Acta Universitatis Apulensis No. 64/2020
ISSN: 1582-5329 pp. 117-125
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.64.08

SOME PROPERTIES OF TUBULAR SURFACES IN $\mathbb{E}^3$

Ali UÇUM

Abstract. In this article, we consider tubular surfaces in Euclidean 3-space. We obtain the necessary and sufficient conditions for tubular surfaces in Euclidean 3-space to be semi-parallel and of the first kind of pointwise 1-type Gauss map. Also we study the tubular surface in Euclidean 3-space such that its mean curvature vector $\vec{H}$ satisfies $\Delta\vec{H} = \lambda H$ for some differentiable functions λ.

2010 *Mathematics Subject Classification*: 53A05, 53B25.

Keywords: Tubular surfaces, Gauss map, semi-parallel, pointwise 1-type.

1. Introduction

The notion of finite type submanifolds introduced by B. Y. Chen during the late 1970's has become an useful tool for investigating and characterizing submanifolds of Euclidean or pseudo-Euclidean space ([1],[3]). Afterwards, the notion was extended to differential maps, in particular, to the Gauss map of submanifolds. Especially, if an oriented submanifold M has 1-type Gauss map G, then G satisfies $\triangle G = \lambda(G + C)$ for a non-zero constant λ and a constant vector C, where $\triangle$ is the Laplace operator. Extending this kind of property which is a typical character valid on helicoids, catenoids and several rotational surfaces, Y. H. Kim defined the notion of submanifolds of Euclidean space with pointwise 1-type Gauss map as follows:

Definition 1. *[10] An oriented submanifold M of Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map G satisfies*

$$\triangle G = f(G + C) \tag{1}$$

for a non-zero smooth function f and a constant vector C.

A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of the second kind. Many interesting submanifolds with pointwise 1-type Gauss map have been studied from different viewpoint and different spaces ([4], [5], [6], [8], [9]).

On the other hand, the submanifold M is called semi-parallel (semi-symmetric [11]) if $\overline{R}\cdot h = 0$ where $\overline{R}$ denotes the curvature tensor of Vander Waerden-Bortoletti connection $\overline{\nabla}$ of M and h is the second fundamental form of M. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\overline{\nabla}h = 0$ [7], [12].

In the present paper, we consider tubular surfaces in Euclidean 3-space to be semi-parallel and to have the first kind of pointwise 1-type Gauss map. We prove the following theorems:

Theorem 1. *Let M be a tubular surface in $\mathbb{E}^3$. Then M has the first kind of pointwise 1-type Gauss map if and only if M is a cylindrical surface.*

Theorem 2. *Let M be a tubular surface in $\mathbb{E}^3$. Then M is semi-parallel if and only if M is a cylindrical surface.*

Also we consider the tubular surface in Euclidean 3-space such that its mean curvature vector $\vec{H}$ satisfies $\Delta\vec{H} = \lambda H$ for some differentiable functions λ and we prove the following theorems:

Theorem 3. *Let M be a tubular surface in $\mathbb{E}^3$. Then the mean curvature vector $\vec{H}$ of M satisfying $\Delta\vec{H} = \lambda H$ for some differentiable functions λ if and only if M is a cylindrical surface.*

2. Preliminaries

We recall some well-known formulas for the surfaces in $\mathbb{E}^3$. Let M be a surface of $\mathbb{E}^3$, the standard connection D on $\mathbb{E}^3$ induces the Levi-Civita connection ∇ on M. We have the following Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y),$$

and the Weingarten formula

$$D_X \xi = -A_\xi X + {}^{\perp}\nabla_X \, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^{\perp}\right)$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^{\perp}\nabla$ is the normal connection. We note that

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The normal curvature tensor ${}^{\perp}R$ is defined by

$$ {}^{\perp}R\left(X,Y\right)\xi = {}^{\perp}\nabla_X\ {}^{\perp}\nabla_Y\ \xi - {}^{\perp}\nabla_Y\ {}^{\perp}\nabla_X\ \xi - {}^{\perp}\nabla_{[X,Y]}\xi\ , $$

where $X, Y \in \Gamma\left(TM\right)$ and $\xi \in \Gamma\left(TM^{\perp}\right)$. Taking the normal part of the following equation

$$ D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi = 0 $$

where $X, Y \in \Gamma\left(TM\right)$ and $\xi \in \Gamma\left(TM^{\perp}\right)$, we get the Ricci equation

$$ \left\langle {}^{\perp}R\left(X,Y\right)\xi, \eta\right\rangle = \left\langle A_\eta X, A_\xi Y\right\rangle - \left\langle A_\xi X, A_\eta Y\right\rangle $$

where $\eta \in \Gamma\left(TM^{\perp}\right)$.

The mean curvature vector field $\overrightarrow{H}$, the mean curvature H and the Gauss curvature of M are given respectively by

$$ \overrightarrow{H} = \frac{1}{2}\text{trace}h \quad \text{and} \quad K = \det A. $$

A surface is called minimal if $H = 0$ identically. A surface is called flat if $K = 0$ identically ([2]).

Let $\overline{R}\cdot h$ be the product tensor of the curvature tensor $\overline{R}$ with the second fundamental form h. The surface M is said to be semi-parallel if $\overline{R}\cdot h = 0$, i.e. $\overline{R}\left(X_i, X_j\right)\cdot h = 0$ ([11]). Now, we give the following result.

Lemma 4. *([7]) Let $M \subset \mathbb{E}^n$ be a smooth surface given with the patch $M\left(u,v\right)$. Then the following equalities are hold:*

$$ \begin{aligned} \left(\overline{R}\left(X_1,X_2\right)\cdot h\right)\left(X_1,X_1\right) &= \left(\sum_{\alpha=1}^{n-2} h_{11}^{\alpha}\left(h_{22}^{\alpha} - h_{11}^{\alpha}\right) + 2K\right) h\left(X_1,X_2\right) \\ &\quad + \sum_{\alpha=1}^{n-2} h_{11}^{\alpha} h_{12}^{\alpha}\left(h\left(X_1,X_1\right) - h\left(X_2,X_2\right)\right), \\ \left(\overline{R}\left(X_1,X_2\right)\cdot h\right)\left(X_1,X_2\right) &= \left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}\left(h_{22}^{\alpha} - h_{11}^{\alpha}\right)\right) h\left(X_1,X_2\right) \\ &\quad + \left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha} h_{12}^{\alpha} - K\right)\left(h\left(X_1,X_1\right) - h\left(X_2,X_2\right)\right), \\ \left(\overline{R}\left(X_1,X_2\right)\cdot h\right)\left(X_2,X_2\right) &= \left(\sum_{\alpha=1}^{n-2} h_{22}^{\alpha}\left(h_{22}^{\alpha} - h_{11}^{\alpha}\right) - 2K\right) h\left(X_1,X_2\right) \\ &\quad + \sum_{\alpha=1}^{n-2} h_{22}^{\alpha} h_{12}^{\alpha}\left(h\left(X_1,X_1\right) - h\left(X_2,X_2\right)\right) \end{aligned} $$

where K is the Gauss curvature of the surface.

The Laplacian $\triangle$ on M is given by

$$\triangle = -\frac{1}{\sqrt{\det\left(g^{ij}\right)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det\left(g^{ij}\right)} g^{ij} \frac{\partial}{\partial x_j} \right) \tag{2}$$

where $\left(g^{ij}\right)$ is the inverse matrix of $\left(g_{ij}\right)$, which is the local components of the metric on M.

3. Tubular surface in $\mathbb{E}^3$

In this section, we study some geometrical properties of tubular surfaces in $\mathbb{E}^3$. We prove the main theorems theorem 1 and theorem 2 and related results.

A canal surface M in $\mathbb{E}^3$ is an immersed surface swept out by a sphere moving along a curve $\alpha = \alpha(s)$ or by a particular circular cross-section of the sphere along the same path ([13]). Due to the generating process of canal surfaces, the parametric formula of M can be given as follows:

$$\begin{aligned} M(s,u) &= \alpha(s) - r'(s)\, r(s)\, T(s) \\ &\quad + r(s) \sqrt{1 - \left(r'(s)\right)^2} \left(\cos u N(s) + \sin u B(s)\right) \end{aligned}$$

where the curve $\alpha(s)$is called the spine curve (center curve) parametrized by arc-length s and $r(s)$ is called the radial function of M. Here $\{T, N, B\}$ is Frenet frame of $\alpha(s)$. In particular, if $r(s)$ is a constant, then M is called a tubular surface.

Let $\alpha : I \to \mathbb{E}^3$ be a unit-speed planar curve satisfying

$$\begin{aligned} T'(s) &= \kappa(s)\, N(s), \\ N'(s) &= -\kappa(s)\, T(s) \end{aligned}$$

and M be a tubular surface whose spine curve is α as follows

$$M(s,u) = \alpha(s) + r\left((\cos u)\, N(s) + (\sin u)\, B\right) \tag{3}$$

where B is constant vector in $\mathbb{E}^3$. Differentiating (3) with respect to s and u, respectively, we get

$$M_s(s,u) = (1 - r\kappa \cos u)\, T, \tag{4}$$

$$M_u(s,u) = -r(\sin u)\, N + r(\cos u)\, B. \tag{5}$$

Here without lost of generality, we assume that $1 - r\kappa \cos u > 0$ for the regularity of the surface M. Thus, an orthonormal tangent bases on M is given by

$$e_1 = \frac{M_s}{\|M_s\|} = T(s), \tag{6}$$

$$e_2 = \frac{M_u}{\|M_u\|} = -(\sin u) N(s) + (\cos u) B. \tag{7}$$

From (6) and (7), we find

$$e_3 = e_1 \times e_2 = -(\cos u) N(s) - (\sin u) B. \tag{8}$$

By covariant differentiation with respect to e_1 and e_2, a straightforward calculation gives

$$D_{e_1} e_1 = \frac{1}{\|M_s\|} D_{M_s} e_1 = \frac{\kappa}{1 - r\kappa \cos u} N,$$

$$D_{e_1} e_2 = \frac{1}{\|M_s\|} D_{M_s} e_2 = \frac{\kappa \sin u}{1 - r\kappa \cos u} T$$

$$D_{e_2} e_2 = \frac{1}{\|M_u\|} D_{M_u} e_2 = \frac{1}{r} (-\cos uN - \sin uB)$$

Then we find,

$$h_{11} = \langle D_{e_1} e_1, e_3 \rangle = \frac{-\kappa \cos u}{1 - r\kappa \cos u}, \quad h_{12} = \langle D_{e_1} e_2, e_3 \rangle = 0$$

$$h_{22} = \langle D_{e_2} e_2, e_3 \rangle = \frac{1}{r}$$

Then we have the following theorem.

Theorem 5. *Let M be a tubular surface given by (3)in $\mathbb{E}^3$. Then the Gauss curvature and mean curvature of M is found as follows*

$$K = \frac{-\kappa \cos u}{r(1 - r\kappa \cos u)} \qquad \text{and} \qquad H = \frac{1 - 2r\kappa \cos u}{2r(1 - r\kappa \cos u)}. \tag{9}$$

Now we define the Gauss map $G(s,u)$ of M by

$$G(s,u) = -(\cos u) N(s) - (\sin u) B. \tag{10}$$

By using (10) and (2), we have

$$\Delta G = \frac{-\kappa' \cos u}{(1 - r\kappa \cos u)^3} T + \frac{-2\cos u + r(1 + 3\cos 2u)\kappa - 4r^2\kappa^2 \cos^3 u}{2r^2 (1 - r\kappa \cos u)^2} N + \frac{-\sin u + r\kappa \sin 2u}{r^2 (1 - r\kappa \cos u)} B. \tag{11}$$

Then we give the following theorem.

Theorem 6. *Let M be a tubular surface given by* (3) *in E^3. Then M has the first kind of pointwise 1-type Gauss map if and only if M is a cylindrical surface.*

Proof. Let M be a tubular surface given by (3) in $\mathbb{E}^3$ and assume that M has the first kind of pointwise 1-type Gauss map, that is, the following equation holds

$$\Delta G = \lambda G \tag{12}$$

where λ is a real valued C^∞ function. By using (10) and (11) in (12) , we get

$$\lambda = \frac{1}{r^2} \text{ and } \kappa = 0,$$

which implies that the surface M is a cylindrical surface.

Conversely, let M be a cylindrical surface. We will show that M has the first kind of pointwise 1-type Gauss map. Let us assume that the following holds

$$\Delta G = \lambda (G + C) \tag{13}$$

where $C = c_1 T(s) + c_2 N(s) + c_3 B$. Substituting (10) and (11) in (13), we obtain

$$\Delta G = -\frac{\cos u}{r^2} N - \frac{\sin u}{r^2} B,$$

$$\lambda (G + C) = \lambda c_1 T + \lambda (-\cos u + c_2) N + \lambda (-\sin u + c_3) B$$

Since the set $\{1, \sin u, \cos u\}$ is linearly independent, we get $\lambda = 1/r^2$ and $C = 0$, which means that M has the first kind of pointwise 1-type Gauss map.

Then we give the following corollaries.

Corollary 7. *Let M be a tubular surface given by* (3) *in $\mathbb{E}^3$. Then M has the first kind of pointwise 1-type Gauss map if and only if the spine curve of M is a straight line.*

Corollary 8. *Let M be a tubular surface given by* (3) *in $\mathbb{E}^3$. Then M does not have a harmonic Gauss map.*

Now, we consider the mean curvature vector $\vec{H}$ of M. The mean curvature vector $\vec{H}$ is given by

$$\vec{H} = \frac{1 - 2r\kappa \cos u}{2r(1 - r\kappa \cos u)} e_3.$$

Then we have

$$\Delta \vec{H} = \frac{(-1 + 4r\kappa \cos u)\, \kappa' \cos u}{2r(1 - r\kappa \cos u)^4} T + P(s,u) N + Q(s,u) B$$

where $P(s,u)$ and $Q(s,u)$ are differentiable functions.

Assume that $\Delta\vec{H} = \lambda H$ for some differentiable functions λ. Then from the coefficients of T, we have

$$\kappa' = 0$$

which implies that

$$P(s,u) = \frac{\left(2+9\kappa^2 r^2\right)\cos u - 8\kappa^3 r^3 \cos^4 u + \kappa r\left(-2 - 8\cos 2u + 5\kappa r \cos 3u\right)}{-4r^3\left(1 - r\kappa\cos u\right)^3}$$

and

$$Q(s,u) = \frac{2\sin u + 2\kappa r\left(\kappa r\left(6 - 4\kappa r\cos^3 u + 5\cos 2u\right)\sin u - 4\sin 2u\right)}{-4r^3\left(1 - r\kappa\cos u\right)^3}.$$

Since $\Delta\vec{H} = \lambda H$, we get

$$\kappa = 0 \qquad \text{and} \qquad \lambda = \frac{1}{r^2}.$$

Then we get the following theorem.

Theorem 9. *Let M be a tubular surface given by (3) in $\mathbb{E}^3$. Then the mean curvature vector $\vec{H}$ of M satisfying $\Delta\vec{H} = \lambda H$ for some differentiable functions λ if and only if M is a cylindrical surface.*

Theorem 10. *Let M be a tubular surface given by (3) in $\mathbb{E}^3$. Then M is semi-parallel if and only if M is a cylindrical surface.*

Proof. Let M be a tubular surface given by (3) in $\mathbb{E}^3$. Assume that M is semi-parallel. Namely, for $(1 \le i,j \le 2)$,

$$\left(\overline{R}(e_1,e_2)\cdot h\right)(e_i,e_j) = 0.$$

By a straightforward calculation, from Lemma 4, we have

$$\begin{aligned}
\left(\overline{R}(e_1,e_2)\cdot h\right)(e_1,e_1) &= \frac{\kappa^2\left(3 - 2r\kappa\cos u\right)\cos^2 u}{r\left(1 - r\kappa\cos u\right)^3}e_3,\\
\left(\overline{R}(e_1,e_2)\cdot h\right)(e_1,e_2) &= \frac{-\kappa\cos u}{r^2\left(1 - r\kappa\cos u\right)^2}e_3,\\
\left(\overline{R}(e_1,e_2)\cdot h\right)(e_2,e_2) &= 0.
\end{aligned}$$

From our assumption, we get $\kappa = 0$ which means that M is a cylindrical surface. The converse of the proof is clear.

As a result, we have the following corollary.

Corollary 11. *Let M be a tubular surface given by* (3)*in* $\mathbb{E}^3$*. Then the followings are equivalent:*
i. *M is a cylindrical surface,*
ii. *M has the first kind of pointwise 1-type Gauss map,*
iii. *The mean curvature vector $\vec{H}$ of M satisfying $\Delta\vec{H} = \lambda H$ for some differentiable functions λ,*
iv. *M is semi-parallel.*

References

[1] B.-Y. Chen, *A report on submanifolds of finite type*, Soochow J. Math., 22 (1996), 117-337.

[2] B.-Y. Chen, *Geometry of Submanifolds*, Dekker, New York, 1973.

[3] B.-Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc., 35 (1987), 161-186.

[4] B.-Y. Chen, M. Choi and Y. H. Kim,*Surfaces of revolution with pointwise 1-type Gauss map,* J. Korean Math. Soc., 42 (2005), 447-455.

[5] M. Choi, Y. H. Kim, L. Huili and D. W. Yoon, *Helicoidal surfaces and their Gauss map in Minkowski* 3*-space*, Bull. Korean Math. Soc., 47 (2010), 859-881.

[6] M. Choi and Y. H. Kim, *Characterization of helicoid as ruled surface with pointwise 1-type Gauss map,* Bull. Korean Math. Soc., 38 (2001), 753-761.

[7] J. Deprez,*Semi-parallel surfaces in Euclidean space*, J. Geom., 25 (1985), 192-200.

[8] U. Dursun, *Hypersurfaces with pointwise 1-type Gauss map*, Taiwanese J. Math., 11 (2007), 1407-1416.

[9] U. Dursun and E. Coskun, *Flat surfaces in Minkowski space E_1^3 with pointwise 1-type Gauss map,* Turk J. Math, 36 (2012), 613-629.

[10] Y. H. Kim and D. W. Yoon, *Ruled surface with pointwise 1-type Gauss map*, J. Geom. Phys., 34 (2000), 191-205.

[11] U. Lumiste, *Classification of two-codimensional semi-symmetric submanifolds*, TRU Toimetised, 803 (1988), 79-84.

[12] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$ I. The Local Version*, J. Differential Geometry, 17 (1982), 531-582.

[13] Z. Q. Xu, R. Z. Feng and J. G. Sun, *Analytic and algebric properties of canal surfaces*, Appl. Math. and Comp., 195 (2006), 220-228.

Ali Uçum
Department of Mathematics, Faculty of Sciences and Arts,
Kırıkkale University,
Kırıkkale, Turkey
email: *aliucum05@gmail.com*