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GENERAL HELICES IN THE LIGHTLIKE CONE

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Abstract. In this paper, we study general helices with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in the lightlike cone $\mathbb{Q}^2$ with respect to the causal character of the slope axis. Furthermore, we get the parametric equations of such curves and give some related examples and their figures.

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1. Introduction

In a theory of space curves, especially, a helix is the most elementary and interesting topic. A helix, moreover, pays attention to natural scientists as well as mathematicians because of its various applications, for example, DNA, carbon nanotube, screws, springs and so on. Also there are many applications of helix curve or helical structures in Science such as fractal geometry, in the fields of computer aided design and computer graphics. Helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc.(see [2, 6, 9, 18]).

From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature (or first curvature of the curve and denote by k_1) and non-vanishing constant torsion (or second curvature of the curve and denote by k_2). Indeed a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 (for details see [12, 13]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of k_1 and k_2 are non-zero constant, it is, of course, a general helix. We call it a circular helix. Its known that straight line and circle are degenerate-helix examples ($k_1 = 0$, if the curve is straight line and $k_2 = 0$, if the curve is a circle). Also, helices or more generally general helices are studied by many authors in different view point in different spaces for example, in Euclidean n-space [4, 10] in Lorentz-Minkowski spaces [1, 7, 8, 11].

In [5], Authors studied general helices with lightlike axis in Minkowski 3-space. Thus a new classifications of general helices with respect to causal character of the constant slope axis is arisen. They show that there is a nice relation between the helices with lightlike axis and biharmonic curves in Minkowski 3-space. Also general helices was studied in [16] and [17].

On the other hand, in [14], the author studies curves in the lightlike cone. the author obtains the conformally invariant arc length in the $(n+1)$-dimensional lightlike cone and then characterize some curves in the 2-dimensional lightlike cone and 3-dimensional lightlike cone.

In this paper, we study general helices with asymptotic orthonormal frame $\{x(s),\ \alpha(s),\ y(s)\}$ in the lightlike cone $\mathbb{Q}^2$, defined in [14], with respect to the causal character of the slope axis. Furthermore, we get the parametric equations of such curves and give some related examples and their figures.

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3 \backslash \{0\}$ can be spacelike if $g(v,v) > 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [15].

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ [3]. On the other hand, a non-null curve α is parametrized by the arc-length parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

The pseudo-Riemann lightlike cone is defined by

$$\mathbb{Q}^2(c) = \left\{ x \in \mathbb{E}_1^3 : g(x-c, x-c) = 0 \right\}.$$

When $c = 0$, we denote $\mathbb{Q}^2(0)$ by $\mathbb{Q}^2$.

([14])For the regular curve $x(s) \subset \mathbb{Q}^2 \subset \mathbb{E}_1^3$ with asymptotic orthonormal frame $\{x(s),\ \alpha(s),\ y(s)\}$ and cone curvature function $\kappa(s)$, the Frenet formulas of the curve $x(s)$ in $\mathbb{Q}^2$ can be written as

$$\begin{cases} x'(s) &= \alpha(s), \\ \alpha'(s) &= \kappa(s)\, x(s) - y(s), \\ y'(s) &= -\kappa(s)\, \alpha(s), \end{cases} \tag{1}$$

where for all s,

$$g(x(s), x(s)) = g(y(s), y(s)) = g(x(s), \alpha(s)) = g(y(s), \alpha(s)) = 0,$$

$$g(\alpha(s), \alpha(s)) = g(x(s), y(s)) = 1.$$

$$g(\alpha(s), \beta(s)) = g(\alpha(s), y(s)) = g(\beta(s), y(s)) = 0.$$

3. General helices in the lightlike cone

In this section, we study general helices with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in the lightlike cone $\mathbb{Q}^2$ with respect to the causal character of the slope axis.

3.1. General helices with spacelike slope axis

Theorem 1. *Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the curvature κ. The slope axis of $x(s)$ is a constant spacelike vector if and only if $\kappa > 0$ constant.*

Proof. Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the curvature κ. We assume that $x(s)$ has a constant spacelike slope axis U and U is given by

$$U = c_1(s)x(s) + c_2(s)\alpha(s) + c_3(s)y(s), \tag{2}$$

Since $x(s)$ is a general helix, we have

$$g(U, x(s)) = c_3(s) = c_3 \text{ (constant} \neq 0). \tag{3}$$

Differentiating (3) with respect to s, and using (1), we easily obtain

$$c_2(s) = g(U, \alpha(s)) = 0 \quad \text{and} \quad c_3\kappa - c_1 = 0. \tag{4}$$

Since $g(U, U) = 1$ and

$$U = \kappa c_3 x(s) + c_3 y(s) \tag{5}$$

we get

$$\kappa = \frac{1}{2c_3^2} \text{ (constant)}$$

Conversely, Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa > 0$. From (5), we find that the slope axis is a spacelike vector. This completes the proof of the theorem.

In the following theorem, we obtain the parametric equations of the general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ and spacelike slope axis.

Theorem 2. *Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa > 0$ and the slope axis of $x(s)$ be a spacelike vector. Then the parametric equation of $x(s) = (x_1(s), x_2(s), x_3(s))$ is given by*

$$\begin{aligned}
x_1(s) &= \sigma(s), \\
x_2(s) &= \frac{c + u_1\sigma(s)}{\sqrt{1+u_1^2}}\cos\varphi - \frac{\sqrt{\sigma^2(s) - c^2 - 2u_1 c\sigma(s)}}{\sqrt{1+u_1^2}}\sin\varphi, \\
x_3(s) &= \frac{c + u_1\sigma(s)}{\sqrt{1+u_1^2}}\sin\varphi + \frac{\sqrt{\sigma^2(s) - c^2 - 2u_1 c\sigma(s)}}{\sqrt{1+u_1^2}}\cos\varphi
\end{aligned}$$

and the constant spacelike slope axis U is given by

$$U = \left(u_1, \sqrt{1+u_1^2}\cos\varphi, \sqrt{1+u_1^2}\sin\varphi\right)$$

where

$$\sigma(s) = \sqrt{c^2\left(1+u_1^2\right)}\cosh\left(\frac{m_1}{\sqrt{c^2}}s + m_2\right) + cu_1,$$

for $m_1, m_2 = \pm 1$ and c, $\varphi \in \mathbb{R}$.

Proof. Assume that $x(s) = (x_1(s), x_2(s), x_3(s))$ is a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa > 0$ and the slope axis of $x(s)$ is a spacelike vector $U = (u_1, u_2, u_3)$ in $\mathbb{E}_1^3$. Then we can write

$$\begin{aligned}
g(x,x) &= -x_1^2 + x_2^2 + x_3^2 = 0, \\
g(U,U) &= -u_1^2 + u_2^2 + u_3^2 = 1.
\end{aligned}$$

Thus, we can set

$$\begin{aligned} x_2 &= m_0 x_1 \cos\theta, \\ x_3 &= m_0 x_1 \sin\theta, \end{aligned} \tag{6}$$

and

$$\begin{aligned} u_2 &= \sqrt{1+u_1^2}\cos\varphi, \\ u_3 &= \sqrt{1+u_1^2}\sin\varphi. \end{aligned} \tag{7}$$

Since $x(s)$ is a general helix, we have

$$g(U,x) = -u_1 x_1(s) + u_2 x_2(s) + u_3 x_3(s) = c \tag{8}$$

where $c \in \mathbb{R}/\{0\}$. By using (6) and (7) in (8), we find,

$$\cos(\theta-\varphi) = \frac{u_1 x_1 + c}{m_0 x_1\sqrt{1+u_1^2}}, \tag{9}$$

and

$$\sin(\theta-\varphi) = \frac{\sqrt{x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1\sqrt{1+u_1^2}} \tag{10}$$

Considering (9) and (10) together, we obtain

$$\begin{aligned} \cos\theta &= \frac{u_1 x_1 + c}{m_0 x_1\sqrt{1+u_1^2}}\cos\varphi - \frac{\sqrt{x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1\sqrt{1+u_1^2}}\sin\varphi, \\ \sin\theta &= \frac{u_1 x_1 + c}{m_0 x_1\sqrt{1+u_1^2}}\sin\varphi + \frac{\sqrt{x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1\sqrt{1+u_1^2}}\cos\varphi. \end{aligned}$$

Thus we get

$$\begin{aligned} x_2(s) &= \frac{u_1 x_1 + c}{\sqrt{1+u_1^2}}\cos\varphi - \frac{\sqrt{x_1^2 - c^2 - 2u_1 x_1 c}}{\sqrt{1+u_1^2}}\sin\varphi, \\ x_3(s) &= \frac{u_1 x_1 + c}{\sqrt{1+u_1^2}}\sin\varphi + \frac{\sqrt{x_1^2 - c^2 - 2u_1 x_1 c}}{\sqrt{1+u_1^2}}\cos\varphi. \end{aligned} \tag{11}$$

Differentiating (11) and using $g(x', x') = 1$, we obtain

$$x_1(s) = \sqrt{c^2\left(1+u_1^2\right)}\cosh\left(\frac{m_1}{\sqrt{c^2}}s + m_2\right) + c u_1.$$

This completes the proof.

Example 1. *If we take* $c = m_1 = 1, m_2 = 0, u_1 = \sqrt{2}, \varphi = \frac{\pi}{6}$ *in theorem 2, we find the curve* $x(s) = (x_1(s), x_2(s), x_3(s))$ *as follows*

$$\begin{aligned} x_1(s) &= \sqrt{2} + \sqrt{3}\cosh s, \\ x_2(s) &= \frac{3 + \sqrt{6}\cosh s - \sinh s}{2}, \\ x_3(s) &= \frac{\sqrt{3} + \sqrt{2}\cosh s + \sqrt{3}\sinh s}{2} \end{aligned}$$

Also the constant spacelike slope axis is $U = \left(\sqrt{2}, \frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ *and* $g(x(s), U) = 1$.

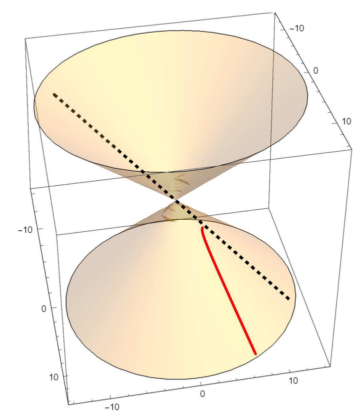

Figure 1: In this graphic, the dashed line represents the slope axis of the helix.

3.2. General helices with timelike slope axis

Theorem 3. *Let* $x(s)$ *be a general helix with asymptotic orthonormal frame* $\{x(s), \alpha(s), y(s)\}$ *in* $\mathbb{Q}^2$ *parametrized by arc-length* s *with the curvature* κ. *The slope axis of* $x(s)$ *is a constant timelike vector if and only if* $\kappa < 0$ *constant.*

Proof. Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the curvature κ. We assume that $x(s)$ has a constant timelike slope axis U and U is given by

$$U = c_1(s)x(s) + c_2(s)\alpha(s) + c_3(s)y(s), \tag{12}$$

Since $x(s)$ is a general helix, we have

$$g(U, x(s)) = c_3(s) = c_3 \text{ (constant} \neq 0). \tag{13}$$

Differentiating (13) with respect to s, and using (1), we easily obtain

$$c_2(s) = g(U, \alpha(s)) = 0 \quad \text{and} \quad c_3\kappa - c_1 = 0. \tag{14}$$

Since $g(U,U) = -1$ and

$$U = \kappa c_3 x(s) + c_3 y(s) \tag{15}$$

we get

$$\kappa = -\frac{1}{2c_3^2} \text{ (constant)}$$

Conversely, Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa < 0$. From (15), we find that the slope axis is a timelike vector. This completes the proof of the theorem.

In the following theorem, we obtain the parametric equations of the general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ and timelike slope axis.

Theorem 4. *Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa < 0$ and the slope axis of $x(s)$ be a timelike vector. Then the parametric equation of $x(s) = (x_1(s), x_2(s), x_3(s))$ is given by*

$$\begin{aligned} x_1(s) &= \sigma(s), \\ x_2(s) &= \frac{c + u_1\sigma(s)}{\sqrt{u_1^2 - 1}} \cos\varphi - \frac{\sqrt{-\sigma^2(s) - c^2 - 2u_1 c\sigma(s)}}{\sqrt{u_1^2 - 1}} \sin\varphi, \\ x_3(s) &= \frac{c + u_1\sigma(s)}{\sqrt{u_1^2 - 1}} \sin\varphi + \frac{\sqrt{-\sigma^2(s) - c^2 - 2u_1 c\sigma(s)}}{\sqrt{u_1^2 - 1}} \cos\varphi \end{aligned}$$

and the constant spacelike slope axis U is given by

$$U = \left(u_1, \sqrt{u_1^2 - 1}\cos\varphi, \sqrt{u_1^2 - 1}\sin\varphi\right)$$

where

$$\sigma(s) = \sqrt{c^2\left(u_1^2 - 1\right)} \sin\left(\frac{m_1}{\sqrt{c^2}} s + m_2\right) - cu_1,$$

for $m_1, m_2 = \pm 1$ and $u_1^2 > 1$, c, $\varphi \in \mathbb{R}$.

Proof. Assume that $x(s) = (x_1(s), x_2(s), x_3(s))$ is a general helix with asymptotic orthonormal frame $\{x(s),\ \alpha(s),\ y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa < 0$ and the slope axis of $x(s)$ is a timelike vector $U = (u_1, u_2, u_3)$ in $\mathbb{E}_1^3$. Then we can write

$$\begin{aligned} g(x,x) &= -x_1^2 + x_2^2 + x_3^2 = 0, \\ g(U,U) &= -u_1^2 + u_2^2 + u_3^2 = -1. \end{aligned}$$

Thus, we can set

$$\begin{aligned} x_2 &= m_0 x_1 \cos\theta, \\ x_3 &= m_0 x_1 \sin\theta, \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_2 &= \sqrt{u_1^2 - 1}\cos\varphi, \\ u_3 &= \sqrt{u_1^2 - 1}\sin\varphi, \end{aligned} \tag{17}$$

where $u_1^2 > 1$. Since $x(s)$ is a general helix, we have

$$g(U,x) = -u_1 x_1(s) + u_2 x_2(s) + u_3 x_3(s) = c \tag{18}$$

where $c \in \mathbb{R}/\{0\}$. By using (16) and (17) in (18), we find,

$$\cos(\theta - \varphi) = \frac{u_1 x_1 + c}{m_0 x_1 \sqrt{u_1^2 - 1}}, \tag{19}$$

and

$$\sin(\theta - \varphi) = \frac{\sqrt{-x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1 \sqrt{u_1^2 - 1}} \tag{20}$$

Considering (19) and (20) together, we obtain

$$\begin{aligned} \cos\theta &= \frac{u_1 x_1 + c}{m_0 x_1 \sqrt{u_1^2 - 1}}\cos\varphi - \frac{\sqrt{-x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1 \sqrt{u_1^2 - 1}}\sin\varphi, \\ \sin\theta &= \frac{u_1 x_1 + c}{m_0 x_1 \sqrt{u_1^2 - 1}}\sin\varphi + \frac{\sqrt{-x_1^2 - c^2 - 2u_1 x_1 c}}{m_0 x_1 \sqrt{u_1^2 - 1}}\cos\varphi. \end{aligned}$$

Thus we get

$$\begin{aligned} x_2(s) &= \frac{u_1 x_1 + c}{\sqrt{u_1^2 - 1}}\cos\varphi - \frac{\sqrt{-x_1^2 - c^2 - 2u_1 x_1 c}}{\sqrt{u_1^2 - 1}}\sin\varphi, \\ x_3(s) &= \frac{u_1 x_1 + c}{\sqrt{u_1^2 - 1}}\sin\varphi + \frac{\sqrt{-x_1^2 - c^2 - 2u_1 x_1 c}}{\sqrt{u_1^2 - 1}}\cos\varphi. \end{aligned} \tag{21}$$

Differentiating (21) and using $g(x', x') = 1$, we obtain

$$x_1(s) = \sqrt{c^2\left(u_1^2 - 1\right)} \sin\left(\frac{m_1}{\sqrt{c^2}} s + m_2\right) - c u_1.$$

This completes the proof.

Example 2. *If we take* $c = m_1 = 1, m_2 = 0, u_1 = \sqrt{2}, \varphi = \frac{\pi}{4}$ *in theorem 4, we find the curve* $x(s) = (x_1(s), x_2(s), x_3(s))$ *as follows*

$$\begin{aligned} x_1(s) &= -\sqrt{2} + \sin s, \\ x_2(s) &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos s + \sin s \\ x_3(s) &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos s + \sin s \end{aligned}$$

Also the constant timelike slope axis is $U = \left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ *and* $g(x(s), U) = 1$.

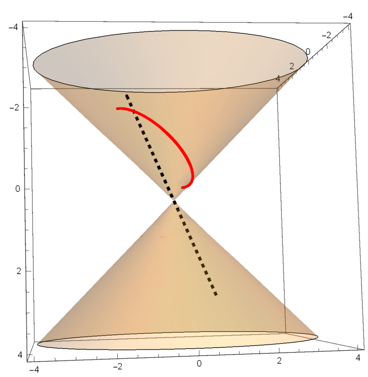

Figure 2: In this graphic, the dashed line represents the slope axis of the helix.

3.3. General helices with lightlike slope axis

Theorem 5. *Let* $x(s)$ *be a general helix with asymptotic orthonormal frame* $\{x(s), \alpha(s), y(s)\}$ *in* $\mathbb{Q}^2$ *parametrized by arc-length* s *with the curvature* κ*. The slope axis of* $x(s)$ *is a constant lightlike vector if and only if* $\kappa = 0$.

Proof. Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the curvature κ. We assume that $x(s)$

has a constant lightlike slope axis U and U is given by

$$U = c_1(s)x(s) + c_2(s)\alpha(s) + c_3(s)y(s), \tag{22}$$

Since $x(s)$ is a general helix, we have

$$g(U, x(s)) = c_3(s) = c_3 \text{ (constant} \neq 0). \tag{23}$$

Differentiating (23) with respect to s, and using (1), we easily obtain

$$c_2(s) = g(U, \alpha(s)) = 0 \quad \text{and} \quad c_3\kappa - c_1 = 0. \tag{24}$$

Since $g(U,U) = 0$ and

$$U = \kappa c_3 x(s) + c_3 y(s) \tag{25}$$

we get

$$\kappa = 0$$

Conversely, Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa = 0$. From (25), we find that the slope axis is a lightlike vector. This completes the proof of the theorem.

In the following theorem, we obtain the parametric equations of the general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ and lightlike slope axis.

Theorem 6. *Let $x(s)$ be a general helix with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa = 0$ and the slope axis of $x(s)$ be a lightlike vector. Then the parametric equation of $x(s) = (x_1(s), x_2(s), x_3(s))$ is given by*

$$\begin{aligned} x_1(s) &= \sigma(s), \\ x_2(s) &= \frac{c + u_1\sigma(s)}{u_1}\cos\varphi - \frac{\sqrt{-c^2 - 2u_1 c\sigma(s)}}{u_1}\sin\varphi, \\ x_3(s) &= \frac{c + u_1\sigma(s)}{u_1}\sin\varphi + \frac{\sqrt{-c^2 - 2u_1 c\sigma(s)}}{u_1}\cos\varphi \end{aligned}$$

and the constant spacelike slope axis U is given by

$$U = (u_1, u_1\cos\varphi, u_1\sin\varphi)$$

where

$$\sigma(s) = \frac{-c^2 - u_1^2(-m_1 s + m_2 c)^2}{2cu_1},$$

for $m_1, m_2 = \pm 1$ and c, $\varphi \in \mathbb{R}$.

Proof. Assume that $x(s) = (x_1(s), x_2(s), x_3(s))$ is a general helix with asymptotic orthonormal frame $\{x(s),\ \alpha(s),\ y(s)\}$ in $\mathbb{Q}^2$ parametrized by arc-length s with the constant curvature $\kappa = 0$ and the slope axis of $x(s)$ is a lightlike vector $U = (u_1, u_2, u_3)$ in $\mathbb{E}_1^3$. Then we can write

$$\begin{aligned} g(x,x) &= -x_1^2 + x_2^2 + x_3^2 = 0, \\ g(U,U) &= -u_1^2 + u_2^2 + u_3^2 = 0. \end{aligned}$$

Thus, we can set

$$\begin{aligned} x_2 &= m_0 x_1 \cos\theta, \\ x_3 &= m_0 x_1 \sin\theta, \end{aligned} \tag{26}$$

and

$$\begin{aligned} u_2 &= u_1 \cos\varphi, \\ u_3 &= u_1 \sin\varphi. \end{aligned} \tag{27}$$

Since $x(s)$ is a general helix, we have

$$g(U,x) = -u_1 x_1(s) + u_2 x_2(s) + u_3 x_3(s) = c \tag{28}$$

where $c \in \mathbb{R}/\{0\}$. By using (26) and (27) in (28), we find,

$$\cos(\theta - \varphi) = \frac{u_1 x_1 + c}{m_0 x_1 u_1}, \tag{29}$$

and

$$\sin(\theta - \varphi) = \frac{\sqrt{-c^2 - 2u_1 x_1 c}}{m_0 x_1 u_1} \tag{30}$$

Considering (29) and (30) together, we obtain

$$\begin{aligned} \cos\theta &= \frac{u_1 x_1 + c}{m_0 x_1 u_1} \cos\varphi - \frac{\sqrt{-c^2 - 2u_1 x_1 c}}{m_0 x_1 u_1} \sin\varphi, \\ \sin\theta &= \frac{u_1 x_1 + c}{m_0 x_1 u_1} \sin\varphi + \frac{\sqrt{-c^2 - 2u_1 x_1 c}}{m_0 x_1 u_1} \cos\varphi. \end{aligned}$$

Thus we get

$$\begin{aligned} x_2(s) &= \frac{u_1 x_1 + c}{u_1} \cos\varphi - \frac{\sqrt{-c^2 - 2u_1 x_1 c}}{u_1} \sin\varphi, \\ x_3(s) &= \frac{u_1 x_1 + c}{u_1} \sin\varphi + \frac{\sqrt{-c^2 - 2u_1 x_1 c}}{u_1} \cos\varphi. \end{aligned} \tag{31}$$

Differentiating (31) and using $g(x', x') = 1$, we obtain

$$x_1(s) = \frac{-c^2 - u_1^2 \left(-m_1 s + m_2 c\right)^2}{2cu_1}$$

where $m_1, m_2 = \pm 1$. This completes the proof.

Example 3. *If we take $c = m_1 = 1, m_2 = 0, u_1 = \sqrt{2}, \varphi = \frac{\pi}{4}$ in theorem 6, we find the curve $x(s) = (x_1(s), x_2(s), x_3(s))$ as follows*

$$\begin{aligned} x_1(s) &= \frac{-1 - 2s^2}{2\sqrt{2}}, \\ x_2(s) &= \frac{1 - 2s^2 - 2\sqrt{2}s}{4}, \\ x_3(s) &= \frac{1 - 2s^2 + 2\sqrt{2}s}{4} \end{aligned}$$

Also the constant lightlike slope axis is $U = \left(\sqrt{2}, 1, 1\right)$ and $g(x(s), U) = 1$.

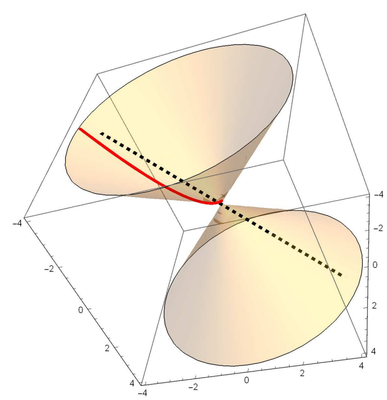

Figure 3: In this graphic, the dashed line represents the slope axis of the helix.

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