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An Absolute Matrix Application of Fourier series on Quasi-f-power Increasing Sequences

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Abstract: The Fourier series play an important role in many areas of applied mathematics. Quite recently, Bor [Absolute weighted arithmetic mean summability factors of infinite series and trigonometric Fourier series, 31:15 (2017), 4963-4968] proved a result dealing with absolute weighted arithmetic mean summability of Fourier series involving quasi-f-power increasing sequence. In our work, we extend his result to $|A, \theta_n|_k$ matrix summability method by using normal matrices.

Keywords: Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality.

MSC2010: 26D15; 42A24; 40F05; 40G99.

1 Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that

$$Mc_n \leq b_n \leq Nc_n \quad (see[1]).$$

A positive sequence $X = (X_n)$ is said to be quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where

$$f = \{f_n(\sigma, \beta)\} = \Big\{n^\sigma (logn)^\beta), \quad \beta \geq 0, \quad 0 < \sigma < 1\Big\} \quad (see\ [25]).$$

If we take $\beta = 0$, then we have a quasi-σ-power increasing sequence. Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [13]). For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum\limits_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

1.1 An application of absolute matrix summability of trigonometric Fourier series

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to

$As = (A_n(s))$, where $A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$ Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \quad (see[15],[23]),$$

where $\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$. If we take $\theta_n = \frac{P_n}{p_n}$, then we have $|A, p_n|_k$ summability (see [24]), if we take $\theta_n = n$, then we have $|A|_k$ summability (see [26]) and if we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we have $\left|\bar{N}, p_n\right|_k$ summability (see [2]). Furthermore, if we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we have $|C, 1|_k$ summability (see [12]). Finally, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we obtain $|R, p_n|_k$ summability (see [3]).
Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. The trigonometric Fourier series of f is defined as

$$f(t) \sim \sum_{n=1}^{\infty} (a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t)$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)cos(nt)dt, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)sin(nt)dt. \end{aligned}$$

We write $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \quad \phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \quad (\alpha > 0)$. It is well known that if $\phi(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$ mean of the sequence $(nC_n(x))$ (see [11]).

Using this fact, Bor has obtained the following main result dealing with the trigonometric Fourier series.

2 Known Results

Theorem 1 [9] *Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences (X_n), (λ_n), and (p_n) satisfy the following conditions*

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{1}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m), \tag{3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} \frac{|t_n(x)|^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{5}$$

then the series $\sum C_n(x) \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Later on, Bor has proved the following theorem by taking a quasi-f-power increasing sequence instead of a quasi-σ-power increasing sequence.

Theorem 2 [10] *Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 1, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \geq 1$.*

3 Main Results

The Fourier series play an important role in many areas of applied mathematics and mechanics. Recently some papers have been done concerning absolute matrix summability of infinite series and Fourier series (see [4]-[5], [7]-[8], [14],[16]-[22],[27]-[35]). The aim of this paper is to generalize Theorem 2 for $|A,\theta_n|_k$ summability method for Fourier series by using quasi-f-power increasing sequences.
Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v=0,1,...$$

and

$$\begin{aligned}\hat{a}_{00} &= a_{00},\\ \hat{a}_{nv} &= \bar{\Delta}\bar{a}_{nv}, \quad n=1,2,...\end{aligned}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{6}$$

and

$$\bar{\Delta}A_n(s) = \textstyle\sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{7}$$

Using the above theorem, we have obtained the following result for $|A,\theta_n|_k$ summability concerning the trigonometric Fourier series.

Theorem 3 *Let $k \geq 1$ and $A=(a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \ n=0,1,..., \tag{8}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for} \ \ n \geq v+1, \tag{9}$$

$$\sum_{v=1}^{n-1} \frac{1}{v}\hat{a}_{n,v+1} = O(a_{nn}). \tag{10}$$

Let (X_n) be a quasi-f-power increasing sequence and let $(\theta_n a_{nn})$ be a non increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$ and (θ_n) is any sequence of positive constants and also the sequences (X_n), (λ_n) and (p_n) satisfy the conditions (1)-(3) of Theorem 1 and the following conditions:

$$\sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

are satisfied, then the series $\sum C_n(x)\lambda_n$ is summable $|A,\theta_n|_k$, $k \geq 1$.

It may be remarked that if we take $A=(\bar{N},p_n)$ and $\theta_n=\frac{P_n}{p_n}$, the conditions (11) and (12) are reduced to (4) and (5), respectively. Also, the conditions (8) and (9) are satisfied automatically and the condition (10) satisfied by the condition (3). Therefore, we get Theorem 2.
We need the following lemma for the proof of our theorem.

Lemma 4 [6] *Under the conditions of Theorem 1 we have that*

$$\begin{aligned} nX_n|\Delta\lambda_n| &= O(1) \quad as \quad n\to\infty, &(13)\\ \sum_{n=1}^{\infty} X_n|\Delta\lambda_n| &< \infty. &(14)\end{aligned}$$

Proof of Theorem

Let $(I_n(x))$ denotes the A-transform of the series $\sum_{n=1}^{\infty} C_n(x)\lambda_n$. Then, by (6) and (7), we have $\bar{\Delta} I_n(x)=\sum_{v=1}^{n}\hat{a}_{nv}C_v(x)\lambda_v$. Applying Abel's transformation to this sum, we have that

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{v=1}^{n-1}\Delta\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)\sum_{r=1}^{v} rC_r(x)+\frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} rC_r(x)=\sum_{v=1}^{n-1}\Delta\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v(x)+\hat{a}_{nn}\lambda_n\frac{n+1}{n}t_n(x)\\ &= \sum_{v=1}^{n-1}\bar{\Delta} a_{nv}\lambda_v t_v(x)\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v t_v(x)\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v(x)}{v}+a_{nn}\lambda_n t_n(x)\frac{n+1}{n}\\ &= I_{n,1}(x)+I_{n,2}(x)+I_{n,3}(x)+I_{n,4}(x).\end{aligned}$$

To complete the proof of Theorem 3, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\theta_n^{k-1} \mid I_{n,r}(x)\mid^k<\infty, \quad \text{for} \quad r=1,2,3,4. \tag{15}$$

By applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, and by virtue of hypotheses of Theorem 3 and Lemma 4, we complete the proof of Theorem 3 (For detail see [34]).

4 Applications

The following results can be easily verified.
1. If we take $\theta_n=\frac{P_n}{p_n}$ in Theorem 3, then we have a result dealing with $|A,p_n|_k$ summability.
2. If we take $\theta_n=n$ in Theorem 3, then we have a result dealing with $|A|_k$ summability.
3. If we take $\theta_n=\frac{P_n}{p_n}$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 3, then we have Theorem 2.
4. If we take $\beta=0$, $\theta_n=\frac{P_n}{p_n}$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 3, then we have Theorem 1.
5. If we take $\theta_n=n$, $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n in Theorem 3, then we have a new result concerning $|C,1|_k$ summability.
6. If we take $\theta_n=n$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 3, then we have $|R,p_n|_k$ summability.

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