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COMPARISON OF VARIOUS RISK MEASURES FOR AN OPTIMAL PORTFOLIO

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Abstract. In this paper, we search for optimal portfolio strategies in the presence of various risk measure that are common in financial applications. Particularly, we deal with the static optimization problem with respect to Value at Risk, Expected Loss and Expected Utility Loss measures. To do so, under the Black-Scholes model for the financial market, Martingale method is applied to give closed-form solutions for the optimal terminal wealths; then via representation problem the optimal portfolio strategies are achieved. We compare the performances of these measures on the terminal wealths and optimal strategies of such constrained investors. Finally, we present some numerical results to compare them in several respects to give light to further studies.

2010 *Mathematics Subject Classification*: 65C60, 65K10, 62-07, 60H30.

Keywords: portfolio optimization, value at risk, expected loss, expected utility loss, black-scholes model, martingale method, risk constraints

1. Introduction

Harry Markowitz, who is the pioneer of the modern portfolio theory, mentioned about trading off the mean return of a portfolio against its variance in his works (see [20, 21]). In order to solve the portfolio optimization problem, Robert C. Merton presented the concept of Itô calculus with methods of continuous-time stochastic optimal control in two works (see [22, 23]) and when the utility function is a power function or the logarithm, he produced solutions to both finite and infinite-horizon models (see [22]). Harrison and Kreps [12] constituted portfolios from martingale representation theorems and started the modern mathematical approach to portfolio management in complete markets, which were built around the ideas of martingale measures. Harrison and Pliska (see [13, 14]) improved this subject much more in the context of the option pricing. The martingale ideas to utility maximization problems were adapted by Pliska [24], Cox and Huang [5, 6], and Karatzas, Lehoczky and, Shreve [15]. You can further examine about these developments in Karatzas and Shreve [17].

In this paper, we investigate optimal strategies for portfolios consisting of only one risky stock and one risk-free bond. This study can easily be generalized to the multi-dimensional Black-Scholes model with $d > 1$ risky stocks. We assume that an investor in this economy has some initial wealth at time zero and there is a finite planning horizon $[0, T]$ that is given. The goal of this investor is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the wealth invested in stock and bond. We assume continuous-time market which allows for permanent trading and re-balancing the portfolio, and we have to find these proportions for every time t to T. Also, we allow the short selling of the stock, which is the selling of a stock that the seller doesn't own, but is promised to be delivered.

Karatzas, Lehoczky, and Shreve [15] and also Cox and Huang [5] solved the utility maximization problem without additional limitations by using martingale approach in the context of the Black-Scholes model of a complete market. Also, the works of Karatzas et al. [16] is an extension of the solution to should be examined for the case of an incomplete market.

We consider shares of a stock and a risk-free bond whose prices follow a geometric Brownian motion in this portfolio. We can obtain the maximum expected utility of the terminal wealth by following the optimal portfolio strategy. However, since the terminal wealth is a random variable with a distribution which is often extremely skew, it shows considerable probability in regions of small values of the terminal wealth. Namely, the optimal terminal wealth may exhibit large shortfall risks. By the term shortfall risk, we indicate the event that the terminal wealth may fall below a given deterministic threshold value, namely, the initial capital or the result of an investment in a pure bond portfolio.

It is necessary to quantify shortfall risks by using appropriate risk measures in order to incorporate such shortfall risks into the optimization. We denote the terminal wealth of the portfolio at time $t = T$ by X_T and let $q > 0$ be threshold value or shortfall level. Then the shortfall risk consists in the random event $\{X_T < q\}$ or $\{Z = X_T - q < 0\}$ and we assign to the random variable (risk) Z the real number $\rho(Z)$ which will be called a *risk measure*.

Therefore, the idea is to restrict the probability of a shortfall:

$$\rho_1(Z) = P(Z < 0) = P(X_T < q).$$

This corresponds to the concept of Value at Risk (VaR) [4], defined by

$$\mathrm{VaR}_\varepsilon(Z) = \inf\{l \in \mathbb{R} : P(Z > l) \leq \varepsilon\},$$

where l can be interpreted such that given $\varepsilon \in (0,1)$, the VaR of the portfolio at the confidence level $1-\varepsilon$ is given by the smallest number l such that the probability that the loss Z exceeds l is at most ε. Although it virtually always represents a loss, VaR is conventionally reported as a positive number. A negative VaR would imply that the portfolio may make a profit. VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. This risk measure is widely used by banks, securities firms, commodity and energy merchants, and other trading organizations. However, VaR risk managers often optimally choose a larger exposure to risky assets than non-risk managers and consequently incur larger losses when losses occur.

In order to remedy the shortcomings of VaR, an alternative risk-management model is suggested, which is based on the expectation of a loss. This alternative model is called as Expected Loss. This risk management maintains limited expected losses when losses occur. You can see risk management objectives which are embedded into utility maximization problem using Value at Risk (VaR) and Expected Loss (EL), for instance in [8, 11]. The EL risk measure is defined by

$$\rho_2(Z) = \mathrm{EL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(X_T - q)^-\right],$$

and it is bounded by a given $\varepsilon > 0$.

As the aim of the portfolio manager is to maximize the expected utility from the terminal wealth, one may also consider the portfolio optimization problem where the portfolio manager is confronted with a risk measured by a constraint of the type

$$\rho_3(Z) = \mathrm{EUL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(u(X_T) - u(q))^-\right] \le \varepsilon,$$

where $\varepsilon > 0$ is a given bound for the Expected Utility Loss (EUL) [10]. Here u denotes the utility function. This risk constraint causes to more explicit calculations for the optimal strategy we are looking for. Also, it allows to the constrained static problem to be solved for a large class of utility functions.

Alternatively, Artzner et al. (1999) [1] and Delbaen (2002) [7] introduced the concept of coherent measures and you can find further risk measures in the class of coherent measures. These measures have the properties of monotonicity, subadditivity, positive homogeneity and the translation invariance property. However, VaR, EL, EUL risk measures do not belong to this class: VaR is not sub-additive, and EL and EUL do not satisfy the translation invariance property.

Here we examine the effects of risk management on optimal terminal wealth choices and on optimal portfolio policies. We consider portfolio managers or investors as expected utility maximizers, who derive utility from wealth at horizon and who must comply with different risk constraints imposed at that horizon.

2. Portfolio optimization under constraints

In this section, we consider the portfolio optimization problem with constraints that are Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) with objective to maximize the expected utility of the terminal wealth. When we discuss these situations, we shall take into account that the terminal wealth X_T may fall below a given deterministic shortfall level q. Also, we will examine the impact of the different risk constraints to the behavior of the portfolio manager.

2.1. Portfolio optimization under Value at Risk constraint

In this section, the portfolio optimization problem is solved by using a Value at Risk constraint, and then the properties of the solution are examined.

The dynamic optimization problem of the VaR investor is solved by using the martingale representation method [5, 15], which allows the problem to be restated as the following static variational problem:

$$\begin{aligned} &\underset{\xi \in B(x)}{\text{maximize}}\ \mathbb{E}\left[u(\xi)\right] \\ &\text{subject to } P(\xi < q) \leq \varepsilon. \end{aligned} \tag{1}$$

The set $B(x)$ contains the budget constraint for the initial capital x. Namely,

$$B(x) = \left\{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T - \text{measurable and } \mathbb{E}\left[H_T \xi\right] \leq x\right\}.$$

The VaR constraint causes to non-concavity for the optimization problem for which the maximization process is more complicated. The following proposition is proved in Basak and Shapiro [2]; it defines the optimal terminal wealth, assuming it exists.

Proposition 1 ([2]). *Time-T optimal wealth of the* VaR *investor is*

$$\xi^{\mathrm{VaR}} = \begin{cases} I(yH_T), & \text{if } H_T < \underline{h}, \\ q, & \text{if } \underline{h} \leq H_T < \overline{h}, \\ I(yH_T), & \text{if } \overline{h} \leq H_T, \end{cases} \tag{2}$$

where I is the inverse function of u', $\underline{h} = \frac{u'(q)}{y}$, $\overline{h}$ is such that $P(H_T > \overline{h}) = \varepsilon$, and $y \geq 0$ solves $\mathbb{E}\left[H_T \xi^{\mathrm{VaR}}\right] = x$.

The VaR *constraint ($P(\xi < q) \leq \varepsilon$) is binding if, and only if, $\underline{h} < \overline{h}$.*

Basak and Shapiro [2] prove that if a terminal wealth satisfies (2) then it is the optimal policy for the VaR portfolio manager. As they note in their proof, to keep the focus, they do not provide general conditions for existence. However, they provide explicit numerical solutions for a variety of parameter values. Their method of proof is applicable to other problems, such as those with non-standard preferences. By the term "non-standard preferences" it means that the optimization problem is not standard because it is non-concave. Also, because the VaR constraint must hold with equality, the definition of $\overline{h}$ is deduced.

We depict in Fig. 1 the optimal terminal wealth of a VaR portfolio manager with $\varepsilon \in (0, 1)$, a benchmark (unconstrained) investor with $\varepsilon = 1$ who does not use a risk constraint in the optimization or ignores large losses, and a portfolio insurer with $\varepsilon = 0$ who does not allow large losses but fully insures himself against large losses.

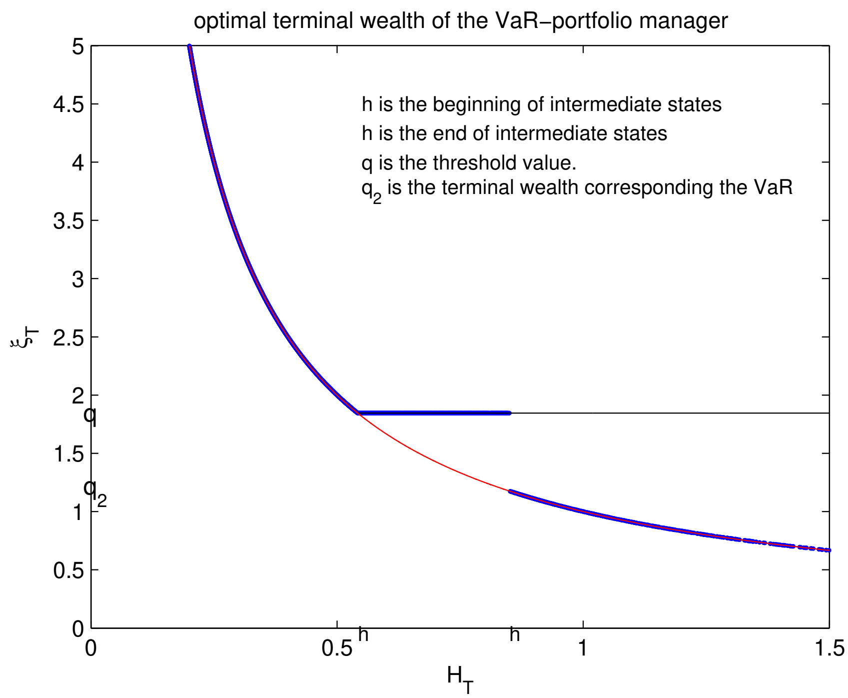

Figure 1: Optimal horizon wealth of the VaR risk manager

The blue curve, in Fig. 1, plots the optimal horizon wealth of the VaR risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor. Furthermore, here we note that q_2 is defined by

$$q_2 = \begin{cases} I(y\overline{h}), & \text{if } \underline{h} < \overline{h}, \\ q, & \text{otherwise.} \end{cases} \tag{3}$$

The VaR portfolio manager's optimal horizon wealth is divided into three distinct regions, where he displays distinct economic behaviors. In the good states, namely low price of consumption $H_T < \underline{h}$, the VaR portfolio manager behaves like a benchmark (unconstrained) investor. In the intermediate states $[\underline{h} \leq H_T < \overline{h}]$, he insures himself against losses by behaving like a portfolio insurer investor, and in the bad states, namely high price of consumption $H_T > \overline{h}$ he is completely uninsured by incurring all losses. Because he is only concerned with the probability (and not the magnitude) of a loss, the VaR portfolio manager chooses to leave the worst states uninsured because they are the most expensive ones to insure against. The measure of these bad states is chosen to comply exactly with the VaR constraint. Consequently, $\overline{h}$ depends solely on ε and the distribution of H_T and is independent of the investor's preferences and initial wealth. The investor can be considered as one who ignores losses in this upper tail of the H_T distribution, where the consumption is the most costly.

When we take into account Fig. 1, we can examine the dependence of the solution on the parameters q and ε. If the threshold value q is increased, more states need to be insured against, and the intermediate region grows at the expense of the good states region. Accordingly, the wealth in both good and bad regions must decrease to meet the bigger threshold value q in the intermediate region. When ε increases, namely, when the investor is allowed to make a loss with higher probability, the intermediate, insured region can shrink, and the good and bad regions both can grow. The investor's horizon wealth can increase in both the good and bad states because he is not required to insure against losses in a large state. The solution reveals that when a large loss occurs, it may be an even larger loss under the VaR constraint, and hence more likely to cause to credit problems. Basak and Shapiro show this situation in [2] and presented by the following proposition.

Proposition 2 ([2]). *Assume* $u(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}$, $\gamma > 0$. *For a given terminal wealth* ξ_T, *define the following two measures of loss:* $L_1(\xi) = \mathbb{E}\left[(q_2 - \xi_T)\mathbb{1}_{\{\xi_T \leq q_2\}}\right]$ *and* $L_2(\xi) = \mathbb{E}\left[\frac{H_T}{H_0}(q_2 - \xi_T)\mathbb{1}_{\{\xi_T \leq q_2\}}\right]$. *Then,*

(i) $L_1(\xi^{\mathrm{VaR}}) \geq L_1(\xi^*)$, *and*

(ii) $L_2(\xi^{\mathrm{VaR}}) \geq L_2(\xi^*)$,

where ξ^* *stands for the solution of the unconstrained (benchmark) problem.*

Proposition 2 shows explicitly that under the VaR constraint the expected extreme losses are higher than those which are incurred by an investor who does not use the VaR constraint ($P(\xi < q) \leq \varepsilon$). The bad states, which are the states of large losses, are considered: $L_1(\xi)$ measures the expected future value of a loss, when there is a large loss, while $L_2(\xi)$ measures its present value.

Although the aim of using VaR approach in the optimization is to prevent large and frequent losses that may cause economic investors out of business, under the VaR constraint losses are not frequent, however, the largest losses are more severe than without the VaR constraint.

Remark 1. *The most frequently used utility function is the power utility function*

$$u(z) = \begin{cases} \frac{z^{1-\gamma}}{1-\gamma}, & \gamma \in (0,\infty) \setminus \{1\}, \\ \ln z, & \gamma = 1. \end{cases} \tag{4}$$

With positive first derivative and negative second derivative, the power utility function (4) *meets the requirement of risk averse investor who prefers more than less wealth. The parameter γ of the power utility function can be interpreted as constant relative risk aversion.*

In his study, Gabih [10] presents explicit expressions for the VaR portfolio manager's optimal wealth and portfolio strategies before the horizon in the following proposition.

Proposition 3 ([10])**.** *Let the assumptions of Proposition 1 be fulfilled, and let u be the utility function given as in* (4)*. Then,*

(i) The VaR*-optimal wealth at time $t < T$ before the horizon is given by*

$$X_t^{\mathrm{VaR}} = F(H_t, t), \tag{5}$$

with

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}}\Phi(-d_1(\underline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\underline{h},z,t))\right] \\ &+ \left[\frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}}\Phi(-d_1(\overline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\overline{h},z,t))\right], \end{aligned}$$

for $z > 0$. Here, Φ is the standard-normal distribution function, y, $\underline{h}$ and $\overline{h}$ are as in Proposition 1. Furthermore,

$$\begin{aligned} \Gamma(t) &= \frac{1-\gamma}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right)(T-t), \\ d_1(u,z,t) &= \frac{\ln\frac{u}{z} + \left(r - \frac{\kappa^2}{2}\right)(T-t)}{\kappa\sqrt{T-t}}, \\ d_2(u,z,t) &= d_1(u,z,t) + \frac{1}{\gamma}\kappa\sqrt{T-t}. \end{aligned}$$

(ii) The VaR*-optimal fraction of wealth invested in stock at time $t < T$ before the horizon is*

$$\theta_t^{\mathrm{VaR}} = \theta^N \Theta(H_t, t),$$

where

$$\begin{aligned} \Theta(z,t) &= 1 - \frac{qe^{-r(T-t)}}{F(z,t)} \left[\Phi(-d_2(\underline{h},z,t)) - \Phi(-d_2(\overline{h},z,t))\right] \\ &+ \frac{\gamma}{\kappa\sqrt{T-t}F(z,t)} \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} \left[\varphi(d_1(\underline{h},z,t)) - \varphi(d_1(\overline{h},z,t))\right] \\ &- \frac{\gamma q e^{-r(T-t)}}{\kappa\sqrt{T-t}F(z,t)} \left[\varphi(d_2(\overline{h},z,t)) - \varphi(d_2(\underline{h},z,t))\right], \end{aligned}$$

for $z > 0$. Here, $\theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}$ denotes the normal strategy, $\Theta(H_t, t)$ is the exposure to risky assets relative to the normal (unconstrained) strategy and φ is the density function of the standard normal distribution.

2.2. Portfolio optimization under Expected Loss constraint

In this section, we consider the Expected Loss (EL) strategy as an alternative to the Value at Risk (VaR) strategy. We then solve the optimization problem of an EL portfolio manager who wants to limit his expected loss and analyze the properties of the solution.

The portfolio manager who uses Value at Risk (VaR) constraint does not concern with the magnitude of a loss and is just interested in controlling the probability of the loss. However, if one wants to control the magnitude of losses, he should control (all or some of the) moments of the loss distribution. Therefore, we now focus on controlling the first moment and examine how one can remedy the shortcomings of VaR constraint. In this case, the investor defines his strategy as follows:

$$\mathrm{EL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(X_T - q)^-\right] \leq \varepsilon, \tag{6}$$

where $Z = X_T - q$ and ε is a given bound for the Expected Loss. This strategy will be called EL strategy. Thus, the aim is to solve the optimization problem constrained by (6). Using the martingale representation approach the dynamic optimization problem of the EL-portfolio manager can be restated as the following static problem

$$\begin{aligned} &\underset{\xi \in B(x)}{\text{maximize}}\ \mathbb{E}\left[u(\xi)\right] \\ &\text{subject to } \mathbb{E}\left[(\xi - q)^-\right] \leq \varepsilon. \end{aligned} \tag{7}$$

The EL-constraint (6) can be interpreted as a risk measure of time-T losses. This measure satisfies the sub-additivity, positive homogeneity, and monotonicity axioms (but not the translation-invariance axiom) defined by Artzner et al. [1]. Hence EL risk measure can be thought that it has an advantage about this issue according to the VaR measure of risk: because the VaR strategy fails to display sub-additivity when combining the risk of two or more portfolios, the VaR of the whole portfolio may be greater than the sum of the VaRs of the individuals.

A. Gabih, R. Wunderlich [11] characterize the optimal terminal wealth ξ^{EL} in the presence of the EL-constraint (6) in the following proposition whose proof is based on the following lemma.

Lemma 1 ([11])**.** *Let $z, y_1, y_2, q > 0$. Then the solution of the optimization problem*

$$\max_{x>0}\{u(x) - y_1 z x - y_2(x-q)^-\}$$

is $x^* = \xi^*(z)$.

Now, the following proposition, Proposition 4, states the optimal solution of the static variational problem, concerning the EL constraint.

Proposition 4 ([11])**.** *The* EL-*optimal terminal wealth is*

$$\xi^{\mathrm{EL}} = \begin{cases} I(y_1 H_T), & \text{if } H_T < \underline{h}, \\ q, & \text{if } \underline{h} \le H_T < \overline{h}, \\ I(y_1 H_T - y_2), & \text{if } \overline{h} \le H_T, \end{cases} \tag{8}$$

where $\underline{h} = \underline{h}(y_1) = \frac{u'(q)}{y_1}, \overline{h} = \overline{h}(y_1, y_2) = \frac{u'(q)+y_2}{y_1}$ and $y_1, y_2 > 0$ solve the system of equations,

$$\begin{aligned} \mathbb{E}\left[H_T \xi^{\mathrm{EL}}(T; y_1, y_2)\right] &= x, \\ \mathbb{E}\left[(\xi^{\mathrm{EL}}(T; y_1, y_2) - q)^-\right] &= \varepsilon. \end{aligned}$$

Moreover, the EL-*constraint* (6) *is binding, if and only if, $\underline{h} < \overline{h}$.*

With the following remark of Gabih (2005) [10], the case of how the EL optimal terminal wealth depends on y_2 is explained:

Remark 2. *For $y_2 \downarrow 0$, the situation of $\xi^{\mathrm{EL}} \to I(y_1 H_T)$ is observed. This limit corresponds to $\varepsilon \uparrow \varepsilon_{\max}$ and the results for the unconstrained problem are derived if $y_2 = 0$ and $\xi^{\mathrm{EL}}(y_1, 0) = I(y_1 H_T)$ are set.*

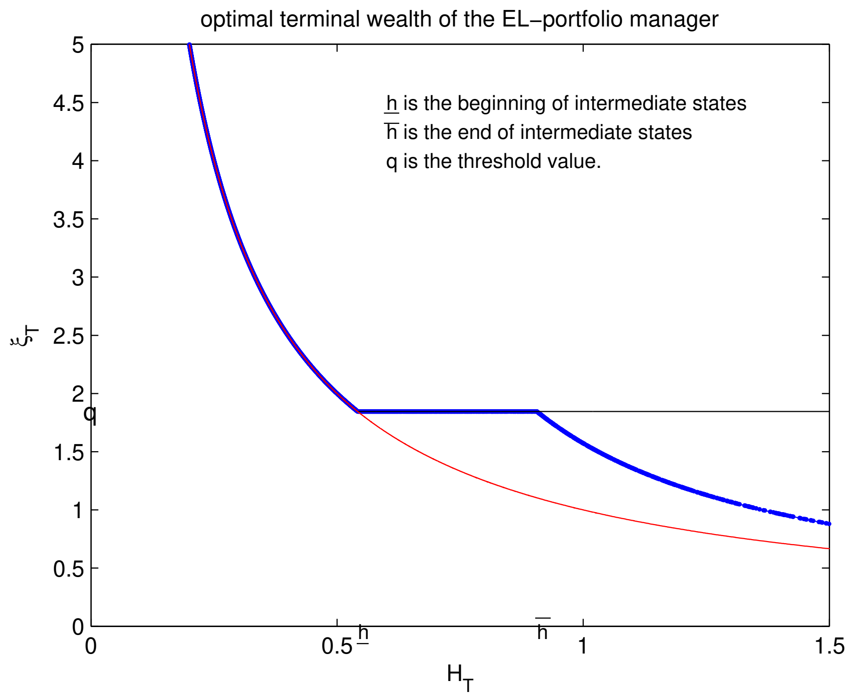

Figure 2: Optimal horizon wealth of the EL risk manager

Fig. 2 depicts the optimal terminal wealth of an EL-portfolio manager [$\varepsilon \in (0, \infty)$], a benchmark (unconstrained) investor ($\varepsilon = \infty$), and a portfolio insurer investor ($\varepsilon = 0$). The blue curve plots the optimal horizon wealth of the EL risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor.

In Fig. 2, we see that the EL portfolio manager's optimal horizon wealth is divided into three distinct regions, where he exhibits distinct economic behaviors: in the so-called "good states" (for low H_T values), the EL portfolio manager behaves like a benchmark (the unconstrained) investor, while in the "intermediate states" (for $\underline{h} \leq H_T < \overline{h}$) the investor fully insures himself against losses by behaving like a portfolio insurer investor (PI), and in the "bad states" (for high H_T values) the investor partially insures himself by incurring partial losses in contrast to the VaR portfolio manager. Here, we see in the bad-states region, $\xi_T^* < \xi_T^{\mathrm{EL}} < \xi_T^{PI}$, where ξ_T^* stands for the solution of the benchmark (unconstrained) problem. This is constituted in contrast to the findings in the VaR case.

Although in some states he wants to settle for a wealth lower than q, he does so while endogenously choosing a higher ξ_T^{EL} than ξ_T^*. The portfolio manager chooses the bad states in which he maintains a loss, because these are the most expensive states to insure against losses, but maintains some level of insurance. Since insuring a terminal wealth at q level is too costly, he sets for less, but enough to comply with the EL constraint. Unlike $\overline{h}$ for VaR strategy, $\overline{h}$ for EL strategy depends on the investor's preferences and the given initial wealth. Another distinction with VaR strategy is that the terminal wealth policy under EL strategy is continuous across the states of the world.

Gabih (2005) [10] presents the explicit expressions for the EL-optimal wealth and portfolio strategy before the horizon via the following proposition.

Proposition 5 ([10])**.** *Let the assumptions of Proposition 4 be fulfilled, and let u be the utility function given in* (4)*. Then,*

(i) The EL*-optimal wealth at time $t < T$ is given by*

$$X_t^{\mathrm{EL}} = F(H_t, t) \tag{9}$$

with

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\left[1 - \Phi(-d_1(\underline{h}, z))\right] \\ &+ q e^{-r(T-t)}\left[\Phi(-d_2(\underline{h}, z)) - \Phi(-d_2(\overline{h}, z))\right] \\ &+ G(z, \overline{h}), \end{aligned}$$

for $z > 0$, where y_1, y_2 are as defined in Proposition 4; $\Gamma(t), d_1, d_2$ are as in Proposition 3; and

$$\begin{aligned} \underline{h} &= \frac{1}{y_1 q^{\gamma}} \text{ and } \overline{h} = \frac{q^{-\gamma} + y_2}{y_1}, \\ G(z, \overline{h}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{c_2(\overline{h}, z)} \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 t e^{a+bu} - y_2)^{\frac{1}{\gamma}}} du, \\ c_2(\overline{h}, z) &= \frac{1}{b}\left(\ln(\frac{\overline{h}}{z}\right) - a), \\ a &= -\left(r + \frac{\kappa^2}{2}\right)(T - t) \text{ and} \\ b &= -\kappa\sqrt{T - t}. \end{aligned}$$

(ii) The EL*-optimal fraction of wealth invested in stock at time $t < T$ is*

$$\theta_t^{\mathrm{EL}} = \theta^N \Theta(H_t, t),$$

where

$$\begin{aligned}\Theta(z,t) &= \frac{1}{F(z,t)}\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\left[1-\Phi(-d_1(\underline{h},z))+\frac{\gamma}{\kappa\sqrt{T-t}}\varphi(d_1(\underline{h},z))\right]\\ &- \frac{q\gamma e^{-r(T-t)}}{F(z,t)\kappa\sqrt{T-t}}\varphi(d_2(\underline{h},z))\\ &+ \frac{y_1 z e^{(\kappa^2-2r)(T-t)}}{F(z,t)}\psi_0\left(c_2(\overline{h},z),b,y_1 z e^{a},y_2,2b,1,1+\frac{1}{\gamma}\right),\end{aligned}$$

for $z>0$ *and*

$$\psi_0(\alpha,\beta,c_1,c_2,m,s,\delta)=\frac{1}{\sqrt{2\pi}s}\int_{-\infty}^{\alpha}\frac{exp(-\frac{(u-m)^2}{2s^2})}{(c_1 e^{\beta u}-c_2)^{\delta}}du.$$

2.3. Portfolio optimization under Expected Utility Loss constraint

In this section, we will be interested in the portfolio optimization problem where the portfolio manager is faced with a risk of loosing expected utility. Here, this risk is measured by a constraint of the type

$$\mathrm{EUL}(Z)=\mathbb{E}\left[Z^-\right]=\mathbb{E}\left[(u(X_T)-u(q))^-\right]\le\varepsilon, \tag{10}$$

where ε is a given bound for the Expected Utility Loss, and $Z=u(X_T)-u(q)$. This risk constraint leads to more explicit calculations for the optimal strategy we are looking for. Also, it allows to the constrained static problem to be solved for a large class of utility functions. Again, we keep the shortfall level or threshold value q to be constant.

The dynamic optimization problem of the EUL-portfolio manager can be restated as the following static variational problem

$$\begin{aligned}&\underset{\xi\in B(x)}{\text{maximize}}\ \mathbb{E}\left[u(\xi)\right]\\ &\text{subject to } \mathbb{E}\left[(u(\xi)-u(q))^-\right]\le\varepsilon.\end{aligned} \tag{11}$$

Gabih (2005) [10] defines the EUL-optimal terminal wealth which is denoted as ξ_T^{EUL} in the following proposition.

Proposition 6 ([10])**.** *The* EUL*-optimal terminal wealth is*

$$\xi^{\mathrm{EUL}}=\begin{cases} I(y_1 H_T), & \text{if } H_T<\underline{h},\\ q, & \text{if } \underline{h}\le H_T<\overline{h},\\ I(\frac{y_1}{1+y_2}H_T), & \text{if } \overline{h}\le H_T,\end{cases}$$

for $H_T > 0$, where

$$\begin{aligned} \underline{h} &= \underline{h}(y_1) = \frac{1}{y_1} u'(q), \\ \overline{h} &= \overline{h}(y_1, y_2) = \frac{1+y_2}{y_1} u'(q) = (1+y_2)\underline{h}, \end{aligned}$$

and y_1, y_2 satisfy the system of equations

$$\begin{aligned} \mathbb{E}\left[H_T \xi^{\mathrm{EUL}}(T; y_1, y_2)\right] &= x, \\ \mathbb{E}\left[(u(\xi^{\mathrm{EUL}}(T; y_1, y_2)) - u(q))^-\right] &= \varepsilon. \end{aligned}$$

With the following remark, Gabih (2005) [10] explains the case of how the EUL optimal terminal wealth depends on y_2 as follows:

Remark 3. *For $y_2 \downarrow 0$, the situation of $\xi^{\mathrm{EUL}} \to I(y_1 H_T)$ is observed. This limit corresponds to $\varepsilon \uparrow \varepsilon_{\max}$ and the results for the unconstrained problem are derived if $y_2 = 0$ and $\xi^{\mathrm{EUL}}(y_1, 0) = I(y_1 H_T)$ are set.*

We depict the optimal terminal wealth of a EUL portfolio manager with $\varepsilon \in (0, \infty)$, a benchmark (the unconstrained) investor ($\varepsilon = \infty$), and a portfolio insurer investor with $\varepsilon = 0$ in Fig. 3. The blue curve plots the optimal horizon wealth of the EUL risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor.

The EUL portfolio manager's optimal horizon wealth is divided into three distinct regions, as before, where he shows distinct economic behaviors. In the good states, namely low price of consumption H_T, the EUL portfolio manager behaves like a benchmark investor. In the intermediate states, where $\underline{h} \leq H_T < \overline{h}$, he fully insures himself against utility losses, and in the bad states, namely high price of consumption H_T he partially insures himself against utility losses. That is, EUL portfolio manager behaves like an EL portfolio manager in the case of insurance according to each states. He just considers about utility losses contrary to the EL portfolio manager who is interested in just losses. That is why, the EUL portfolio manager chooses the cases of insurance, like the one above, may be based on the reasons presented for EL portfolio manager. However, here the rules of EUL risk constraint are valid. The measure of bad states is chosen to comply exactly with the EUL constraint. Here $\overline{h}$ for EUL strategy depends on the investor's preferences and initial wealth. As before, another distinction with VaR strategy is that the terminal wealth policy under EUL strategy is continuous across the states of the world.

Gabih (2005) [10] characterizes the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon in the following proposition.

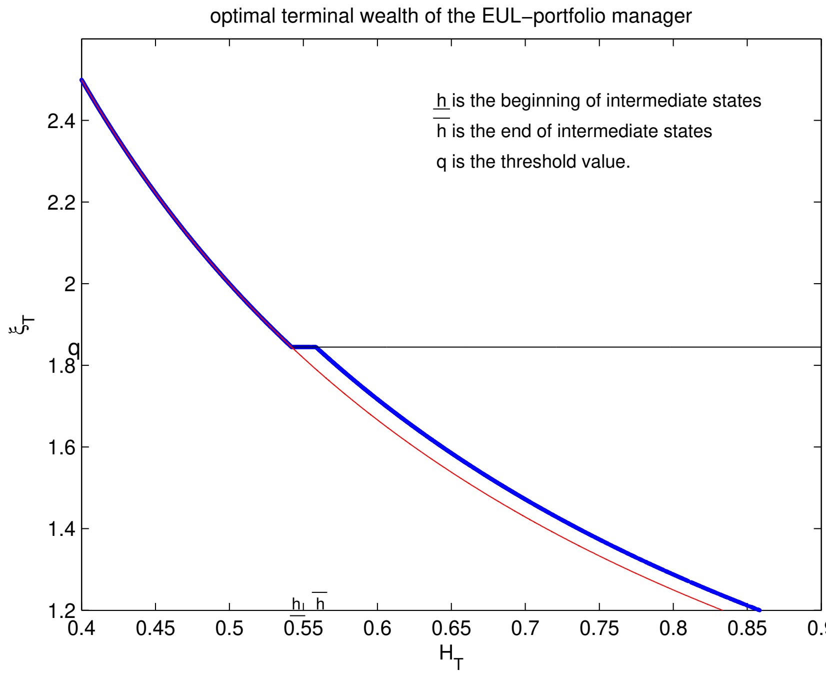

Figure 3: Optimal horizon wealth of the EUL risk manager

Proposition 7 ([10]). *Let the assumptions of Proposition 6 be fulfilled, and let u be the utility function given in* (4)*. Then,*

(i) The EUL*-optimal wealth at time $t < T$ before the horizon is given by*

$$X_t^{\mathrm{EUL}} = F(H_t, t), \tag{12}$$

where

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\Phi(-d_1(\underline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\underline{h},z,t))\right] \\ &+ \left[\frac{(1+y_2)^{\frac{1}{\gamma}}e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\Phi(-d_1(\overline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\overline{h},z,t))\right], \end{aligned}$$

for $z > 0$*, where* y_1, y_2 *and* $\underline{h}, \overline{h}$ *are as defined in Proposition 6; and*

$$\begin{aligned}\Gamma(t) &= \frac{1-\gamma}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right)(T-t),\\ d_2(u,z,t) &= \frac{\ln\frac{u}{z} + \left(r - \frac{\kappa^2}{2}\right)(T-t)}{\kappa\sqrt{T-t}},\\ d_1(u,z,t) &= d_2(u,z,t) + \frac{1}{\gamma}\kappa\sqrt{T-t}.\end{aligned}$$

(ii) The EUL*-optimal fraction of wealth invested in stock at time* $t < T$ *is*

$$\theta_t^{\mathrm{EUL}} = \theta^N \Theta(H_t, t),$$

where

$$\Theta(z,t) = 1 - \frac{qe^{-r(T-t)}}{F(z,t)}\left[\Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\overline{h}, z, t))\right]$$

for $z > 0$*.*

Gabih [10] also presented the two special properties of the function $\Theta(z,t)$ appearing in the definition of the above representation of the EUL-optimal strategy:

Proposition 8 ([10])**.** *Let the assumptions of Proposition 6 be fulfilled, and let* u *be the utility function given in* (4)*. Then, for the function* $\Theta(z,t)$*, defined in Proposition 7, we have,*

(i) $0 < \Theta(z,t) < 1$ *for all* $z > 0$ *and* $t \in [0,T)$*,*

(ii) $\lim_{t\to T} \Theta(z,t) = \begin{cases} 1, & \text{if } z < \underline{h} \text{ or } z > \overline{h},\\ 0, & \text{if } \underline{h} < z < \overline{h},\\ \frac{1}{2}, & \text{if } z = \underline{h}, \overline{h}\end{cases}$

Based on Proposition 8, Gabih [10] makes the following statement about the boundaries of $\Theta(z,t)$:

Remark 4. *The second assertion of Proposition 8 shows that the lower and upper bounds for* $\Theta(z,t)$ *given in the first assertion can not be improved. The given bounds are reached (depending on the value of z) asymptotically if time* t *approaches the horizon* T*.*

From the proposition we can deduce that the EUL-optimal fraction of wealth θ_T^{EUL} invested in the stock at the horizon is equal to the normal (unconstrained) strategy θ^* in the bad and good states, and equal to zero in the intermediate states of the market, which are described by H_T. Before the horizon T, the optimal EUL strategy, θ_t^{EUL}, is always strictly positive and never exceeds the normal (unconstrained) strategy θ^*.

3. Numerical results

In this section, we wish to examine the findings of the previous sections with examples of the portfolio optimization under Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) constraints. For the sake of comparison, we also give the corresponding behaviors of the unconstrained investor, and investors who invest in pure stock and pure bond portfolio, separately. First, we examine the probability density functions of the optimal terminal wealth of each of the above investors, and next, the optimal portfolio strategies.

We use Table 1 which shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. Our aim is to maximize the expected logarithmic utility ($\gamma = 1$) of the terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years in this example. The shortfall level or threshold value q is chosen to be 75% of the terminal wealth of a pure bond portfolio,namely, $q = 0.75xe^{rT}$, where x is the initial wealth. In the optimization with the VaR constraint, we bound the shortfall probability $P(\xi_T < q)$ by $\varepsilon = 0.06$. In the optimization with the Expected Loss constraint, we bound the expected loss $\mathrm{EL}(\xi_T < q)$ by $\varepsilon = 0.06$ and bound the expected utility loss $\mathrm{EUL}(u(\xi_T) - u(q))$ by $\varepsilon = 0.06$ in the optimization with the Expected Utility Loss.

Table 1: Parameters of the optimization problems

stock	$\mu = 9\%, \sigma = 20\%$
bond	$r = 6\%$
horizon	$T = 15$
initial wealth	$x = 1$
utility function	$u(x) = \ln x \ (\gamma = 1)$
shortfall level	$q = 0.75xe^{rT} = 1.8447$
shortfall probability (VaR)	$P(\xi_T < q) < \varepsilon = 0.06$
EL constraint	$\mathrm{EL}(\xi_T - q) \leq \varepsilon = 0.06$
EUL constraint	$\mathrm{EUL}(u(\xi_T) - u(q)) \leq \varepsilon = 0.06$

We consider the solutions of the static problems which leads to the optimal terminal wealths $\xi_T^{\mathrm{VaR}}, \xi_T^{\mathrm{EL}}$ and ξ_T^{EUL}. At first, we show the probability density functions of these random variables, belonging to VaR strategy, EL strategy, EUL strategy, unconstrained strategy, pure stock strategy and pure bond strategy, separately. On the horizontal axes of depicted figures, the expected terminal wealths $\mathbb{E}[\xi_T]$ for the considered portfolios are marked. Next, we examine the solution of the representation problem, that is, we depict the optimal strategy θ_t for each type of investors that we deal with.

3.1. Probability density function of VaR based optimal terminal wealth and the VaR-optimal wealth and strategy at time $t < T$ before the horizon

In this section, firstly we examine the probability density function of the optimal terminal wealth which the portfolio manager manages by using Value at Risk (VaR) strategy. Also, for the sake of comparison we give the probability density functions of the terminal wealth of portfolios managed by the pure bond strategy, whose fraction of wealth invested in stock is 0, the pure stock strategy, whose fraction of wealth invested in stock is 1, and the optimal strategy of the unconstrained (benchmark) problem, whose fraction of wealth invested in stock is $\theta_t = \theta^* = \frac{\mu - r}{\gamma\sigma^2} = 0.75$.

Fig. 4 depicts the shape of the probability density functions of the terminal wealths in the VaR, pure stock, benchmark(unconstrained) and pure bond solutions. The blue curve plots the shape of the probability density function of the VaR portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. Also, the expected terminal wealths $\mathbb{E}\left[\xi_T\right]$ for the considered portfolios are marked on the horizontal axes.

In the density plot, in the case of the pure bond portfolio strategy, denoted by $\xi_T^{\theta^0}$, there is a probability mass built up in the single point xe^{rT}. The probability of the terminal wealth of the pure stock portfolio strategy, denoted by $\xi_T^{\theta^1}$, and the probability of the terminal wealth of the unconstrained (benchmark) portfolio strategy $\xi_T^{\theta^*}$ are absolutely continuous. When we compute the expected values of terminal wealth of above strategies and also expected value of terminal wealth of VaR strategy $\xi_T^{\theta^{\mathrm{VaR}}}$, we see

$$\begin{aligned} \mathbb{E}\left[\xi_T^{\theta^*}\right] &= 3.4469, \\ \mathbb{E}\left[\xi_T^{\theta^0}\right] = e^{rT} &= 2.4596, \\ \mathbb{E}\left[\xi_T^{\theta^{\mathrm{VaR}}}\right] &= 8.7437 \text{ and} \\ \mathbb{E}\left[\xi_T^{\theta^1}\right] = e^{\mu T} &= 3.8574. \end{aligned}$$

This shows that the following comparison is true:

$$\mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right] < \mathbb{E}\left[\xi_T^{\theta^{\mathrm{VaR}}}\right].$$

Recall that $\xi^* = \xi_T^{\theta^*}$ maximizes the expected utility $\mathbb{E}\left[u(\xi_T^{\theta^*})\right]$, but not the

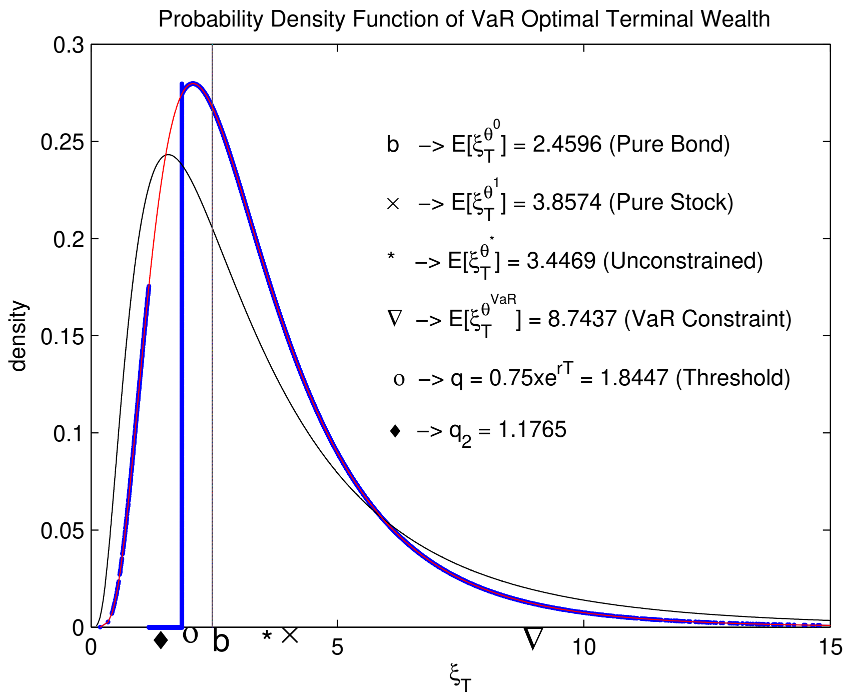

Figure 4: Probability density of the optimal horizon wealth belonging to the VaR portfolio manager

expected terminal wealth $\mathbb{E}\left[\xi_T^{\theta^*}\right]$ itself: thus, the inequalities above is not really a contradiction nor a surprise.

The VaR portfolio manager has a discontinuity, with no states having wealth between the benchmark value of $q = 0.75xe^{rT} = 1.8447$ and $q_2 = 1.1765$. q_2 is the VaR terminal wealth that consists of equation (3). However, states with wealth below q_2 have probability $\varepsilon = 6\%$. In these bad states, the VaR portfolio manager has more loss with higher probability than the portfolio manager who does not use any constraint in the portfolio optimization. The VaR portfolio manager allows 6% probability for losses in these bad states, whereas the unconstrained manager allows less probability for these losses. For example, while the probability of VaR optimal terminal wealth whose value is in the interval of (0,1.0807), which is less than $q_2 = 1.1765$, is 6%, the probability of unconstrained terminal wealth whose value is in the interval of (0,1.0807) is 4.56%. The probability mass built up at the shortfall level $q = 1.8447$ is marked by a vertical line at q in Fig. 4. The gap which we mentioned above is due to an interval $(q_2, q) = (1.1765, 1.8447)$ of values below the shortfall level or threshold value q (small losses) which carries no probability while the interval $(0, q_2] = (0, 1.1765]$ (large losses) carries the maximum allowed probability of $\varepsilon = 6\%$. Due to this situation, we encounter a serious drawback of the VaR constraint, which bounds only the probability of the losses, but does not consider the magnitude of losses.

The solution of the representation problem, in other words, the optimal strategy θ_t^{VaR} performed by the VaR portfolio manager is shown in Fig. 5. The blue curve plots the shape of the VaR portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy.

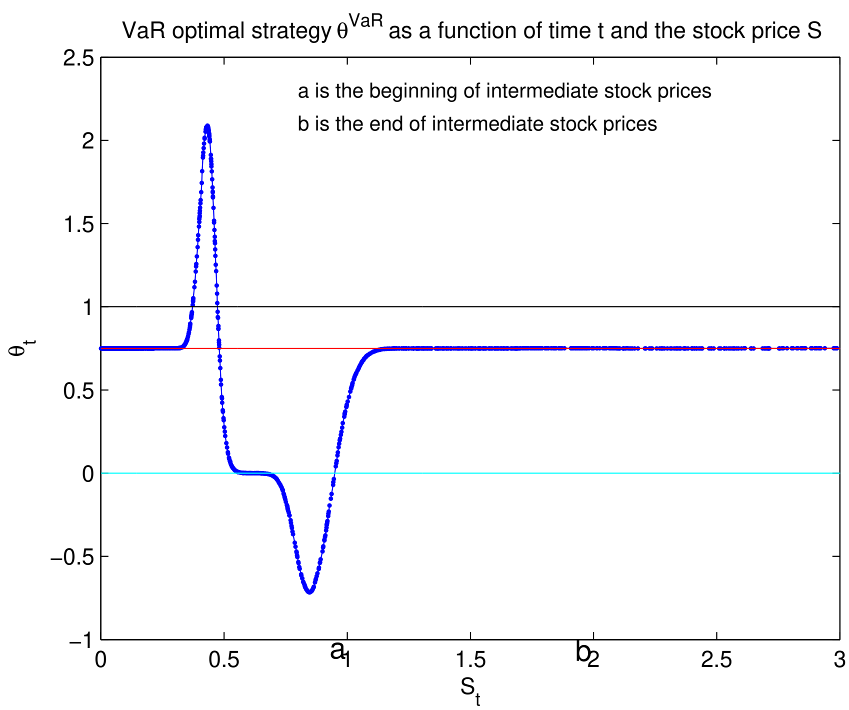

Figure 5: The VaR-optimal strategy θ^{VaR} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

For being an example of before the horizon, we take the time to be $t = 5 < T = 15$. Notice also that we allow short selling in the present applications. For the sake of comparison, in Fig. 5 we depict the strategies of the trivial portfolios, namely, the ones with the pure bond strategy ($\theta^0 \equiv 0$) and the pure stock strategy ($\theta^1 \equiv 1$), as well as and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$).

As stated before, indeed in Proposition 3 (ii), an equivalent representation of θ_t^{VaR} which is a function of time t and, consequently, the state price density H_t. However, on the other hand, because H_t can be expressed in terms t and the stock prices S_t, the optimal strategy θ_t^{VaR} can also be interpreted as a function of time t and the stock prices S_t. Hence, the dependence of θ_t^{VaR} on the stock price S_t for time $t = 5$, before the horizon, is shown in Fig. 5.

For time $t = 5$ before the horizon $T = 15$, in the case of very small stock prices, that is, in the case of $S_t \in (0, 0.9282)$ computed accordingly by the values of the parameters in Table 1, we can see that the investor invests more in risky stock under VaR constraint than without risk management or does short selling the risky stock whose fraction is very close to the investment without risk management. In case of intermediate and large stock prices, the portfolio manager or the investor behaves like an unconstrained investor in terms of fractions of wealth invested in risky stock.

3.2. Probability density function of EL based optimal terminal wealth and the EL-optimal wealth and strategy at time $t < T$ before the horizon

In this section, we examine the probability density function of the optimal terminal wealth which the portfolio manager follows the Expected Loss (EL) strategy. Also, for the sake of comparison, we give the probability density functions of the terminal wealth of portfolios which we mentioned in Section 3.1: the trivial portfolios we will use for comparison are the pure bond portfolio ($\theta^0 \equiv 0$), whose fraction of wealth invested in stock is 0, the pure stock portfolio ($\theta^1 \equiv 1$), whose fraction of wealth invested in stock is 1, and the unconstrained (benchmark) portfolio ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$), whose fraction of wealth invested in stock is 0.75.

Again, in this example, the aim is to maximize the expected logarithmic utility ($\gamma = 1$) of terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years. We will use the parameters of Table 1 for our applications. Having examined the probability density functions of these above mentioned portfolios, we will try to understand the dynamics of the optimal Expected Loss (EL) strategy at time $t < T$, for instance, by choosing the time to be $t = 5$ before the horizon, as before. Comparison with the pure bond as well as pure stock portfolios, and the unconstrained (benchmark) portfolio will be made.

We consider the solution of the static problem which leads to the optimal terminal wealth ξ^{EL}. Fig. 6 shows the probability density function of this random variable, and the probability density functions of pure stock, unconstrained (benchmark) and pure bond portfolios. The blue curve plots the shape of the probability density function of the EL portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. In addition, the expected terminal wealth $\mathbb{E}[\xi_T]$ for the considered portfolios are marked on the horizontal axes.

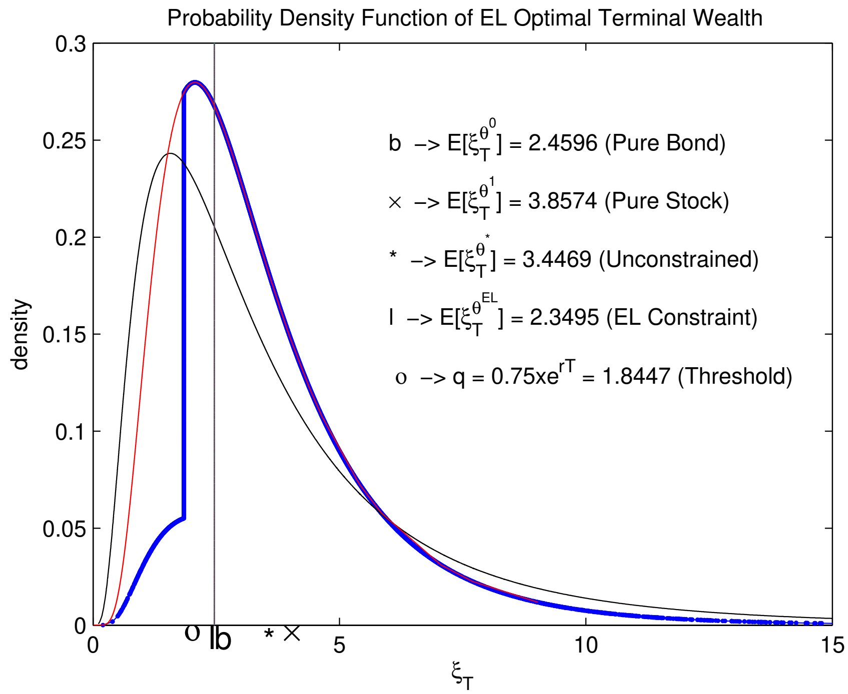

Figure 6: Probability density of the optimal horizon wealth belonging to the EL portfolio manager

When Fig. 6 is closely examined, we see that there is a probability mass build-up in the EL investor's or portfolio manager's horizon wealth, at the floor $q = 0.75xe^{rT} = 1.8447$. However, optimal EL terminal wealth's probability density has no discontinuous across states, unlike that of the optimal VaR terminal wealth. Moreover, contrary to VaR strategy, in the bad states, EL portfolio manager has less loss with higher probability; or we may say that in the bad states EL portfolio manager's probability of large losses is less than the VaR portfolio manager's probability of large losses. For example, while the probability of the EL optimal terminal wealth whose value is in the interval of (0,1.0807), which is less than $q_2 = 1.1765$ and $q = 1.8447$, is 1.14%, the probability of the VaR optimal terminal wealth whose value is in the interval of (0,1.0807) is 6%. Again while in the case of the pure bond portfolio strategy $\xi_T^{\theta^0}$ there is a probability mass built up in the single point xe^{rT}, the probability of the terminal wealth $\xi_T^{\theta^1}$ and the probability of the terminal wealth $\xi_T^{\theta^*}$ are absolutely continuous. That is to say that the probability of the terminal wealth of pure stock portfolio and the probability of the terminal wealth of unconstrained portfolio, respectively, are absolutely continuous.

When the expected terminal wealths are examined, the following equalities are easily deduced:

$$\begin{aligned} \xi_T^{\theta^0} = e^{rT} = \mathbb{E}\left[\xi_T^{\theta^0}\right] &= 2.4596, \\ e^{\mu T} = \mathbb{E}\left[\xi_T^{\theta^1}\right] &= 3.8574, \\ \mathbb{E}\left[\xi_T^{\theta^*}\right] &= 3.4469, \\ \text{and we also obtain } \mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right] &= 2.3495. \end{aligned}$$

These equalities ensure

$$\mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right] < \mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right].$$

Likewise, as in the VaR strategy of Section 3.1, $\xi^{\mathrm{EL}} = \xi_T^{\theta^{\mathrm{EL}}}$ maximizes the expected utility $\mathbb{E}\left[u(\xi_T^{\theta^{\mathrm{EL}}})\right]$ and not the expected terminal wealth $\mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right]$ itself, therefore above inequalities is not at all contradicting the general belief.

On the other hand, solution of the representation problem, namely, the path of the optimal strategy θ_t^{EL} is shown in Fig. 7 together with the paths of the trivial strategies: The blue curve plots the shape of the EL portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy.

As for an illustrative example for time t before the horizon T, we take $t = 5 < T = 15$. Also, we allow the short selling in our applications as usual. For the sake of comparison, in Fig. 7 we present the strategies of the other trivial portfolios considered before and depicted in Fig. 6: the pure bond strategy ($\theta^0 \equiv 0$), the pure stock strategy ($\theta^1 \equiv 1$) and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$).

In Proposition 5 (ii), on the other hand, we have examined an equivalent representation of θ_t^{EL}, represented in terms of t and the state price density H_t. Thence, as before, one can depict this dependence of θ_t^{EL} on the stock price S_t for time $t = 5$. See Fig. 7.

For time $t = 5$, before the horizon $T = 15$, in the beginning of very small stock prices, $S_t \in (0, 0.9282)$ calculated according to parameters in Table 1, the EL

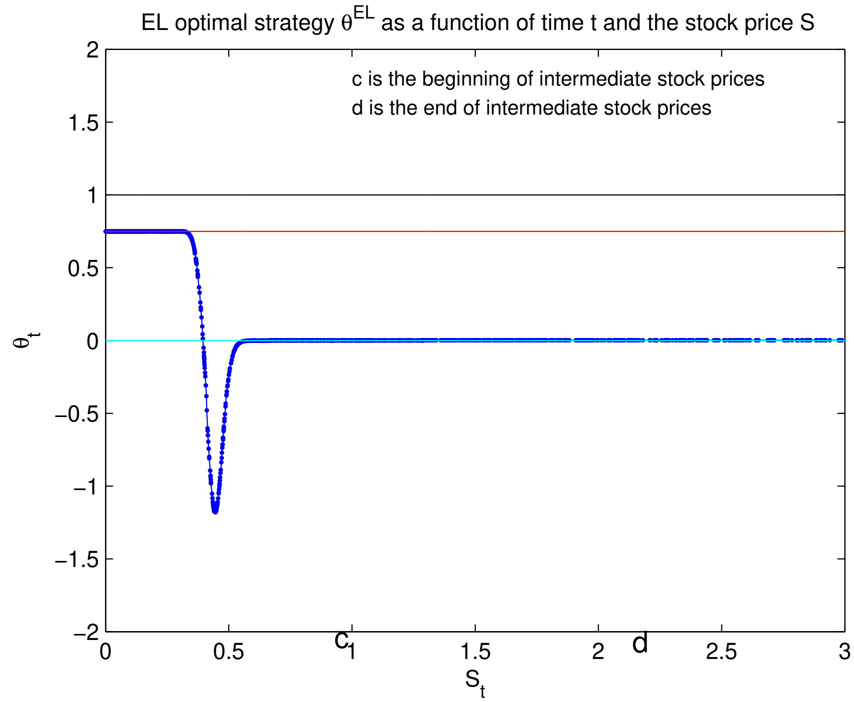

Figure 7: The EL-optimal strategy θ^{EL} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

portfolio manager behaves like an unconstrained (benchmark) investor by investing 75% of his wealth in risky stock. At the middle of small stock prices, he starts the short selling, whose fraction is larger than the fraction of the unconstrained portfolio manager when the stock price is approximately 0.5. Then, the manager starts to reduce the proportion of short selling, and towards the end of the small stock prices, as the prices increase, investor does not spend on the risky asset by behaving like an investor who only invests in the bond. In the cases of intermediate and large stock prices, that is, in the intervals of $S_t \in (0.9282, 2.1373)$ and $S_t \in (2.1373, \infty)$, respectively, he carries on with this behavior. In these states of stock prices, the optimal strategies θ_t^{EL} and θ^0 of the constrained and pure bond portfolio strategy coincide, which indicates that in these cases the complete capital is invested in the riskless bond, in order to ensure that the terminal wealth exceeds the given threshold value q.

3.3. Probability density function of EUL based optimal terminal wealth and the EUL-optimal wealth and strategy at time $t < T$ before the horizon

In this section, we examine the probability density function of the optimal terminal wealth which the portfolio manager manages by using Expected Utility Loss (EUL) strategy. Also, for the sake of comparison, we plot the probability density functions of the terminal wealth of portfolios which were discussed in Section 3.1 and Section 3.2: the portfolios we will use for comparison are the pure bond portfolio ($\theta^0 \equiv 0$), whose fraction of wealth invested in stock is 0, the pure stock portfolio ($\theta^1 \equiv 1$), whose fraction of wealth invested in stock is 1, and the unconstrained (benchmark) portfolio ($\theta^* \equiv \frac{\mu-r}{\gamma\sigma^2} = 0.75$), whose fraction of wealth invested in stock is 0.75.

The aim is again to maximize, in this time, the expected logarithmic utility ($\gamma = 1$) of terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years, and we will be using the values of the parameters of Table 1. Having examined the probability density functions of these above mentioned portfolios, we try to extract the Expected Utility Loss (EUL)-optimal strategy at time $t < T$ before the horizon: we choose the time to be $t = 5$, while knowing that our horizon is $T = 15$ years. We will also be considering the pure bond portfolio, pure stock portfolio and the unconstrained (benchmark) portfolio within the context.

To start with, we consider the solution of the static problem which leads to the optimal terminal wealth ξ^{EUL}. Fig. 8 shows the probability density function of this random variable, and the probability density functions of pure stock, unconstrained (benchmark) and pure bond portfolios for comparison. The blue curve plots the shape of the probability density function of the EUL portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. In addition, the expected terminal wealth $\mathbb{E}[\xi_T]$ for the considered portfolios are marked on the horizontal axes.

When Fig. 8 is examined, we see immediately that there is a probability mass build-up in the EUL investor's or portfolio manager's horizon wealth, at the floor q. However, this mass is smaller than the mass of that we see in Fig. 6 due to the definition of EL risk strategy. Similarly, the probability density of the terminal wealth for EUL constrained problem has no discontinuous across states: bad, intermediate, and good ones. In the bad states, EUL portfolio manager has loss with higher probability than EL portfolio manager. However, the probability of that the terminal wealth may fall below the value of $q_2 = 1.1765$ is much more bigger in the VaR strategy than in the EL and EUL strategies. For instance, while the probability of the EUL optimal terminal wealth whose value is in the interval of (0,1.0807), which

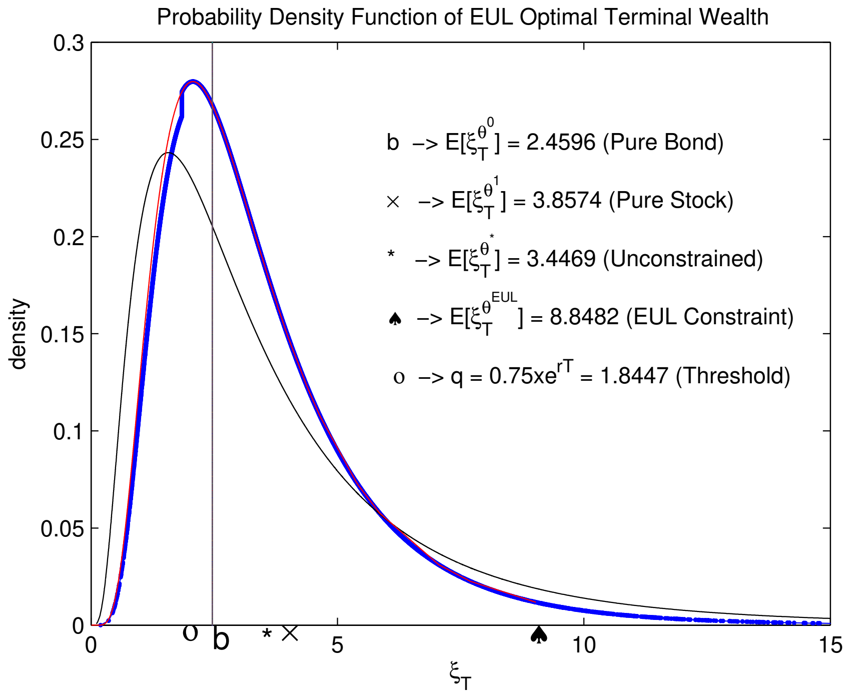

Figure 8: Probability density of the optimal horizon wealth belonging to the EUL portfolio manager

is less than $q_2 = 1.1765$ and $q = 1.8447$ is 3.93%; the probability of the VaR optimal terminal wealth whose value is in the interval of (0,1.0807) is 6%, and the probability of the EL optimal terminal wealth whose value is in the interval of (0,1.0807) is 1.14%. Again while in the case of the pure bond portfolio strategy $\xi_T^{\theta^0}$ there is a probability mass built up in the single point xe^{rT}, the probability of the terminal wealth $\xi_T^{\theta^1}$ and the probability of the terminal wealth $\xi_T^{\theta^*}$ are absolutely continuous. In other words, the probability of the terminal wealth of pure stock portfolio and the probability of the terminal wealth of unconstrained portfolio, respectively, are absolutely continuous.

Calculations of the expected terminal wealths as,

$$
\begin{aligned}
\xi_T^{\theta^0} = e^{rT} = \mathbb{E}\left[\xi_T^{\theta^0}\right] &= 2.4596, \\
e^{\mu T} = \mathbb{E}\left[\xi_T^{\theta^1}\right] &= 3.8574,
\end{aligned}
$$

$$\mathbb{E}\left[\xi_T^{\theta^*}\right] = 3.4469,$$

$$\text{and we also obtain } \mathbb{E}\left[\xi_T^{\theta^{\text{EUL}}}\right] = 8.8482,$$

immediately yields the following inequalities:

$$\mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right] < \mathbb{E}\left[\xi_T^{\theta^{\text{EUL}}}\right],$$

which is neither contradicting the previous results, nor surprising.

Accordingly, by the help of the representation problem, the optimal strategy θ_t^{EUL} for the EUL constrained problem is depicted in Fig. 9 along with the trivial portfolio strategies: The blue curve plots the shape of the EUL portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy.

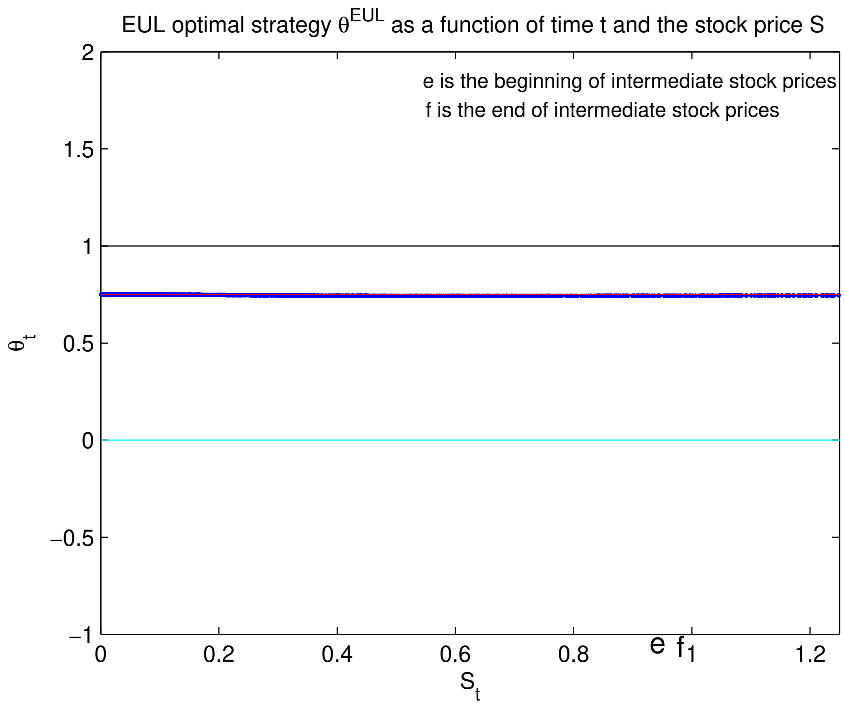

Figure 9: The EUL-optimal strategy θ^{EUL} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

Concerning the case before the horizon, we take the time to be $t = 5 < T = 15$. For the sake of comparison, in Fig. 9 we present the strategies of the other portfolios considered previously: the pure bond strategy ($\theta^0 \equiv 0$), the pure stock strategy ($\theta^1 \equiv 1$) and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$). Note that, as before, the optimal strategies are plotted as a function of the stock prices, as the optimal strategies can also be written also as a function of the stock price S_t, and hence, t only. In Fig. 9, we also show the dependence of θ_t^{EUL} on the stock price S_t for time $t = 5$, before the horizon.

As is clear in Fig. 9, the fraction of wealth invested in risky stock is very close to the unconstrained fraction, which is 0.75 in this example, in almost every states of the world although there are some little changes in fractions in some states. Thus we can deduce that before the horizon $T = 15$, the EUL-optimal fraction of wealth θ_t^{EUL} is always strictly positive and does not exceed the normal strategy $\theta^* = 0.75$. Refer to Proposition 8.

4. Conclusion and outlook

Harry Markowitz, who is the pioneer of the modern portfolio theory, considers an investor who would (or should) select one of efficient portfolios which are those with minimum variance for given expected return or more and maximum expected return for given variance or less. However, in Markowitz's model short selling is not allowed, namely the fractions of wealth invested in the securities can not be negative, because necessary portfolios are chosen from inside of the attainable set of portfolios. The attainable set of portfolios consists of all portfolios which satisfy constraints $\sum_{i=0}^{n} \theta_i = 1$ and $\theta_i \geq 0$ for $i = 1, 2, 3, ..., n$. However in this paper, short selling is allowed. We use the martingale representation approach to solve the optimization problem in continuous time.

Merton presented the method of continuous-time stochastic optimal control when the utility function is a power function or the logarithm [22]. While the static problem is necessary for the martingale approach, in the stochastic optimal control method the dynamic problem is used. However, martingale approach is much easier than the dynamic programming approach. Martingale technique characterizes optimal consumption-portfolio policies simply when there exist non-negativity constraints on consumption and on final wealth [5]. On the other hand, when there is the non-negativity constraint on consumption, the stochastic dynamic programming is more difficult. Also in the dynamic programming, it is in general difficult to construct a solution.

The goal of this work is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the wealth invested in stock and bond, respectively. As we examine in this paper, when we do not use any risk limitations, the optimal terminal wealth may not exceed the initial capital with a high probability. So we quantify such shortfall risks by using appropriate risk measures and then we add them into the optimization as constraints. Hence, we use Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) risk constraints in order to reduce such shortfall risks. By the term shortfall risk, we mean the event that the terminal wealth may fall below threshold value, namely, the initial capital or the result of an investment in a pure bond portfolio. In this work, portfolio optimization under VaR constraint, EL constraint, and EUL constraint are separately examined with their own numerical results. An investor may benefit separately from each strategy by choosing carefully constraint bound ε and the threshold value q for each strategy: ε and q are given and deterministic, and one can choose them in accordance to his risk tolerance for each strategy.

Here, we assume that all investors are risk averse and use the logarithmic utility function for meeting the requirements of these investors. We examine the numerical results of VaR, EL and EUL strategies and, for the sake of comparison, give the results of unconstrained, pure bond and pure stock strategies, and try to understand which is more suitable to risk averse investors and whether these measures are good enough to meet exactly all requirements.

Starting with the portfolio optimization problem under VaR constraint, we choose the shortfall probability as $\varepsilon = 6\%$ and the shortfall level or threshold value as $q = 0.75xe^{rT} = 1.8447$. At the beginning of very small stock prices, before the horizon, the VaR portfolio manager behaves like a benchmark (unconstrained) investor by investing as the fraction of unconstrained strategy. However, towards the middle of very small stock prices he increases the fraction and this fraction exceeds the fraction of unconstrained strategy. In this states, the behavior of VaR agent does not appear as an desirable one because it is risky and not rational. Although in good states unconstrained and VaR agent's optimal fractions which are invested in risky stock result in similar optimal terminal wealth, VaR agent exposures to more risk by investing much more in the risky stock than the unconstrained agent. In the case of intermediate and high stock prices, before the horizon VaR agent's behavior turns to the behavior of the unconstrained agent by investing as the unconstrained fraction of wealth in the risky stock. However, in this case, while the interval (q_2, q) does not carry probability, the interval $(0, q_2)$ carries the maximum allowed probability of ε. That is, while the interval of small losses does not carry probability, the interval of large losses carries the maximum allowed probability of ε. Here, q is the threshold value, and q_2 is the VaR terminal wealth that consists of the equation (3) and the maximum allowed probability that the terminal wealth falls below this value (q_2) is ε. This is a serious drawback of the VaR constraint which bounds only the probability of the losses but does not take care of the magnitude of losses. This may cause to credit problems, defeating the purpose of using the VaR constraint in real world applications. A regulatory requirement to manage risk using the VaR approach is designed, in principle, to prevent large and frequent losses that may drive economic investors out of business. It is true that under the VaR constraint losses are not frequent, however, the largest losses are more severe than without the VaR constraint.

In addition to the shortcomings of VaR constraint, we can consider the case of the property of sub-additivity, which is the diversification principle to reduce risk by investing in a variety of assets. Since VaR constraint does not satisfy this property, diversification can lead to an increase of VaR.

In order to remedy the shortcomings of VaR constraint, especially in bad states, as in the case of large losses expected losses are higher in the VaR strategy than those the investor would have incurred if he had not engaged in VaR constraint, Expected Loss (EL) strategy is presented as an alternative risk measure in this work. In contrary to the VaR agent who interests in controlling just the probability of the loss, which causes undesirable situations in the bad states as indicated, EL agent concerns with the magnitude of a loss in order to maintain limited expected losses when losses occur. Hence, if one wants to control the magnitude of losses, he should control all moments of the loss distribution, and in this paper, we focus on controlling the first moment of the loss distribution in the EL strategy. For the EL strategy, in our example concerning this strategy, we choose the bound ε such that $\mathrm{EL}(\xi_T - q) \leq \ \varepsilon = 0.06$. That is, when losses occur, we maintain limited expected losses such that these losses can be at most 6% of our initial capital, and again we choose the threshold value such that $q = 0.75xe^{rT} = 1.8447$.

At the beginning of very small stock prices, before the horizon, the EL portfolio manager behaves like an unconstrained investor by investing of 75% ($\theta^* = \frac{\mu - r}{\gamma\sigma^2} = 0.75$) of wealth in the risky stock of our example in Section 3.2. However, towards the middle of very small stock prices he reduces the fraction and then starts the short selling. When cases of intermediate and high stock prices reached, he stays fixed at the fraction of pure bond strategy, namely $\theta^0 = 0$, in order to ensure that the terminal wealth exceeds the threshold value q. In fact, in the case of small stock prices, the short selling may be considered as a desirable situation since borrowing the low-value stock and selling it when the stock prices increase may lead to the profit for the investor who uses the approximately fraction of unconstrained agent in the short selling case. When the EL optimal terminal wealth is reached, in the bad states EL portfolio manager's probability of large losses becomes less than the VaR portfolio manager's probability of large losses.

Also contrary to the VaR strategy, EL strategy has no discontinuous across states. In the EL strategy, in the bad states, i.e. in the states of large losses, the investor partially insures himself for maintaining limited expected losses, incurring partial losses in contrary to the VaR investor. However, maintaining some level of insurance requires from the investor a cost, too; it is necessary to think well about how much cost is to spent for insurance and whether it is worth leaving bad states completely uninsured.

In addition, contrary to the VaR constraint, EL constraint satisfies the sub-additivity property of coherent risk measures. However, it does not satisfy the translation-invariance axiom: For a given $a \in \mathbb{R}$ we should have $\rho(Z_1+a) = \rho(Z_1)-a$. This might be considered as a disadvantage of EL constraint since when cash which has the value a is added to the portfolio, the risk of $Z_1 + a$ is more than the risk of Z_1 and this risk is as much as the cash which has the value a.

Since one of the goals of a portfolio manager is to maximize the expected utility from the terminal wealth, it is interesting to deal with another risk measure called Expected Utility Loss (EUL), which we investigate in this paper. EUL risk constraint leads to more explicit calculations for the optimal strategy that we are looking for and allows us to solve the constrained static problem for a large class of utility functions. Thus it might be a convenient risk measure.

In the case of EUL optimal horizon wealth, similar to the EL constraint, in the bad states, namely the high price of consumption H_T, he partially insures himself against losses and therefore in this partially insured states EUL agent may keep the EUL optimal terminal wealth above the optimal terminal wealths of other strategies mentioned. This is achieved by shrinking the insured region in the intermediate states, but by settling for a wealth lower than q so that it is enough to comply with the EUL constraint in the bad states. However, again, since insurance is very costly in these bad states, here EUL agent prefers partially insurance.

For the EUL strategy, in our example, we choose the EUL bound ε such that $\mathrm{EUL}(u(\xi_T) - u(q)) \leq \ \varepsilon = 0.06$. That is, when losses occur, we maintain limited expected utility losses such that those utility losses can be at most 0.06, and again we choose the threshold value such that $q = 0.75xe^{rT} = 1.8447$. As we examine in Section 3.3, before the horizon, in all states of stock prices, the EUL portfolio manager invests in risky stock as a value of fraction that is very close to the fraction of unconstrained strategy. We also infer that the EUL optimal fraction θ_t^{EUL}, before the horizon, is always strictly positive and never exceeds the normal strategy θ^* as is examined in Proposition 8. Hence, we understand that if we use the EUL constraint in our optimization problem, when we take drift term μ bigger than r, short selling will not be allowed here, in contrary to the VaR and EL strategies. Finally, to point out that, neither EL nor EUL risk measures not coherent risk measures, unfortunately.

Consequently, each of risk measures in this work, which are Value at Risk (VaR), Expected Loss (EL) and Expected Utility Loss (EUL) risk measures, has various advantages and disadvantages separately as mentioned in the above discussions. When a portfolio manager wants to use risk constraints in the optimization problem, it is too significant to choose the bounds and threshold values rationally for each risk constraint and examine in details the advantages and disadvantages of these risk measures before performing an investment in order to be able to achieve the desired results. However, a very serious deficiency of VaR, EL and EUL risk measures is that all of them are not coherent risk measures: the VaR risk measure does not satisfy the sub-additivity property and, the EL and EUL risk measures do not satisfy the translation-invariance property. Sub-additivity property reflects the idea that risk can be reduced by diversification, so non-subadditive measures of risk in portfolio optimization may create portfolios with high risk.

As an outlook, thanks to the translation-invariance property of a risk measure, the risk of a portfolio can be reduced by simply adding a certain amount of riskless money. So, when the shortcomings of these non-coherent risk measures are to be avoided, it appears that, in the constrained portfolio optimization problems, using coherent risk measures may be much more rational and it may be necessary to search coherent risk measures for being alternative to the VaR, EL and EUL risk measures.

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