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A CLASS OF $\beta-$UNIFORMLY UNIVALENT FUNCTIONS DEFINED BY SALAGEAN TYPE $Q-$DIFFERENCE OPERATOR

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Abstract. In this paper, using the Salagean $q-$difference operator, we define a class of $\beta-$uniformly functions and obtain coefficient estimates, distortion theorems, radii of close -to- convexity, starlikeness and convexity for functions in this class. Further we determine partial sums results for the functions class.

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1. Introduction

Let S be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}= \{z : z\in \mathbb{C} : |z| < 1\} \tag{1.1}$$

and T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0;\ z \in \mathbb{U}). \tag{1.2}$$

Also let $S^*(\alpha)$ and $C(\alpha)$ denote the subclasses of S which are, respectively, starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$, satisfying

$$S^*(\alpha) = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ 0 \leq \alpha < 1 \tag{1.3}$$

and

$$C(\alpha) = \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ 0 \leq \alpha < 1. \tag{1.4}$$

For convenience, we write $S^*(0) = S^*$ and $C(0) = C$ (see Robertson [19] and Srivastava and Owa [29]).

From (1.3) and (1.4) we have

$$f(z) \in C(\alpha) \Longleftrightarrow zf^{'}(z) \in S^*(\alpha). \tag{1}$$

Let

$T^*(\alpha) = S^*(\alpha) \cap T$ and $K(\alpha) = C(\alpha) \cap T$ (see Silverman [29]).

Goodman ([11] and [12]) defined the following subclasses of $S^*(C)$.

Definition 1. A function $f(z)$ is uniformly starlike (convex) in $\mathbb{U}$ if $f(z)$ is in $S^*(C)$ and has the property that for every circular are γ contained in $\mathbb{U}$, with center ζ also in $\mathbb{U}$, the arc $f(\gamma)$ is starlike (convex) with respect to $f(\zeta)$. The classes of uniformly starlike and convex functions are denoted by UST and UCV, respectively (for details see [11] and [12])).

$$f(z) \in UCV \Leftrightarrow \Re\left\{1 + (z-\zeta)\frac{f^{''}(z)}{f^{'}(z)}\right\} \geq 0,\ (z,\zeta) \in \mathbb{U}\times\mathbb{U} \tag{1.5}$$

and

$$f(z) \in UST \Leftrightarrow \Re\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f^{'}(z)}\right\} \geq 0,\ (z,\zeta) \in \mathbb{U}\times\mathbb{U}. \tag{1.6}$$

It is well known (see [17, 21]) that

$$f(z) \in UCV \Longleftrightarrow \left|\frac{zf^{''}(z)}{f^{'}(z)}\right| \leq \Re\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\},\ z \in \mathbb{U}. \tag{1.7}$$

In [21], Ronning introduced the new class of starlike functions related to UCV by

$$f(z) \in S_p \Longleftrightarrow \left|\frac{zf^{'}(z)}{f(z)} - 1\right| \leq \Re\left\{\frac{zf^{'}(z)}{f(z)}\right\},\ z \in \mathbb{U}. \tag{1.8}$$

Further Ronning [20], generalized the class S_p by introducing a parameter α by:

Definition 2. [20] A function $f(z)$ of the form (1.1) is in the class $S_p(\alpha)$ if it satisfies

$$\left|\frac{zf^{'}(z)}{f(z)} - 1\right| \leq \Re\left\{\frac{zf^{'}(z)}{f(z)} - \alpha\right\}\ (-1 \leq \alpha < 1,\ z \in \mathbb{U}) \tag{1.9}$$

and $f(z) \in UCV(\alpha)$ if and only if $zf^{'}(z) \in S_p(\alpha)$.

By $\beta - UCV$ ($0 \leq \beta < \infty$), we denote the class of all $\beta-$uniformly convex functions introduced by Kanas and Wisniowska [15]. Recall that a function $f(z) \in S$

is said to be $\beta-$uniformly convex in $\mathbb{U}$ if the image of every circular *arc* contained in $\mathbb{U}$ with center at ζ, where $|\zeta| \leq \beta$, is convex. Note that the class $1-UCV$ coincides with the class UCV.

It is known that $f(z) \in \beta-UCV$ if and only if it satisfies the following condition:

$$\Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\} > \beta\left|\frac{zf^{''}(z)}{f^{'}(z)}\right| \quad (z \in \mathbb{U},\ 0 \leq \beta < \infty). \tag{1.10}$$

The class $\beta-UST$ ($0 \leq \beta < \infty$), of $\beta-$uniformly starlike functions (see [16]) is associated with $\beta-UCV$ by the relation

$$f(z) \in \beta-UCV \Leftrightarrow zf^{'}(z) \in \beta-UST. \tag{1.11}$$

Thus, the class $\beta-UST$, with $0 \leq \beta < \infty$, is ths subclass of S satisfies the following condition:

$$\Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \beta\left|\frac{zf^{'}(z)}{f(z)}-1\right| \quad (z \in \mathbb{U},\ 0 \leq \beta < \infty). \tag{1.12}$$

For $f(z) \in S$, Salagean [23] (see also [3]) defined the operator:

$$\begin{aligned} D^{1}f(z) &= Df(z) = zf^{'}(z), \\ D^{n}f(z) &= D(D^{n-1}f(z)) \end{aligned}$$

$$= z+\sum_{k=2}^{\infty} k^{n}a_{k}z^{k} \quad (n \in \mathbb{N}_{0} = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1,2,...\}). \tag{1.13}$$

For $0 < q < 1$, the Jackson's $q-$derivative of a function $f(z) \in S$ is given by (see [1, 4, 7, 10, 14, 24, 25])

$$D_{q}f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0, \end{cases} \tag{1.14}$$

and $D_{q}^{2}f(z) = D_{q}(D_{q}f(z))$. From (1.14), we have

$$D_{q}f(z) = 1+\sum_{k=2}^{\infty}[k]_{q}\, a_{k}z^{k-1}, \tag{1.15}$$

where

$$[k]_{q} = \frac{1-q^{k}}{1-q} \quad (0 < q < 1). \tag{1.16}$$

If $q \to 1^-$, $[k]_q \to k$. For a function $h(z) = z^k$, we obtain $D_q h(z) = D_q z^k = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$ and $\lim\limits_{q \to 1^-} D_q h(z) = kz^{k-1} = h'(z)$, where h' is the ordinary derivative of h.

Recently for $f \in S$, Govindaraj and Sivasubramanian [13] (also see [18]) defined the Salagean $q-$difference operator by:

$$\begin{aligned} D_q^0 f(z) &= f(z), \\ D_q^1 f(z) &= zD_q f(z), \\ &\vdots \end{aligned}$$

$$D_q^n f(z) = zD_q(D_q^{n-1} f(z) = z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k \quad (0 < q < 1,\ z \in \mathbb{U}). \tag{1.17}$$

We note that $\lim\limits_{q \to 1^-} D_q^n f(z) = D^n f(z)$, where $D^n f(z)$ is defined by (1.13).

For $\beta \geq 0$, $-1 \leq \alpha < 1$, $0 < q < 1$ and $n \in \mathbb{N}_0$, denote by $S_q^n(\alpha, \beta)$ the subclass of S satisfying

$$\Re\left\{ \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - \alpha \right\} > \beta \left| \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right|,\ z \in \mathbb{U}. \tag{1.18}$$

Let $T_q(n, \alpha, \beta) = S_q^n(\alpha, \beta) \cap T$. We note that

(i) $\lim\limits_{q \to 1^-} T_q(n, \alpha, \beta) = T(n, \alpha, \beta)$ (see Aouf [2]),

(ii) $T_q(0, \alpha, \beta) = T_q(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{zD_q f(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zD_q f(z)}{f(z)} \right| \right\}$;

(iii) $T_q(1, \alpha, \beta) = C_q(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{zD_q(D_q^1 f(z))}{D_q^1 f(z)} - \alpha \right\} > \beta \left| \frac{zD_q(D_q^1 f(z))}{D_q^1 f(z)} \right| \right\}$;

(iv) $\lim\limits_{q \to 1^-} T_q(\alpha, \beta) = T(\alpha, \beta) = \left\{ f \in T : \Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}$;

(v) $\lim\limits_{q \to 1^-} C_q(\alpha, \beta) = C(\alpha, \beta) = \left\{ f \in T : \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right| \right\}$;

(vi) $\lim\limits_{q \to 1^-} T_q(n, \alpha, \beta) = C(n, \alpha, \beta) = \left\{ f \in T : \Re\left\{ 1 + \frac{z(D^n f(z))''}{(D^n f(z))'} - \alpha \right\} > \beta \left| \frac{z(D^n f(z))''}{(D^n f(z))'} \right| \right\}$;

(vii) $T_q(0, \alpha, 0) = T_q^*(\alpha) = \Re\left\{ \frac{zD_q f(z)}{f(z)} \right\} > \alpha$;

$(viii)$ $T_q(1, \alpha, 0) = K_q(\alpha) = \Re\left\{ \frac{zD_q(D_q f(z))}{D_q f(z)} \right\} > \alpha$;

(ix) $\lim\limits_{q \to 1^-} T_q^*(\alpha) = T^*(\alpha)$;

(x) $\lim\limits_{q \to 1^-} K_q(\alpha) = K(\alpha)$.

2. COEFFICIENT ESTIMATES

Unless indicated, we assume that $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < q < 1$, $n \in \mathbb{N}_0$, $f(z) \in T$ and $z \in \mathbb{U}$.

Theorem 1. A function $f(z) \in T_q(n, \alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] a_k \leq 1-\alpha. \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds. Then it is suffices to show that

$$\beta \left| \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right| - \Re \left\{ \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right\} \leq 1-\alpha.$$

We have

$$\begin{aligned} & \beta \left| \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right| - \Re \left\{ \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right\} \\ \leq \;& (1+\beta) \left| \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} - 1 \right| \\ \leq \;& \frac{(1+\beta)\sum_{k=2}^{\infty} [k]_q^n \left([k]_q - 1 \right) a_k}{1 - \sum_{k=2}^{\infty} [k]_q^n a_k}. \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ since (2.1) holds.

Conversely we show that if $f(z) \in T_q(n, \alpha, \beta)$ and z is real, then

$$\frac{1 - \sum\limits_{k=2}^{\infty} [k]_q^n ([k]_q) a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} [k]_q^n a_k z^{k-1}} - \alpha \geq \beta \left| \frac{\sum\limits_{k=2}^{\infty} [k]_q^n ([k]_q - 1) a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} [k]_q^n a_k z^{k-1}} \right|.$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality (2.1).

Hence the proof of Theorem 1 is completed.

Corollary 1. Let the function $f(z) \in T_q(n, \alpha, \beta)$. Then

$$a_k \leq \frac{1-\alpha}{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right]} \quad (k \geq 2). \tag{2.2}$$

The result is sharp for

$$f(z) = z - \frac{1-\alpha}{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right]} z^k \quad (k \geq 2). \tag{2.3}$$

3. Growth and Distortion Theorems

Theorem 2. Let $f(z) \in T_q(n, \alpha, \beta)$. Then for $0 \leq i \leq n$,

$$\left|D_q^i f(z)\right| \geq |z| - \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}|z|^2 \tag{3.1}$$

and

$$\left|D_q^i f(z)\right| \leq |z| + \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}|z|^2.$$

The equalities in (3.1) and (3.2) are attained for the function

$$f(z) = z - \frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}z^2. \tag{3.3}$$

Proof. Note that $f(z) \in T_q(n, \alpha, \beta)$ if and only if $D_q^i f(z) \in T_q(n-i, \alpha, \beta)$, where

$$D_q^i f(z) = z - \sum_{k=2}^{\infty} [k]_q^i a_k z^k. \tag{3.4}$$

Using Theorem 1, we have

$$\begin{aligned} &[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]\sum_{k=2}^{\infty}[k]_q^i a_k \\ \leq\ & \sum_{k=2}^{\infty}[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]a_k \leq 1-\alpha, \end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty}[k]_q^i a_k \leq \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\left|D_q^i f(z)\right| \geq |z| - |z|^2\sum_{k=2}^{\infty}[k]_q^i a_k \geq |z| - \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}|z|^2, \tag{3.6}$$

and

$$\left|D_q^i f(z)\right| \leq |z| + |z|^2\sum_{k=2}^{\infty}[k]_q^i a_k \leq |z| - \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}|z|^2. \tag{3.7}$$

Finally, we note that the bounds in (3.1) and (3.2) are attained for $f(z)$ defined by

$$D_q^i f(z) = z - \frac{1-\alpha}{[2]_q^{n-i}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} z^2 \quad (z \in \mathbb{U}). \tag{3.8}$$

This completes the proof of Theorem 2.

Corollary 2. Let $f(z) \in T_q(n, \alpha, \beta)$. Then

$$|f(z)| \geq |z| - \frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} |z|^2, \tag{3.9}$$

and

$$|f(z)| \leq |z| + \frac{1-\alpha}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} |z|^2. \tag{3.10}$$

The sharpenss are attained for the function $f(z)$ given by (3.3).

Proof. Taking $i = 0$ in Theorem 2, we can easily obtain (3.9) and (3.10).

Corollary 3. Let $f(z) \in T_q(n, \alpha, \beta)$. Then

$$\left|D_q^1 f(z)\right| \geq |z| - \frac{1-\alpha}{[2]_q^{n-1}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} |z|^2 \quad (z \in \mathbb{U}), \tag{3.11}$$

and

$$\left|D_q^1 f(z)\right| \leq |z| + \frac{1-\alpha}{[2]_q^{n-1}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} |z|^2 \quad (z \in \mathbb{U}). \tag{3.12}$$

The equalities in (3.11) and (3.12) are attained for the function $f(z)$ given by (3.3).

Proof. Note that $D_q^1 f(z) = z\ D_q f(z)$. Hence taking $i = 1$ in Theorem 2, we have the corollary.

Corollary 4. Let $f(z) \in T_q(n, \alpha, \beta)$. Then the unite disc $\mathbb{U}$ is mapped onto a domain that contains the disc

$$|w| < \frac{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]-(1-\alpha)}{[2]_q^{n}\left[[2]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{3.13}$$

The result is sharp with the extremal function $f(z)$ given by (3.3).

4. Closure Theorem

Let the functions $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0,\ z \in \mathbb{U}). \tag{4.1}$$

Theorem 3. Let the function $f_j(z)$ defined by (4.1) be in the class $T_q(n, \alpha, \beta)$ for every $j = 1, 2, ..., m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} c_j f_j(z), \tag{4.2}$$

is also in the same class, where $c_j \geq 0$, $\sum_{j=1}^{m} c_j = 1$.

Proof. According to (4.2), we can write

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k,j} \right) z^k. \tag{4.3}$$

Further, since $f_j(z) \in T_q(n, \alpha, \beta)$, we get

$$\sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] a_{k,j} \leq 1 - \alpha. \tag{4.4}$$

Hence

$$\begin{aligned} & \sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] \left(\sum_{j=1}^{m} c_j a_{k,j} \right) \\ = & \sum_{j=1}^{m} c_j \left[\sum_{k=2}^{\infty} [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right] a_{k,j} \right] \\ & \leq \left(\sum_{j=1}^{m} c_j \right) (1-\alpha) = 1 - \alpha, \end{aligned} \tag{4.5}$$

which implies that $h(z) \in T_q(n, \alpha, \beta)$. Thus we have the theorem.

Corollary 5. The class $T_q(n, \alpha, \beta)$ is closed under convex linear combination.

Proof. Let $f_j(z)$ defined by (4.1) be in the class $T_q(n,\alpha,\beta)$. It is sufficient to show that if

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \le \mu \le 1), \tag{4.6}$$

then $h(z) \in T_q(n,\alpha,\beta)$. By, taking $m = 2$, $c_1 = \mu$ and $c_2 = 1-\mu$ $(0 \le \mu \le 1)$ in Theorem 3, we have the corollary.

As a consequence of Corollary 5, there exist extreme points of the class $T_q(n,\alpha,\beta)$.

Theorem 4. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{1-\alpha}{[k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta)\right]} z^k \quad (k \ge 2;\ 0 \le \alpha < 1). \tag{4.7}$$

Then $f(z) \in T_q(n,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.8}$$

where $\mu_k \ge 0$ $(k \ge 1)$ and $\sum\limits_{k=1}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[k]_q^n \left[[k]_q(1+\beta)-(\alpha+\beta)\right]} \mu_k z^k. \tag{4.9}$$

Then

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left[[k]_q(1+\beta)-(\alpha+\beta)\right]}{1-\alpha} \cdot \frac{1-\alpha}{[k]_q^n \left[[k]_q(1+\beta)-(\alpha+\beta)\right]} \mu_k$$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \le 1. \tag{4.10}$$

So by Theorem 1, $f(z) \in T_q(n,\alpha,\beta)$.

Conversely, assume that the function $f(z) \in T_q(n,\alpha,\beta)$. Then

$$a_k \le \frac{1-\alpha}{[k]_q^n \left[[k]_q(1+\beta)-(\alpha+\beta)\right]} \quad (k \ge 2). \tag{4.11}$$

Setting

$$\mu_k = \frac{[k]_q^n \left[[k]_q(1+\beta)-(\alpha+\beta)\right]}{1-\alpha} a_k \quad (k \ge 2), \tag{4.12}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.13}$$

we can see that $f(z)$ can be expressed in the form (4.8). This completes the proof of Theorem 4.

Corollary 6. The extreme points of the class $T_q(n, \alpha, \beta)$ are the functions $f_k(z)$ $(k \geq 1)$ given in Theorem 4.

5. RADII OF CLOSE -TO- CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 5. Let $f(z) \in T_q(n, \alpha, \beta)$. Then $f(z)$ is close -to- convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where

$$r_1 = r_1(n, \alpha, \beta, \rho, q) := \inf_k \left[\frac{(1-\rho)\left[[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\right]}{k(1-\alpha)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \tag{5.1}$$

The result is sharp, for $f(z)$ given by (2.3).

Proof. We must show that

$$\left| f^{'}(z) - 1 \right| \leq 1 - \rho \quad for \quad |z| < r_1(n, \alpha, \beta, \rho, q).$$

From (1.2), we have

$$\left| f^{'}(z) - 1 \right| \leq \sum_{k=2}^{\infty} k a_k \, |z|^{k-1}.$$

Thus

$$\left| f^{'}(z) - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k \, |z|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho} \right) |z|^{k-1} \leq \frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left[[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\right]}{k(1-\alpha)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \tag{5.3}$$

Theorem 5 follows from (5.3).

Theorem 6. If $f(z) \in T_q(n, \alpha, \beta)$, then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where

$$r_2 = r_2(n, \alpha, \beta, \rho, q) = \inf_k \left[\frac{(1-\rho)\left[[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\right]}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \tag{5.4}$$

The result is sharp, with $f(z)$ given by (2.3).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad for \ \ |z| < r_2(n, \alpha, \beta, \rho).$$

We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k-\rho}{1-\rho} \right) a_k \, |z|^{k-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{k-\rho}{1-\rho} \right) |z|^{k-1} \leq \frac{[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left[[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\right]}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \tag{5.6}$$

Theorem 6 follows from (5.6).

Corollary 7. If $f(z) \in T_q(n,\alpha,\beta)$, then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where

$$r_3 = r_3(n,\alpha,\beta,\rho,q) = \inf_k \left[\frac{(1-\rho)\left[[k]_q^n\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\right]}{k(k-\rho)(1-\alpha)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \tag{5.7}$$

The result is sharp, with $f(z)$ given by (2.3).

6. PARTIAL SUMS

For $f(z)$ of the form (1.1), the sequence of partial sums is given by

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N}\backslash\{1\}).$$

Silverman [27] determined sharp lower bounds for the real part of each of $\frac{f(z)}{f_m(z)}$, $\frac{f_m(z)}{f(z)}$, $\frac{f'(z)}{f'_m(z)}$ and $\frac{f'_m(z)}{f'(z)}$, when $f \in S^*$ or $f \in C$.

We will follow the work of Silverman [27] and also the works cited in [5, 6, 8, 9, 22 and 26] on partial sums of analytic functions, to obtain our results of this section. We let

$$\Psi_{q,k}^n = \Psi_q^n(k,\alpha,\beta) = [k]_q^n \left[[k]_q (1+\beta) - (\alpha+\beta) \right]. \tag{6.1}$$

Theorem 7. If $f \in S$ satisfies the condition (2.1), then

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{\Psi_{q,m+1}^n - 1 + \alpha}{\Psi_{q,m+1}^n} \quad (z \in \mathbb{U}), \tag{6.2}$$

where

$$\Psi_{q,k}^n \geq \begin{cases} 1-\alpha, & if \quad k = 2,3,...,m \\ \Psi_{q,m+1}^n, & if \quad k = m+1, m+2, ... \,. \end{cases} \tag{6.3}$$

The result (6.2) is sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{\Psi_{q,m+1}^n} z^{m+1} \quad (m \in \mathbb{N}). \tag{6.4}$$

Proof. Define the function $g(z)$ by

$$\frac{1+g(z)}{1-g(z)} = \frac{\Psi_{q,m+1}^n}{1-\alpha}\left[\frac{f(z)}{f_m(z)} - \frac{\Psi_{q,m+1}^n - 1 + \alpha}{\Psi_{q,m+1}^n}\right]$$

$$= \frac{1+\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{m} a_k z^{k-1}}. \tag{6.5}$$

It suffices to show that $|g(z)| \leq 1$. Now from (6.5) we can write

$$g(z) = \frac{\left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{2+2\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|g(z)| \leq \frac{\left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} |a_k|}{2-2\sum\limits_{k=2}^{m} |a_k| - \left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum\limits_{k=m+1}^{\infty} |a_k|}.$$

Now $|g(z)| \leq 1$ if and only if

$$2\left(\frac{\Psi_{q,m+1}^{n}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k| \leq 2-2\sum_{k=2}^{m} |a_k|.$$

or, equivalently,

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Psi_{q,m+1}^{n}}{1-\alpha} |a_k| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Psi_{q,m+1}^{n}}{1-\alpha} |a_k| \leq \sum_{k=2}^{\infty} \frac{\Psi_{q,k}^{n}}{1-\alpha} |a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m} \left(\frac{\Psi_{q,k}^{n}-1+\alpha}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty} \left(\frac{\Psi_{q,k}^{n}-\Psi_{q,m+1}^{n}}{1-\alpha}\right)|a_k| \geq 0. \tag{6.6}$$

To see that the function given by (6.4) gives the sharp result, we observe that for $z = re^{i\pi/m}$

$$\frac{f(z)}{f_m(z)} = 1 + \frac{1-\alpha}{\Psi_{q,m+1}^{n}} z^k \to 1 - \frac{1-\alpha}{\Psi_{q,m+1}^{n}} = \frac{\Psi_{q,m+1}^{n}-1+\alpha}{\Psi_{q,m+1}^{n}} \ \textit{where } r \to 1^-.$$

This completes the proof of Theorem 7.

We next determine bounds for $\frac{f_m(z)}{f(z)}$.

Theorem 8. If $f \in S$ of the form (1.1) satisfies the condition (2.1), then

$$\Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\Psi_{q,m+1}^n}{\Psi_{q,m+1}^n + 1 - \alpha} \quad (z \in \mathbb{U}), \tag{6.7}$$

where $\Psi_{q,m+1}^n \geq 1 - \alpha$ and

$$\Psi_{q,k}^n \geq \begin{cases} 1-\alpha, & if \quad k = 2, 3, ..., m \\ \Psi_{q,m+1}^n, & if \quad k = m+1, m+2, ... \,. \end{cases} \tag{6.8}$$

The result (6.7) is sharp with the function given by (6.4).

Proof. The proof follows by defining

$$\frac{1+g(z)}{1-g(z)} = \frac{\Psi_{q,m+1}^n + 1 - \alpha}{1-\alpha}\left[\frac{f_m(z)}{f(z)} - \frac{\Psi_{q,m+1}^n}{\Psi_{q,m+1}^n + 1 - \alpha}\right]$$

and much akin are to similar arguments in Theorem 7. So, we omit it.

We next turns to ratios involving derivatives.

Theorem 9. If $f \in S$ satisfies the condition (2.1), then

$$\Re\left(\frac{f'(z)}{f_m'(z)}\right) \geq \frac{\Psi_{q,m+1}^n - (m+1)(1-\alpha)}{\Psi_{q,m+1}^n} \quad (z \in \mathbb{U}), \tag{6.9}$$

and

$$\Re\left(\frac{f_m'(z)}{f'(z)}\right) \geq \frac{\Psi_{q,m+1}^n}{\Psi_{q,m+1}^n + (m+1)(1-\alpha)} \quad (z \in \mathbb{U}), \tag{6.10}$$

where $\Psi_{q,m+1}^n \geq (m+1)(1-\alpha)$ and

$$\Psi_{q,k}^n \geq \begin{cases} k(1-\alpha), & if \quad k = 2, 3, ..., m \\ k\left(\frac{\Psi_{q,m+1}^n}{(m+1)}\right), & if \quad k = m+1, m+2, ... \,. \end{cases} \tag{6.11}$$

The results are sharp with the function given by (6.4).

Proof. We write

$$\frac{1+g(z)}{1-g(z)} = \frac{\Psi_{q,m+1}^n}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f_m'(z)} - \left(\frac{\Psi_{q,m+1}^n - (m+1)(1-\alpha)}{\Psi_{q,m+1}^n}\right)\right],$$

where

$$g(z)=\frac{\left(\frac{\Psi^n_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty}a_kz^{k-1}}{2+2\sum\limits_{k=2}^{m}ka_kz^{k-1}+\left(\frac{\Psi^n_{m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty}ka_kz^{k-1}}.$$

Now $|g(z)|\leq 1$ if and only if

$$\sum_{k=2}^{m}k\left|a_k\right|+\left(\frac{\Psi^n_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}k\left|a_k\right|\leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{k=2}^{m}k\left|a_k\right|+\left(\frac{\Psi^n_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}k\left|a_k\right|\leq\sum_{k=2}^{\infty}\frac{\Psi^n_{q,k}}{1-\alpha}\left|a_k\right|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Psi^n_{q,k}-k(1-\alpha)}{1-\alpha}\right)\left|a_k\right|+\sum_{k=m+1}^{\infty}\left(\frac{(m+1)\Psi^n_{q,k}-k\Psi^n_{q,m+1}}{(m+1)(1-\alpha)}\right)\left|a_k\right|\geq 0.$$

To prove the result (6.10), define the function $g(z)$ by

$$\frac{1+g(z)}{1-g(z)}=\frac{(m+1)(1-\alpha)+\Psi^n_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f_m'(z)}{f'(z)}-\frac{\Psi^n_{q,m+1}}{(m+1)(1-\alpha)+\Psi^n_{q,m+1}}\right],$$

and by similar arguments in first part we get desired result.

Remark 1.
(i) Putting $n=\beta=0$ in our results we get the results for the class $T^*_q(\alpha)$,
(ii) Putting $n=1$ and $\beta=0$ in our results we get the results for the class $K_q(\alpha)$.

Remark 2. Our results in Theorems 7, 8 and 9, respectively, modified the results obtained by Vijaya et al. [30, Theorems 4.1, 4.2 and 4.3 with $\mu=1$, respectively].

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