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FEKETE-SZEGÖ PROPERTIES FOR QUASI-SUBORDINATION CLASS OF COMPLEX ORDER DEFINED BY Q−DERIVATIVE OPERATOR

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ABSTRACT. In this paper, we introduce the class $W^{\lambda}_{Q,b,q}(\varphi,\psi)$ of quasi-subordination defined by $q-$derivative operator, and obtain the Fekete-Szegö coefficient bounds for functions belonging to this class.

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1. Introduction

Denote by $\mathcal{A}$ the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ \ (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) . \tag{1}$$

For $g(z) \in \mathcal{A}$, given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (convolution) of $f(z)$ and $g(z)$ is defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{2}$$

Definition 1 ([22],[30]). The the function $f(z)$ is said to be quasi subordinate to $g(z)$ if there exists two analytic functions $\varphi(z)$ and $\omega(z)$, where $|\varphi(z)| \leq 1, |\omega(z)| < 1$ and $\omega(0) = 0$ such that $f(z) = \varphi(z)g(\omega(z))$ and written by

$$f(z) \prec_Q g(z).$$

We note that:

$f(z)$ is subordinate to $g(z)$ and written $f(z) \prec g(z)$, when $\varphi(z) = 1$, and $f(z) = g(\omega(z))$ (see [9],[24]). Also, $f(z)$ is majorized by $g(z)$ and written $f(z) << g(z)$, when $\omega(z) = z$, and $f(z) = \varphi(z)g(z)$ (see [13], [14], [27], [35]).

It is observed that quasi subordination is a generalization of subordination as well as majorization see ([3],[19],[29]) for works related to quasi subordination.

Let $\psi(z)$ be univalent starlike function with positive real part on $\mathbb{U}$, which satisfies $\psi(0) = 1$ and $\psi'(0) > 0$. Denote by $\wp$ the class of these functions.

For $f(z) \in \mathcal{A}$, given by (1) and $0 < q < 1$, the Jackson's $q-$derivative of a function f is given by [16] (see also [4],[5],[7],[8],[31],[34],[36],[37])

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)\,z}, (z \in \mathbb{U}, 0 < q < 1)\,, \tag{3}$$

$D_q f(0) = f^{'}(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (3) we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}(0 < q < 1). \tag{5}$$

As $q \to 1^-, [k]_q \to k$ and, so $D_q f(z) = f'(z)$.

According to the definition of quasi subordination and $q-$derivative operator, we introduce the following class:

Definition 1. If $f(z) \in \mathcal{A}$, $\lambda \geq 0, b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $0 < q < 1$, then f is said to be in the class $W^{\lambda}_{Q,b,q}(\varphi, \psi)$, if it satisfies the following quasi subordination condition:

$$1 + \frac{1}{b}\left[(1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q(zD_q f(z))}{D_q f(z)} - 1\right] \prec_Q \psi(z). \tag{6}$$

Specializing the parameters $\lambda, b, q,$ and $\psi(z)$, we obtain the following subclasses:

(i)$W^{\lambda}_{Q,b,q}(1, \psi) = H^{\lambda}_{q,b}(\psi)$ (see Aouf et al. [6]),

(ii)$W^{\lambda}_{Q,1,q}(\varphi, \psi) = W^{\lambda}_{Q,q}(\varphi, \psi)$:

$$\left\{ f \in \mathcal{A} : (1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q(zD_q f(z))}{D_q f(z)} \prec_Q \psi(z) \right\},$$

(iii)$W^{0}_{Q,b,q}(\varphi, \psi) = W_{Q,b,q}(\varphi, \psi)$:

$$\left\{ f \in \mathcal{A} : 1 + \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} - 1\right] \prec_Q \psi(z) \right\},$$

(iv)$W^{1}_{Q,b,q}(\varphi,\psi)=\mathfrak{W}_{Q,b,q}(\varphi,\psi)$:

$$\left\{ f\in\mathcal{A}: 1+\frac{1}{b}\left[\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]\prec_Q \psi(z)\right\},$$

(v)$\lim_{q\to1^-} W^{0}_{Q,b,q}(\varphi,\psi)=S^{*}_{Q,b}(\varphi,\psi)$ and $\lim_{q\to1^-} W^{1}_{Q,b,q}(\varphi,\psi)=C^{*}_{Q,b}(\varphi,\psi)$ (see [11]),

(vi) $\lim_{q\to1^-} W^{\lambda}_{Q,1,q}(\varphi,\psi)=W^{\lambda}_{Q}(\varphi,\psi)$:

$$\left\{ f\in\mathcal{A}: (1-\lambda)\frac{zf'(z)}{f(z)}+\lambda(1+\frac{zf''(z)}{f'(z)})\prec_Q \psi(z)\right\},$$

(vii)$\lim_{q\to1^-} W^{0}_{Q,1,q}(\varphi,\psi)=S^{*}_{Q}(\varphi,\psi)$ and $\lim_{q\to1^-} W^{1}_{Q,1,q}(\varphi,\psi)=C_{Q}(\varphi,\psi)$ (see [25]),

(viii)$\lim_{q\to1^-} W^{0}_{Q,1,q}(1,\psi)=S^{*}(\psi)$ and $\lim_{q\to1^-} W^{1}_{Q,1,q}(1,\psi)=C^{*}(\psi)$ (see [20]),

(x)$\lim_{q\to1^-} W^{0}_{Q,(1-\rho)\cos\gamma e^{-i\gamma},q}(\varphi,\psi)=S_Q(\rho,\gamma,\varphi,\psi)(0<\rho\leq1,|\gamma|<\frac{\pi}{2})$ and $\lim_{q\to1^-} W^{1}_{Q,(1-\rho)\cos\gamma e^{-i\gamma},q}(\varphi,\psi)=C_Q(\rho,\gamma,\varphi,\psi)$ (see [11]),

(xi) $\lim_{q\to1^-} W^{\lambda}_{Q,1,q}(1,\psi)=W^{\lambda}(\psi)$ (see [2]),

(xii)$W^{0}_{Q,b,q}(1,\psi)=\mathfrak{F}_{b,q}(\psi)$ and $W^{1}_{Q,b,q}(1,\psi)=\mathfrak{T}_{b,q}(\psi)$ (see [31]),

(xiii)$\lim_{q\to1^-} W^{0}_{Q,b,q}(1,\psi)=S^{*}_{b}(\psi)$ and $\lim_{q\to1^-} W^{1}_{Q,b,q}(1,\psi)=C_{b}(\psi)$ (see [28]),

(xiv)$\lim_{q\to1^-} W^{\lambda}_{Q,b,q}(1,\psi)=W^{\lambda}_{b}(\psi)$:

$$\left\{ f\in\mathcal{A}: 1+\frac{1}{b}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda(1+\frac{zf''(z)}{f'(z)})-1\right]\prec \psi(z)\right\},$$

(xv)$W^{\lambda}_{Q,1,q}(1,\psi)=W^{\lambda}_{q}(\psi)$:

$$\left\{ f\in\mathcal{A}: (1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}\prec \psi(z)\right\},$$

(xvi)$W^{\lambda}_{Q,(1-\rho)\cos\gamma e^{-i\gamma},q}(\varphi,\psi)=W^{\lambda}_{Q,q}(\rho,\gamma,\varphi,\psi)(0\leq\rho<1,|\gamma|<\frac{\pi}{2})$:

$$\left\{ f\in\mathcal{A}: e^{i\gamma}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}\right]\prec_Q \psi(z)(1-\rho)\cos\gamma+\rho\cos\gamma+i\sin\gamma\right\},$$

(xvii)$\lim_{q\to1^-} W^{\lambda}_{Q,(1-\rho)\cos\gamma e^{-i\gamma},q}(\varphi,\psi)=W^{\lambda}_{Q}(\rho,\gamma,\varphi,\psi)(0\leq\rho<1,|\gamma|<\frac{\pi}{2})$:

$$\left\{ f\in\mathcal{A}: e^{i\gamma}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda(1+\frac{zf''(z)}{f'(z)})\right]\prec_Q \psi(z)(1-\rho)\cos\gamma+\rho\cos\gamma+i\sin\gamma\right\}.$$

Let Ω be the class of analytic functions of the form

$$\omega(z) = \omega_1 z + \omega_2 z^2 + ..., \qquad |\omega| < 1. \tag{7}$$

In order to prove our results, we need the following lemma.

Lemma 1. [18] Let $\omega \in \Omega$, then for any complex number τ we have

$$|\omega_1| \le 1, \quad \left|\omega_2 - \tau\omega_1^2\right| \le 1 + (|\tau| - 1)\,|\omega_1|^2 \le \max\{1, |\tau|\}. \tag{8}$$

The result is sharp for the function

$$\omega(z) = z \text{ when } |\tau| \ge 1 \text{ and } \omega(z) = z^2 \text{ when } |\tau| < 1 .$$

Srivastava et al. [32] discussed Fekete-Szegö properties for classes of starlike, convex, and close-to convex functions. see also ([1],[12],[15],[17],[23],[21],[26]). The purpose of this paper is to obtain the Fekete-Szegö properties for the class $W^{\lambda}_{Q,b,q}(\varphi, \psi)$.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $\omega(z)$ of the form (7), $\varphi(z) = c_0 + c_1 z + c_2 z^2 + ...$, $\psi(z) = 1 + B_1 z + B_2 z^2 + ...$, $\lambda \ge 0, 0 < q < 1, \psi \in \wp, b \in \mathbb{C}^*, B_1 > 0, \mu \in \mathbb{C}, c_0 \ne 0$ and $z \in \mathbb{U}$.

Using Lemma 1, we have the following theorem:

Theorem 1. Let $f(z) \in W^{\lambda}_{Q,b,q}(\varphi, \psi)$. Then

$$|a_2| \le \frac{|b|B_1}{([2]_q - 1)[\lambda([2]_q - 1) + 1]}, \tag{9}$$

$$|a_3| < \frac{|b|B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{1 + \max\left[1, \frac{|B_2|}{B_1} + \frac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}|b|B_1\right]\right\} \tag{10}$$

and

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1|b|}{(1-[3]_q)[\lambda(1-[3]_q)-1]}\left\{1 + \max\left[1, \frac{|B_2|}{B_1} + \left(\frac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}\right)|b|B_1\right]\right\}. \tag{11}$$

The result is sharp.

Proof. If $f(z) \in W^{\lambda}_{Q,b,q}(\varphi, \psi)$, then there exist analytic functions $\varphi(z)$ and $\omega(z)$ with $|\varphi(z)| < 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \tfrac{1}{b}\left[(1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda\frac{D_q(zD_q f(z))}{D_q f(z)} - 1\right] = \varphi(z)[\psi(\omega(z)) - 1]. \tag{12}$$

We have

$$\begin{aligned} & \tfrac{1}{b}\left[(1-\lambda)\tfrac{zD_qf(z)}{f(z)}+\lambda\tfrac{D_q(zD_qf(z))}{D_qf(z)}-1\right] \\ = & \tfrac{1}{b}([2]_q-1)[\lambda([2]_q-1)+1]a_2z+\tfrac{1}{b}\{([3]_q-1)[\lambda([3]_q-1)+1]a_3 \\ & -\{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]\}a_2^2\}z^2+... \end{aligned} \tag{13}$$

Also,

$$\psi(\omega(z))-1=B_1\omega_1z+(B_1\omega_1+B_2\omega_1^2)z^2+...$$

and

$$\varphi(z)[\psi(\omega(z))-1]=c_0B_1\omega_1z+[c_0B_1\omega_2+c_0B_2\omega_1^2+c_1B_1\omega_1]z^2+... \tag{14}$$

Substituting (13) and (14) in (12) and equating the coefficients of z and z^2 , we get

$$a_2=\frac{bc_0B_1\omega_1}{([2]_q-1)[\lambda([2]_q-1)+1]},$$

$$a_3=\tfrac{bc_0B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{\omega_2+\omega_1\frac{c_1}{c_0}+\omega_1^2\left[\tfrac{|B_2|}{B_1}+\tfrac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}bc_0B_1\right]\right\}$$

and

$$\begin{aligned} a_3-\mu a_2^2 = & \tfrac{bc_0B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{\omega_2+\omega_1\tfrac{c_1}{c_0}+\omega_1^2\left[\tfrac{|B_2|}{B_1}\right.\right. \\ & \left.\left.+\tfrac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}bc_0B_1\right]\right\}, \end{aligned}$$

then

$$\begin{aligned} |a_3-\mu a_2^2| \leq & \tfrac{|b|B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{\omega_1|\tfrac{c_1}{c_0}|+\left|\omega_2+\omega_1^2\left[\tfrac{|B_2|}{B_1}\right.\right.\right. \\ & \left.\left.\left.+\tfrac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}bc_0B_1\right]\right|\right\}. \end{aligned}$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{U}$, we have (see [28])

$$|c_n|\leq 1-|c_0|^2\leq 1 \quad (n>0),$$

and from (8), $|\omega_1|\leq 1$, then

$$\begin{aligned} |a_3-\mu a_2^2| \leq & \tfrac{|b|B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{1+\left|\omega_2+\omega_1^2\left[\tfrac{|B_2|}{B_1}\right.\right.\right. \\ & \left.\left.\left.+\tfrac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}bc_0B_1\right]\right|\right\}. \end{aligned}$$

Our result now follows by using Lemma 1. The result is sharp for the functions

$$\frac{1}{b}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]=\varphi(z)[\psi(z)-1]$$

and

$$\frac{1}{b}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]=\varphi(z)[\psi(z^2)-1].$$

This completes the proof of Theorem 1. ∎

Putting $\lambda=0$ in Theorem 1, we get the following corollary:

Corollary 1. Let $f(z)\in W_{Q,b,q}(\varphi,\psi)$, Then

$$|a_2|\leq\frac{|b|B_1}{([2]_q-1)},$$

$$|a_3|<\tfrac{|b|B_1}{([3]_q-1)}\left\{1+\max\left[1,\frac{|B_2|}{B_1}+([2]_q-1)|b|B_1\right]\right\}$$

and

$$\left|a_3-\mu a_2^2\right|\leq\tfrac{B_1|b|}{([3]_q-1)}\left\{1+\max\left[1,\frac{|B_2|}{B_1}+\left(\tfrac{([2]_q-1)-\mu([3]_q-1)}{([2]_q-1)^2}\right)|b|B_1\right]\right\}.$$

Putting $\lambda=1$ in Theorem 1, we get the following corollary:

Corollary 2. Let $f(z)\in\mathfrak{W}_{Q,b,q}(\varphi,\psi)$, Then

$$|a_2|\leq\frac{|b|B_1}{([2]_q-1)[2]_q},$$

$$|a_3|<\tfrac{|b|B_1}{([3]_q-1)[3]_q}\left\{1+\max\left[1,\frac{|B_2|}{B_1}+\tfrac{([2]_q-1)-[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[2]_q^2}|b|B_1\right]\right\}$$

and

$$\left|a_3-\mu a_2^2\right|\leq\tfrac{B_1|b|}{([3]_q-1)[3]_q}\left\{1+\max\left[1,\tfrac{|B_2|}{B_1}+\tfrac{|([2]_q-1)-[([2]_q-[2]_q^2+1)[2]_q-1]-\mu[3]_q([3]_q-1)|}{([2]_q-1)^2[2]_q^2}|b|B_1\right]\right\}.$$

Theorem 2. If $f(z)$ satisfies

$$\frac{1}{b}\left[(1-\lambda)\frac{zD_qf(z)}{f(z)}+\lambda\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]<<\psi(z)-1.$$

Then

$$|a_2|\leq\frac{|b|B_1}{([2]_q-1)[\lambda([2]_q-1)+1]},$$

$$|a_3| < \frac{|b|B_1}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{1+\frac{|B_2|}{B_1}+\frac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}|b|B_1\right\}$$

and

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1|b|}{([3]_q-1)[\lambda([3]_q-1)+1]}\left\{1+\frac{|B_2|}{B_1}+\frac{\left|([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]\right|}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}|b|B_1\right\}.$$

The result is sharp.

Proof. The result follows by taking $\omega(z)=z$ in the proof of Theorem 1. ∎

Putting $\lambda=0$ in Theorem 2, we get the following corollary:
Corollary 3. If $f(z)$ satisfies

$$\frac{1}{b}\left[\frac{zD_qf(z)}{f(z)}-1\right] << \psi(z)-1.$$

Then

$$|a_2| \le \frac{|b|B_1}{([2]_q-1)},$$

$$|a_3| < \frac{|b|B_1}{([3]_q-1)}\left\{1+\frac{|B_2|}{B_1}+([2]_q-1)|b|B_1\right\}$$

and

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1|b|}{([3]_q-1)}\left\{1+\frac{|B_2|}{B_1}+\frac{|([2]_q-1)-\mu([3]_q-1)|}{([2]_q-1)^2}|b|B_1\right\}.$$

Putting $\lambda=1$ in Theorem 2, we get the following corollary:
Corollary 4. If $f(z)$ satisfies

$$\frac{1}{b}\left[\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right] << \psi(z)-1.$$

Then

$$|a_2| \le \frac{|b|B_1}{([2]_q-1)[2]_q},$$

$$|a_3| < \frac{|b|B_1}{([3]_q-1)[3]_q}\left\{1+\frac{|B_2|}{B_1}+\frac{([2]_q-1)-[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[2]_q^2}|b|B_1\right\}$$

and

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1|b|}{([3]_q-1)[3]_q}\left\{1+\frac{|B_2|}{B_1}+\frac{\left|([2]_q-1)-[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[3]_q\right|}{([2]_q-1)^2[2]_q^2}|b|B_1\right\}.$$

Taking $\varphi(z)=1$, in Theorem 1. we obtain

Theorem 3. If $f(z) \in W_{b,q}^{\lambda}(\psi)$, Then

$$|a_2| \leq \frac{|b|B_1}{([2]_q - 1)[\lambda([2]_q - 1) + 1]},$$

$$|a_3| < \frac{|b|B_1}{([3]_q-1)[\lambda([3]_q-1)+1]} \max\left\{1, \frac{|B_2|}{B_1} + \frac{([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}|b|B_1\right\}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1|b|}{([3]_q-1)[\lambda([3]_q-1)+1]} \max\left\{1, \frac{|B_2|}{B_1} + \frac{\left|([2]_q-1)-\lambda[([2]_q-[2]_q^2+1)[2]_q-1]-\mu([3]_q-1)[\lambda([3]_q-1)+1]\right|}{([2]_q-1)^2[\lambda([2]_q-1)+1]^2}|b|B_1\right\}.$$

The result is sharp.

Remarks:

(1) For $b = 1, q \to 1^-$ and $\lambda = 0$ in Theorem 1, we obtain the result obtained by (see [[25] ,Theorem 2.1]);

(2) For $b = \lambda = 1$ and $q \to 1^-$ in Theorem 1, we obtain the result obtained by (see [[25] ,Theorem 2.4]);

(3) For $\varphi(z) = 1, q \to 1^-$ and $\lambda = 0$ in Theorem 1, we obtain the result obtained by (see [[28] ,Theorem 4.1]);

(4) For $b = 1, q \to 1^-$ and $\varphi(z) = 1$ in Theorem 3, we obtain the result obtained by (see [[10] ,Theorem 2.10, with $k = 1$]);

(5) For $b = 1$ and $q \to 1^-$ in Theorem 3, we obtain the result obtained by (see [[25] ,Theorem 2.12]) and Ali et al. (see [[2] , Theorem 2.9, with $k = 1$]);

(6) For $q \to 1^-$ and $\lambda = 0$ in Theorem 3, we obtain the result obtained by Suchithra et al. (see [[33] ,Theorem 2.1, with $\alpha = 0$]).

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