{z} + \sum_{k=2}^{\infty} a_k z^k \\
&= \delta_1 f_1(z) + \sum_{k=2}^{\infty} \delta_k f_k(z) \\
&= \sum_{k=1}^{\infty} \delta_k f_k(z).
\end{aligned}
$$

This completes the proof.

Corollary 6. The extreme points of concave meromorphic functions $f(z)$ of the form *(1.4)* are given by $f_1(z) = \frac{1}{z}$ and $f_k(z) = \frac{1}{z} + \frac{z^k}{k^2[1+\lambda(k-1)]^n[\gamma+(1-\gamma)C(\alpha,k)]}$, $(k = 1, 2, 3, ...)$.

Proof. The proof follows by condition (2.2).

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