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Rotary Hypermaps of Genus 2

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Abstract. We classify the rotary hypermaps (sometimes called regular hypermaps) on an orientable surface of genus 2. There are 43 of them, of which 10 are maps (classified by Threlfall), 20 more can be obtained from the 10 maps by applying Machì's operations, and the remaining 13 may be obtained from the maps by using Walsh's bijection between maps and hypermaps. As a corollary, we deduce that there are no non-orientable reflexible hypermaps of characteristic -1.

1. Introduction

An orientable hypermap $\mathcal{H}$ is said to be *rotary* if its rotation group, that is, its orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{H}$, acts transitively on the set of brins of $\mathcal{H}$. (Such hypermaps have often been called *regular*, but we will avoid this term since it is sometimes used for the stronger condition that the full automorphism group $\mathrm{Aut}\,\mathcal{H}$ (including orientation-reversing automorphisms) should act transitively on the blades; following [5] we will call this condition *reflexibility*). The rotary hypermaps on the sphere and the torus have been determined by Corn and Singerman in [4]; in each case, there are infinitely many, whereas on a surface of genus $g \geq 2$ the number must always be finite. Our aim here is to treat the simplest case, and classify the rotary hypermaps of genus 2. Much of the preliminary work on this problem has already been done: hypermaps include maps, and Threlfall [8] has determined the rotary maps $\mathcal{M}$ of this genus; there are 10 of them, listed by Coxeter

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and Moser in Table 9 of [5] with their types and rotation groups $\mathrm{Aut}^+\mathcal{M}$. Similarly Corn and Singerman, in Table 2 of [4], have determined the possible types and rotation groups of the remaining rotary hypermaps of genus 2. We shall complete their results by enumerating, describing and constructing all these hypermaps $\mathcal{H}$ and specifying their automorphism groups $\mathrm{Aut}\,\mathcal{H}$.

The 10 rotary maps $\mathcal{M}$ classified by Threlfall and listed in [5] are summarised in Table 1 and are described in detail in §§5–9. We need this information in order to describe the remaining hypermaps. First one can form another 20 rotary hypermaps as associates of these maps, that is, simply by regarding the maps as hypermaps and then applying Machì's operations of renaming hypervertices, hyperedges and hyperfaces. There remain 13 rotary hypermaps which are not associates of maps: these form 5 sets of associates corresponding to the 5 rows in Table 2 of [4], each set containing a hypermap $\mathcal{H}_r\,(1 \le r \le 5)$ whose Walsh map $W(\mathcal{H}_r)$ (see §2, also [4, 9]) is one of the 10 rotary maps $\mathcal{M}$ of genus 2. These 13 hypermaps are summarised in Table 2, and are described in detail in §§11–15. Thus there are, in all, $10 + 20 + 13 = 43$ rotary hypermaps of genus 2. By inspection, we find that they are all reflexible, that is, each has an additional orientation-reversing automorphism. (It follows easily from [4, §4(D)] that the same happens for genus 0, whereas on the torus most rotary maps and hypermaps are chiral – not isomorphic to their mirror-images [5, §§8.3, 8.4].)

As an immediate corollary of our classification, we show (in §16) that there are no reflexible hypermaps on a non-orientable surface of characteristic -1; this extends the result of Coxeter and Moser [5, §8.8] on the non-existence of reflexible maps on such a surface.

2. Reflexible and rotary hypermaps

First, we briefly review some facts we need from the theory of hypermaps; see [2] or [6] for a more general account, and [3] for the orientable case.

We define a *hypermap* $\mathcal{H}$ to be a transitive permutation representation $\theta : \Delta \to G$ of the free product

$$\Delta = \langle R_0, R_1, R_2 \mid R_i^2 = 1\rangle \cong C_2 * C_2 * C_2$$

onto a group G of permutations of a set Ω; the elements of Ω are called *blades*. The i-faces of $\mathcal{H}$ $(i = 0, 1, 2)$, that is, the hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, are the orbits in Ω of the dihedral subgroups $\langle R_1, R_2\rangle, \langle R_2, R_0\rangle, \langle R_0, R_1\rangle$ of Δ, with incidence given by non-empty intersection.

For a combinatorial model of $\mathcal{H}$, we can take the permutation graph $\mathcal{G}$ for Δ on Ω with respect to the generators R_i: this is a trivalent graph with vertex-set Ω; it has edges labelled i corresponding to the 2-cycles of R_i on Ω, and free edges corresponding to fixed points. For a topological model, we take a set of 2-simplexes σ_α, one for each blade $\alpha \in \Omega$, with their vertices arbitrarily labelled $i = 0, 1$ and 2; whenever $(\alpha\beta)$ is a 2-cycle of R_i we join σ_α to σ_β by identifying the sides opposite their vertices labelled i. This results in a triangulated surface $\mathcal{S}$ (possibly with boundary), the vertices labelled $i = 0, 1, 2$ so that adjacent vertices have different labels. The dual of this triangulation is an imbedding of $\mathcal{G}$ in $\mathcal{S}$, with faces labelled $i = 0, 1, 2$ corresponding to the i-faces of $\mathcal{H}$. Each edge separates faces with different labels, so if we give it the third available label we recover the edge-labelling of $\mathcal{G}$. We define the *orientability*, *characteristic* and *genus* of $\mathcal{H}$ to be those of $\mathcal{S}$. The *type* of $\mathcal{H}$ is the triple (l_0, l_1, l_2), where l_i is the order of the permutation of Ω induced by the element $X_i = R_{i+1}R_{i+2}$ (subscripts mod (3)). A *map* is simply a hypermap with $l_1 = 1$ or 2; it is usual to represent a map topologically by contracting each hypervertex to a point (called a *vertex*), and each hyperedge to a 1-simplex (called an *edge*).

The automorphism group $\mathrm{Aut}\,\mathcal{H}$ of $\mathcal{H}$ is the group of all permutations of Ω which commute with G; being the centraliser of a transitive group, it acts semi-regularly on Ω. It is isomorphic to the quotient group $N_\Delta(H)/H$, where H is the stabiliser in Δ of a blade (called a *hypermap subgroup*). We say that $\mathcal{H}$ is *reflexible* if $\mathrm{Aut}\,\mathcal{H}$ acts transitively on Ω; this is equivalent to G acting regularly on Ω, in which case we can identify Ω with G so that G acts by right-multiplication. Then H is normal in Δ, with $\Delta/H \cong G$, and $\mathcal{G}$ is just the Cayley graph for G with respect to the triple $\mathbf{r} = (r_0, r_1, r_2)$ of generators $r_i = R_i\theta$ of G (called a Δ-*basis* of G); we will call $\mathcal{H}$ a *reflexible G-hypermap*.

When $\mathcal{H}$ is reflexible its automorphisms (or equivalently those of $\mathcal{G}$) are induced by the left-multiplications $g \mapsto x^{-1}g$ where $x \in G$, so

$$\mathrm{Aut}\,\mathcal{H} \cong \mathrm{Aut}\,\mathcal{G} \cong G \cong \Delta/H.$$

Two reflexible G-hypermaps $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic if and only if their hypermap subgroups are equal, that is, their corresponding edge-labelled graphs $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic, so the reflexible G-hypermaps $\mathcal{H}$ with automorphism group $\mathrm{Aut}\,\mathcal{H} \cong G$ are in bijective correspondence with the normal subgroups $H \triangleleft \Delta$ with $\Delta/H \cong G$, or equivalently with the orbits of $\mathrm{Aut}\,G$ on the Δ-bases of G.

A hypermap $\mathcal{H}$ is orientable and without boundary if and only if its hypermap subgroup H is contained in the even subgroup

$$\Delta^+ = \langle X_0, X_1, X_2 \mid X_0X_1X_2 = 1\rangle$$

of index 2 in Δ. In these circumstances the cycles of R_2 in Ω, all of length 2, correspond to the "brins" of $\mathcal{H}$ in [3, 4]. The orientation-preserving automorphism group (or *rotation group*) $\mathrm{Aut}^+\mathcal{H}$, isomorphic to $N_{\Delta^+}(H)/H$, permutes these brins, and we say that $\mathcal{H}$ is *rotary* if it does so transitively; this is equivalent to H being normal in Δ^+, in which case

$$\mathrm{Aut}^+\mathcal{H} \cong \Delta^+/H$$

and $\mathrm{Aut}^+\mathcal{H}$ is generated by a Δ^+-*basis*, a triple $\mathbf{x} = (x_0, x_1, x_2)$ of elements satisfying $x_0x_1x_2 = 1$. If $\mathcal{H}$ is reflexible then it is rotary, but the converse is false: a rotary hypermap is reflexible if and only if it has an orientation-reversing automorphism, or equivalently $\mathrm{Aut}^+\mathcal{H}$ has an automorphism inverting two of the terms in $\mathbf{x}$. A rotary hypermap $\mathcal{H}$ of type (l_0, l_1, l_2) has $N_i = N/l_i$ i-faces, all of valency l_i, where $N = |\mathrm{Aut}^+\mathcal{H}|$; its Euler characteristic is $\chi = \sum N_i - N = N(\sum l_i^{-1} - 1)$, and its genus is $g = 1 - \frac{1}{2}\chi$.

In [2] we defined the seven 2-blade hypermaps $\mathcal{B} = \mathcal{B}^+, \mathcal{B}^i$ and $\mathcal{B}^{\hat{i}}$, where $i = 0, 1, 2$; these are reflexible hypermaps with $\mathrm{Aut}\,\mathcal{B} \cong C_2 = \{\pm 1\}$. The Δ-basis for C_2 corresponding to $\mathcal{B}^+$ is $\mathbf{r} = (-1, -1, -1)$; in the case of $\mathcal{B}^i$ it is given by $r_i = 1, r_j = r_k = -1$, while for $\mathcal{B}^{\hat{i}}$ we have $r_i = -1, r_j = r_k = 1$. When $\mathcal{B} = \mathcal{B}^+$, the corresponding hypermap subgroup $B \triangleleft \Delta$

is Δ^+; the subgroups $B \lhd \Delta$ corresponding to $\mathcal{B}^i$ and $\mathcal{B}^{\hat{i}}$ are denoted by Δ^i and $\Delta^{\hat{i}}$. If $\mathcal{H}$ is a hypermap which does not cover one of these hypermaps $\mathcal{B}$ (that is, whose map subgroup H is not contained in B), then we can form the double covering $\mathcal{H} \times \mathcal{B} = \mathcal{H}^+, \mathcal{H}^i$ or $\mathcal{H}^{\hat{i}}$ of $\mathcal{H}$, the hypermap corresponding to the hypermap subgroup $H \cap B$ of Δ, as described in [2]. When $\mathcal{B} = \mathcal{B}^+$, for example, this is the (unbranched) orientable double covering $\mathcal{H}^+$ of $\mathcal{H}$; when $\mathcal{B} = \mathcal{B}^i$ or $\mathcal{B}^{\hat{i}}$ however, we obtain a double covering branched over those j- and k-faces of $\mathcal{H}$ with odd valency.

The *Walsh map* $W(\mathcal{H})$ of a hypermap $\mathcal{H}$ is the dual of the tessellation of $\mathcal{S}$ obtained by contracting each hyperface of $\mathcal{H}$ to a point; it is a bipartite map on $\mathcal{S}$, its two sets of vertices (conventionally coloured black and white) corresponding to the hypervertices and hyperedges of $\mathcal{H}$, its edges to the brins, and its faces to the hyperfaces of $\mathcal{H}$. As Walsh showed in [9], W gives a bijection between hypermaps and bipartite maps on the same surface.

Machì's group $S \cong S_3$ of hypermap operations [7] transforms one hypermap $\mathcal{H}$ to another (called an *associate* $\mathcal{H}^\pi$ of $\mathcal{H}$) by renaming hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, that is, by applying a permutation $\pi \in S_3$ to the edge-labels $i = 0, 1, 2$ of $\mathcal{G}$. These operations preserve the underlying surface, and if $\mathcal{H}$ is reflexible or rotary then so are all its associates, with the same automorphism and rotation groups. In classifying the rotary hypermaps of genus 2 it is therefore sufficient for us to find one representative from each S-orbit.

3. The rotary maps of genus 2

The rotary maps $\mathcal{M}$ of genus 2 were classified by Threlfall [8] in 1932, completing earlier work of Brahana [1] (see Table 9 of [5]). They are the maps $\mathcal{M}_0, \ldots, \mathcal{M}_5$ described in Table 1 and illustrated in Figure 1, together with the duals of $\mathcal{M}_1, \mathcal{M}_3, \mathcal{M}_4$ and $\mathcal{M}_5$, which are denoted by $\mathcal{M}_1^{(02)}$, etc., to indicate a transposition of 0- and 2-faces (vertices and faces).

Map	Notation in [5]	Hyp. type	σ	$N_0\, N_1\, N_2$	$Aut^+\mathcal{M}$	$Aut\mathcal{M}$
$\mathcal{M}_0$	$\{8,8\}_{1,0}$	8 2 8	3	1 4 1	C_8	D_8
$\mathcal{M}_1$ $\mathcal{M}_1^{(02)}$	$\{10,5\}_2$ $\{5,10\}_2$	5 2 10 10 2 5	6	2 5 1 1 5 2	C_{10}	D_{10}
$\mathcal{M}_2$	$\{6,6\}_2$	6 2 6	3	2 6 2	$C_6 \times C_2$	$D_6 \times C_2$
$\mathcal{M}_3$ $\mathcal{M}_3^{(02)}$	$\{8,4\}_{1,1}$ $\{4,8\}_{1,1}$	4 2 8 8 2 4	6	4 8 2 2 8 4	$\langle -2,4 \mid 2\rangle$	$\mathrm{Hol}(C_8)$
$\mathcal{M}_4$ $\mathcal{M}_4^{(02)}$	$\{6,4 \mid 2\}$ $\{4,6 \mid 2\}$	4 2 6 6 2 4	6	6 12 4 4 12 6	$(4,6 \mid 2,2)$	$D_3 \times D_4$
$\mathcal{M}_5$ $\mathcal{M}_5^{(02)}$	$\{4+4,3\}$ $\{3,4+4\}$	3 2 8 8 2 3	6	16 24 6 6 24 16	$GL_2(3)$	$GL_2(3) \rtimes C_2$

Table 1

The entries in each of the six rows are explained as follows: the first two columns give our notation for $\mathcal{M}$ and that in [5, Table 9]. The third column gives the type $(l_0, l_1, l_2) = (l, m, n)$ of $\mathcal{M}$ as a hypermap: thus each i-face of $\mathcal{M}$ is incident with l_i j-faces and l_i k-faces, where $\{i, j, k\} = \{0, 1, 2\}$, so in the notation of [5], $\mathcal{M}$ is a map of type $\{l_2, l_0\}$. The next column gives the number σ of non-isomorphic associates $\mathcal{M}^\pi$ of $\mathcal{M}$ ($\pi \in S_3$); this is the length of the S-orbit containing $\mathcal{M}$. The fifth column gives the number N_i of i-faces of $\mathcal{M}$, the sixth describes the rotation-group $\mathrm{Aut}^+\mathcal{M}$ (of order $N = N_i l_i$), and the final column gives the full automorphism group $\mathrm{Aut}\,\mathcal{M}$ (of order $2N$ since each map $\mathcal{M}$ is reflexible); these groups will be explained in more detail in §§5–9.

The maps $\mathcal{M}_0, \ldots, \mathcal{M}_5$ are illustrated in Figure 1. In each case we have indicated a pair of sides to be identified; the remaining identifications can be deduced by symmetry about the centre, since the maps are rotary. Where a map is bipartite, we have indicated this by 2-colouring the vertices; such maps $\mathcal{M}_r$ will reappear in §§11–15 as Walsh maps of other rotary hypermaps $\mathcal{H}_r$ of genus 2, with $\mathrm{Aut}\,\mathcal{H}_r$ corresponding to the subgroup $\mathrm{Aut}^{\hat{0}}\mathcal{M}_r$ of $\mathrm{Aut}\,\mathcal{M}_r$ preserving the vertex-colours, so we will determine these subgroups in §§5–9.

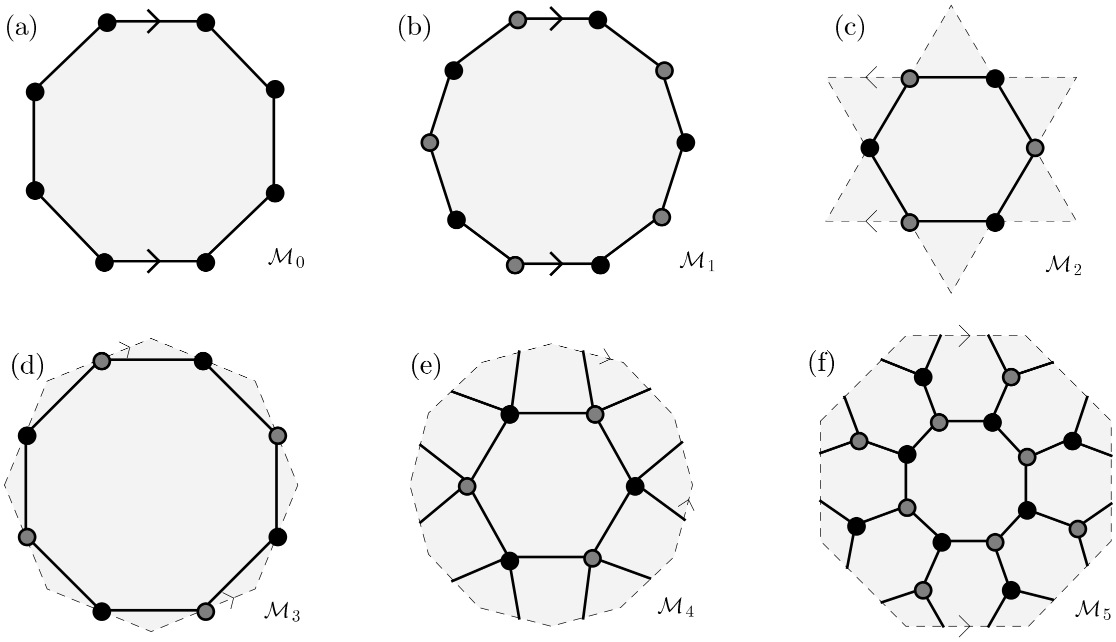

Figure 1. The rotary maps $\mathcal{M}_r$

For each $\mathcal{M} = \mathcal{M}_0, \ldots, \mathcal{M}_5$ in Figure 1, $\mathrm{Aut}\,\mathcal{M}$ is generated by three automorphisms a, b, c; we take a to be the rotation (in the anticlockwise direction) by $2\pi/n$ about the central face, b to be the reflection in the vertical axis (so that $\langle a, b\rangle = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle \cong D_n$ is the subgroup preserving that face), and c to be the rotation by π about the midpoint of the lowest edge of that face (so that $bc = cb$ and $\langle a, c\rangle = \mathrm{Aut}^+\mathcal{M}$). Alternatively, $\mathrm{Aut}\,\mathcal{M}$ can be generated by the reflections $a_0 = b$, $a_1 = ab$ and $a_2 = bc$ in the sides of a triangle, satisfying

$$a_i^2 = (a_1a_2)^m = (a_2a_0)^2 = (a_0a_1)^n = 1$$

(see Figure 2).

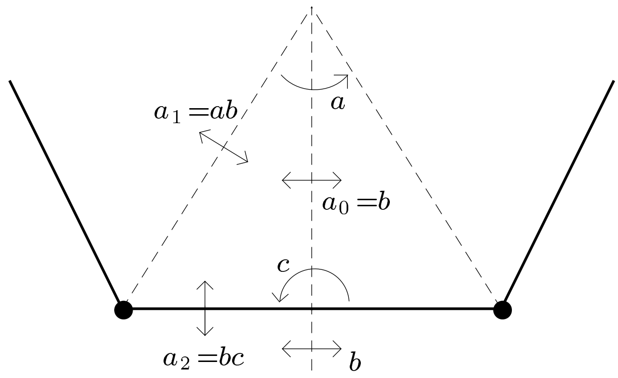

Figure 2. Generators for Aut $\mathcal{M}$

4. Some dihedral maps

Before discussing the maps $\mathcal{M}_r$ in detail, we need to describe some maps with dihedral automorphism group

$$D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle,$$

since some of these are isomorphic to maps $\mathcal{M}_r$ or arise as their direct factors.

The Δ-basis $\mathbf{r} = (r_0, r_1, r_2) = (b, ab, b)$ of D_n corresponds to a reflexible hypermap $\mathcal{D}_n^*$ of type $(n, 1, n)$ on the sphere; this is a map with one vertex, one face, and n free edges (Figure 3):

Figure 3. $\mathcal{D}_6^*$ as a hypermap and as a map

Taking $\mathbf{r} = (b, ab, 1)$ we obtain a reflexible hypermap $\mathcal{D}_n^\circ$ of type $(2, 2, n)$ on a closed disc; this is a map with n vertices and n edges (forming the boundary) and one face (Figure 4):

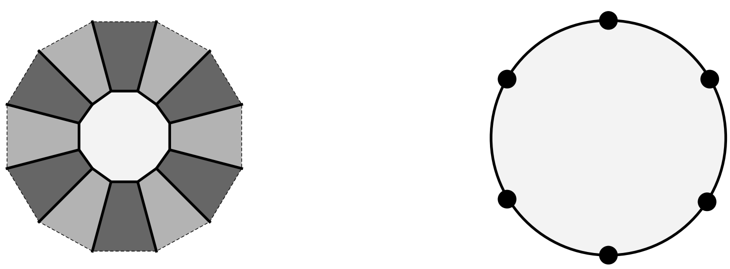

Figure 4. $\mathcal{D}_6^\circ$ as a hypermap and as a map

If $n = 2m$ is even and we take $\mathbf{r} = (b, ab, a^m b)$ we obtain a reflexible orientable map $\mathcal{D}_n^\diamond$ with one face, m edges, and one or two vertices as m is even or odd; $\mathcal{D}_n^\diamond$ has characteristic

$\chi = 2 - m$ or $3 - m$, and hence has genus $g = m/2$ or $(m-1)/2$ respectively. One can form $\mathcal{D}_n^\diamond$ by identifying opposite sides of a regular n-gon (see Figure 5 for the cases $n = 4$ and $n = 6$).

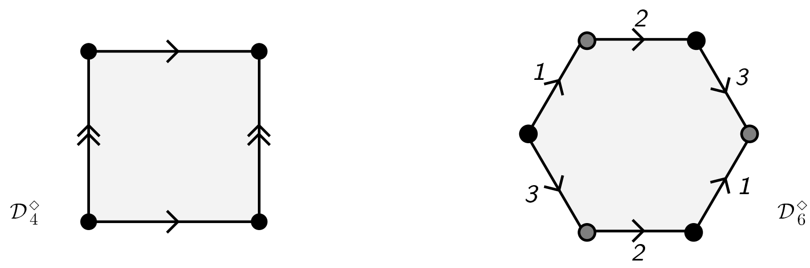

Figure 5. $\mathcal{D}_4^\diamond$ and $\mathcal{D}_6^\diamond$

If $n = 2m$ with m odd, and $\mathbf{r} = (b, a^2b, a^m)$, we obtain a reflexible spherical map $\mathcal{D}_n^\ominus$ with two faces and an equatorial belt of m vertices and m edges (see Figure 6 for $\mathcal{D}_6^\ominus$).

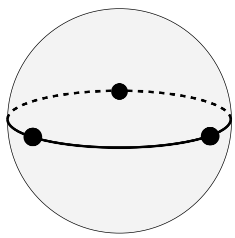

Figure 6. $\mathcal{D}_6^\ominus$

5. The maps $\mathcal{M}_0$ and $\mathcal{M}_1$

The maps $\mathcal{M}_0$ and $\mathcal{M}_1$ each consist of a single n-gon, where $n = 8$ or 10 respectively, with opposite sides identified to form 4 or 5 edges and 1 or 2 vertices, so $\mathcal{M}_0 \cong \mathcal{D}_8^\diamond$ and $\mathcal{M}_1 \cong \mathcal{D}_{10}^\diamond$. If a, b and c are as in Figure 2, then since the identifications imply that $c = a^{n/2}$ we have

$$\mathrm{Aut}^+\mathcal{M} = \langle a \mid a^n = 1\rangle \cong C_n$$

and

$$\mathrm{Aut}\,\mathcal{M} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle \cong D_n.$$

For $n = 8$ and 10, D_n has a Δ-basis $\mathbf{r} = (r_0, r_1, r_2) = (b, ab, a^{n/2}b)$ of type $(8, 2, 8)$ (resp. $(5, 2, 10)$) which is unique up to automorphisms, so $\mathcal{M}_0$ and $\mathcal{M}_1$ are the unique reflexible D_n-maps of their given types. In particular, $\mathcal{M}_0$ is self-dual, since the automorphism $a \mapsto a^5, b \mapsto a^4b$ of D_8 sends $\mathbf{r}$ to the reverse Δ-basis $\mathbf{r}^{(02)} = (r_2, r_1, r_0)$. However, $\mathcal{M}_1$ is not self-dual: its dual $\mathcal{M}_1^{(02)}$ is another reflexible D_{10}-map of genus 2, with two pentagonal faces, one vertex and five edges (all loops). As hypermaps, $\mathcal{M}_0$ and $\mathcal{M}_1$ therefore lie in S-orbits of lengths $\sigma = 3$ and 6 respectively.

The subgroup $\mathrm{Aut}^{\hat{0}}\mathcal{M}_1$ of $\mathrm{Aut}\,\mathcal{M}_1$ fixing the two vertices is $\langle a^2, ab\rangle \cong D_5$; since $\mathrm{Aut}\,\mathcal{M}_1 \cong \mathrm{Aut}^{\hat{0}}\mathcal{M}_1 \times \langle a^5\rangle \cong D_5 \times C_2$, it follows that $\mathcal{M}_1$ decomposes as a disjoint product

$$\mathcal{M}_1 \cong \mathcal{D}_5^* \times \mathcal{B}^{\hat{0}} = (\mathcal{D}_5^*)^{\hat{0}},$$

where $\mathcal{M}_1/\mathrm{Aut}^{\hat{0}}\mathcal{M}_1$ is the 2-blade disc map $\mathcal{B}^{\hat{0}}$, and $\mathcal{M}_1/\langle a^5\rangle$ is the reflexible spherical D_5-map $\mathcal{D}_5^*$ (see Figure 7); thus $\mathcal{M}_1$ is a double covering of $\mathcal{D}_5^*$ with branch-points on its five edges. This decomposition of $\mathrm{Aut}\,\mathcal{M}_1$ is not unique: one can replace the first factor $\mathrm{Aut}^{\hat{0}}\mathcal{M}_1$ with $\langle a^2, b\rangle \cong D_5$, giving

$$\mathcal{M}_1 \cong \mathcal{D}_5^* \times \mathcal{B}^0 = (\mathcal{D}_5^*)^0.$$

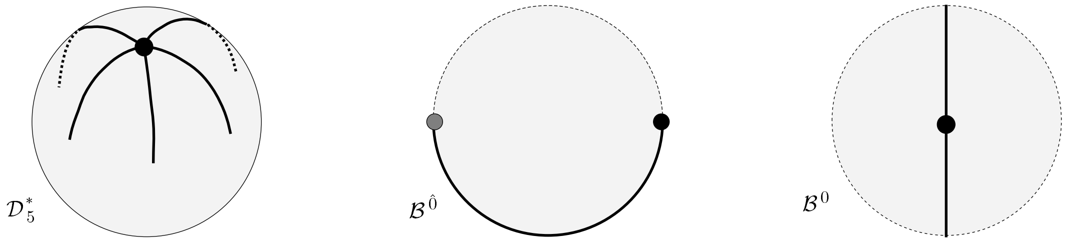

Figure 7. $\mathcal{D}_5^*$, $\mathcal{B}^{\hat{0}}$ and $\mathcal{B}^0$

6. The map $\mathcal{M}_2$

In Figure 1(c), one hexagonal face of $\mathcal{M}_2$ is obvious, while the six small triangles make up the other face. The automorphism group $\mathrm{Aut}\,\mathcal{M}_2$ is

$$\langle a, b, c \mid a^6 = b^2 = (ab)^2 = c^2 = 1, a^c = a, b^c = b\rangle \cong D_6 \times C_2,$$

where a is a rotation of Figure 8 by $2\pi/6$ and b is the reflection in the horizontal axis (these generate the factor D_6, the subgroup preserving the two faces), while c is a half-turn reversing each edge and transposing the two faces and the two vertices.

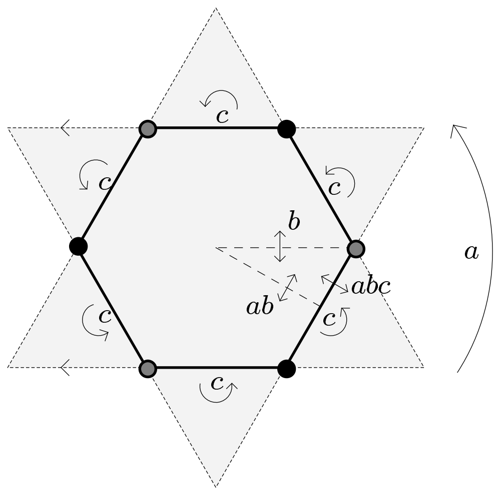

Figure 8. Generators for $\mathrm{Aut}\,\mathcal{M}_2$

It follows that $\mathrm{Aut}^+\mathcal{M}_2 = \langle a, c\rangle \cong C_6 \times C_2$. The direct decomposition $\mathrm{Aut}\,\mathcal{M}_2 = \langle a, b\rangle \times \langle c\rangle \cong D_6 \times C_2$ gives

$$\mathcal{M}_2 \cong \mathcal{D}_6^* \times \mathcal{B}^0 = (\mathcal{D}_6^*)^0,$$

where $\mathcal{M}_2/\langle c\rangle \cong \mathcal{D}_6^*$ (see Figure 3) and $\mathcal{M}_2/\langle a,b\rangle \cong \mathcal{B}^0$ (Figure 7). These decompositions of $\mathrm{Aut}\,\mathcal{M}_2$ and of $\mathcal{M}_2$ are not unique: one can replace the direct factor $\langle c\rangle$ with $\langle a^3c\rangle \cong C_2$, so that

$$\mathcal{M}_2 \cong \mathcal{D}_6^\diamond \times \mathcal{B}^0 = (\mathcal{D}_6^\diamond)^0$$

where $\mathcal{M}_2/\langle a^3c\rangle$ is the torus map $\mathcal{D}_6^\diamond$ shown in Figure 5; one can also replace $\langle a,b\rangle$ with $\langle ac,b\rangle \cong D_6$ (the subgroup $\mathrm{Aut}^{\hat{0}}\mathcal{M}_2$ of $\mathrm{Aut}\,\mathcal{M}_2$ fixing the two vertices), thus replacing $\mathcal{B}^0$ with $\mathcal{B}^{\hat{0}}$, so that

$$\mathcal{M}_2 \cong (\mathcal{D}_6^*)^{\hat{0}} \cong (\mathcal{D}_6^\diamond)^{\hat{0}}.$$

Further decompositions of $\mathcal{M}_2$ can be obtained by using the isomorphism $D_6 \cong D_3 \times C_2$.

As a Δ-basis for $D_6 \times C_2$ corresponding to $\mathcal{M}_2$ one can take $\mathbf{r} = (ab, b, abc)$, reflections in the sides of the triangle in Figure 8. The automorphism $a \mapsto ac, b \mapsto b, c \mapsto c$ reverses r, thus confirming that $\mathcal{M}_2$ is self-dual.

7. The map $\mathcal{M}_3$

Like $\mathcal{M}_2$, $\mathcal{M}_3$ has two faces, one made up of the outer triangles in Figure 1(d). It has automorphism group

$$\mathrm{Aut}\,\mathcal{M}_3 = \langle a,b,c \mid a^8 = b^3 = c^2 = 1, a^b = a^{-1}, a^c = a^3, b^c = b\rangle,$$

where a is a rotation through $2\pi/8$, b is a horizontal reflection, and c is a rotation about the midpoint of an edge. The rotation group is

$$\mathrm{Aut}^+\mathcal{M}_3 = \langle a,c \mid a^8 = c^2 = 1, a^c = a^3\rangle,$$

a group of order 16 isomorphic to $\langle -2,4 \mid 2\rangle$ in [5, §6.6], while the subgroup preserving each of the two faces is

$$\langle a,b \mid a^8 = b^2 = 1, a^b = a^{-1}\rangle \cong D_8.$$

It is clear from the presentation that $\langle a\rangle$ is a normal subgroup of $\mathrm{Aut}\,\mathcal{M}_3$, complemented by the Klein 4-group $\langle b,c\rangle$ which induces all four automorphisms $a \mapsto a^{\pm 1}, a^{\pm 3}$ of $\langle a\rangle$; thus $\mathrm{Aut}\,\mathcal{M}_3$ is isomorphic to the holomorph $\mathrm{Hol}\,C_8$ of C_8. The map $\mathcal{M}_3$ is bipartite, the subgroup $\mathrm{Aut}^{\hat{0}}\mathcal{M}_3$ preserving the vertex-colours being the "even subgroup" $\langle a^2, ac, bc\rangle$. This is a central product

$$\mathrm{Aut}^{\hat{0}}\mathcal{M}_3 \cong Q_8 \cdot C_4,$$

of

$$\mathrm{Aut}^{\hat{0}}\mathcal{M}_3 \cap \mathrm{Aut}^+\mathcal{M}_3 = \langle a^2, ac\rangle \cong Q_8$$

(a quaternion group of order 8), and

$$\langle a^2bc\rangle \cong C_4,$$

amalgamating their central subgroups $\langle a^4 = (ac)^2\rangle$ and $\langle (a^2bc)^2 = a^4\rangle$ (both isomorphic to C_2).

Like $\mathrm{Aut}\,\mathcal{M}_0$, $\mathrm{Aut}\,\mathcal{M}_3$ is indecomposable (as a direct product), so $\mathcal{M}_3$ does not arise as a disjoint product of simpler maps. As a Δ-basis for $\mathrm{Aut}\,\mathcal{M}_3$, we can take $\mathbf{r} = (b, ab, bc)$; by reversing $\mathbf{r}$ we obtain the dual of $\mathcal{M}_3$, a reflexible map $\mathcal{M}_3^{(02)}$ with four quadrilateral faces.

8. The map $\mathcal{M}_4$

$\mathcal{M}_4$ has four hexagonal faces: the central face in Figure 1(e) meets two of the others alternately across its six edges, and meets the fourth face at the six vertices. The automorphism group is a direct product

$$\mathrm{Aut}\,\mathcal{M}_4 = \langle a, b\rangle \times \langle c, d\rangle \cong D_3 \times D_4,$$

where a and c are rotations by $2\pi/3$ and π about the central face, and b and d are reflections in the vertical axis and in a horizontal edge (Figure 9).

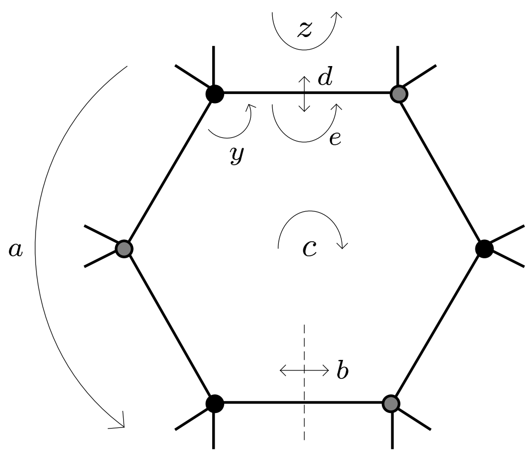

Figure 9. Generators for Aut $\mathcal{M}_4$

(One can easily check that a and b commute with c and d, and that they satisfy $a^3 = b^2 = (ab)^2 = c^2 = d^2 = (cd)^4 = 1$.)

This decomposition of Aut $\mathcal{M}_4$ yields

$$\mathcal{M}_4 \cong \mathcal{D}_3^\circ \times (\mathcal{D}_4^\circ)^{(02)},$$

where $\mathcal{M}_4/\langle c, d\rangle$ and $\mathcal{M}_4/\langle a, b\rangle$ are the disc maps $\mathcal{D}_3^\circ$ and $(\mathcal{D}_4^\circ)^{(02)}$ shown in Figure 10.

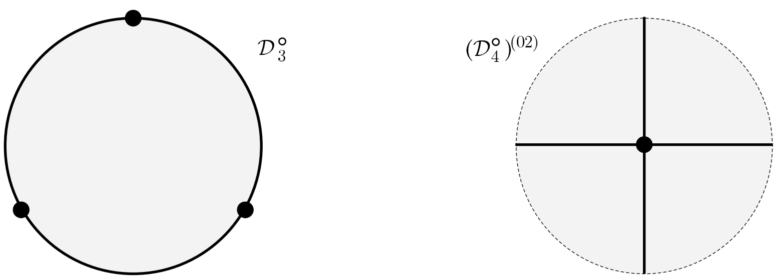

Figure 10. $\mathcal{D}_3^\circ$ and $(\mathcal{D}_4^\circ)^{(02)}$

The rotation group $\mathrm{Aut}^+\mathcal{M}_4$ is a split extension of $\langle a, c, c^d\rangle \cong C_3 \times C_2 \times C_2$ by $\langle e\rangle \cong C_2$, where $e = bd$ is a half-turn about the midpoint of a horizontal edge, inverting a and transposing c and c^d. This is isomorphic to the group

$$(4, 6 \mid 2, 2) = \langle y, z \mid y^4 = z^6 = (yz)^2 = (y^{-1}z)^2 = 1\rangle$$

of [5, §8.5]: we can take $y = ace$ and $z = ac^d$ (rotations about a vertex and a face).

The subgroup $\mathrm{Aut}^{\hat{0}}\mathcal{M}_4$ of $\mathrm{Aut}\,\mathcal{M}_4$ preserving the two vertex-colours is isomorphic to (but distinct from) $\mathrm{Aut}^+\mathcal{M}_4$, being a split extension of $\langle a, d, d^c\rangle \cong C_3 \times C_2 \times C_2$ by $\langle bc\rangle \cong C_2$, with bc inverting a and transposing d and d^c. An isomorphism with $(4, 6 \mid 2, 2)$ is given by putting $y = adbc$ and $z = ad^c$.

As a Δ-basis for $\mathrm{Aut}\,\mathcal{M}_4$ we can take $r_0 = b, r_1 = abc$ and $r_2 = d$. Transposing r_0 and r_2 we obtain the dual map

$$\mathcal{M}_4^{(02)} \cong (\mathcal{D}_3^0)^{(02)} \times \mathcal{D}_4^0,$$

a reflexible map with six quadrilateral faces.

9. The map $\mathcal{M}_5$

$\mathcal{M}_5$ has six octagonal faces. The central face in Figure 1(f) meets four others, each across two edges, but does not meet the sixth face.

It is simplest to start with the rotation group

$$\mathrm{Aut}^+\mathcal{M}_5 = \langle x, y, z \mid x^2 = y^3 = z^8 = xyz = (xz^4)^2 = 1\rangle,$$

where x, y and z are rotations about an edge, vertex and face (all incident), as in Figure 11.

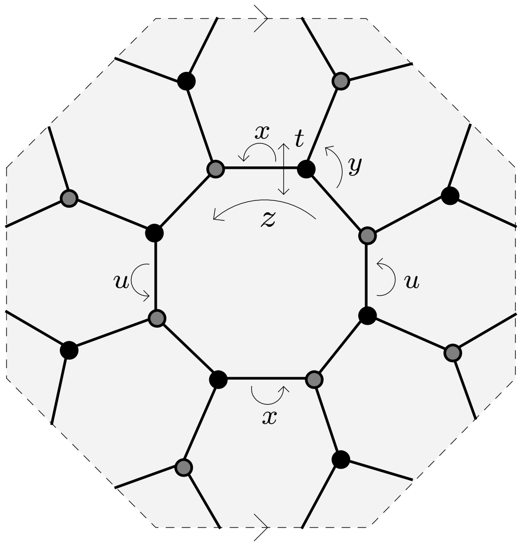

Figure 11. Generators for $\mathrm{Aut}\,\mathcal{M}_5$

This group can be identified with $GL_2(3)$ by putting

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

so that z^4 is the central involution $-I$. It is also isomorphic to the group

$$\langle -3, 4 \mid 2\rangle = \langle r, s \mid r^{-3} = s^4,\ (rs)^2 = 1\rangle$$

of [5, §6.6]: we can take $r = -y$ and $s = z$ (so that $r^{-3} = s^4 = -I$).

Now $\mathrm{Aut}\, GL_2(3) \cong PGL_2(3) \times C_2$, with $PGL_2(3)$ ($= GL_2(3)/\langle -I\rangle \cong S_4$) the group of inner automorphisms, and the factor C_2 generated by the outer automorphism $\alpha : g \mapsto \det(g).g$ of $GL_2(3)$. The full automorphism group $\mathrm{Aut}\,\mathcal{M}_5$ is generated by $\mathrm{Aut}^+\mathcal{M}_5$ together with the reflection t shown in Figure 11. This acts on $\mathrm{Aut}^+\mathcal{M}_5$ by $x^t = x, y^t = y^{-1}$, so t induces the automorphism $\tau : g \mapsto \det(g).u^{-1}gu$ of $GL_2(3)$ where $u = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, that is, $\tau = \alpha \circ i_u = i_u \circ \alpha$ where i_u is the inner automorphism induced by u. (The rotation u is shown in Figure 11.) If we identify $\mathrm{Aut}^+\mathcal{M}_5$ with $GL_2(3)$ as above, and define $v = tu \in \mathrm{Aut}\,\mathcal{M}_5$, then $\mathrm{Aut}\,\mathcal{M}_5$ is also generated by $\mathrm{Aut}^+\mathcal{M}_5 = GL_2(3)$ and v, with $v^2 = (tu)^2 = u^t.u = -u.u = -I$ and v inducing the automorphism α of $GL_2(3)$ by conjugation. The subgroup of $\mathrm{Aut}\,\mathcal{M}_5$ preserving the vertex-colours is $\mathrm{Aut}^{\hat{0}}\mathcal{M}_5 = \langle SL_2(3), t\rangle \cong GL_2(3)$, isomorphic to but distinct from $\mathrm{Aut}^+\mathcal{M}_5$. As a Δ-basis for $\mathrm{Aut}\,\mathcal{M}_5$ we can take $r_0 = xt,\ r_1 = ty$ and $r_2 = t$; transposing r_0 and r_2 we obtain $\mathcal{M}_5^{(02)}$, a reflexible map with 16 triangular faces.

10. The remaining hypermaps

We have now described the 10 rotary maps of genus 2. By regarding them as hypermaps, and by taking their associates (under the action of S), we obtain 30 of the 43 rotary hypermaps of genus 2; we will now consider the remaining 13 hypermaps, whose properties are summarised in Table 2, each row describing a representative $\mathcal{H}_r$ ($r = 1, \ldots, 5$) of an S-orbit of length σ.

Hypermap $\mathcal{H}$	Hyp. type	σ	$N_0\ N_1\ N_2$	$Aut^+\mathcal{H}$	$Aut\mathcal{H}$
$\mathcal{H}_1 = W^{-1}(\mathcal{M}_1)$	5 5 5	3	1 1 1	C_5	D_5
$\mathcal{H}_2 = W^{-1}(\mathcal{M}_2)$	6 6 3	3	1 1 2	C_6	D_6
$\mathcal{H}_3 = W^{-1}(\mathcal{M}_3)$	4 4 4	1	2 2 2	Q_8	$Q_8 \cdot C_4$
$\mathcal{H}_4 = W^{-1}(\mathcal{M}_4)$	4 4 3	3	3 3 4	$\hat{D}_3$	$(4, 6 \mid 2, 2)$
$\mathcal{H}_5 = W^{-1}(\mathcal{M}_5)$	3 3 4	3	8 8 6	$SL_2(3)$	$GL_2(3)$

Table 2

This table is an extension of Table 2 of [4], where Corn and Singerman determined the possible types and rotation groups $\mathrm{Aut}^+\mathcal{H}$ of the rotary hypermaps $\mathcal{H}$ of genus 2. (Note that the presentation immediately following Table 2 in [4] should read $\langle a, b, c \mid a^r = b^s = c^t = abc\rangle$, and not as given.) We shall continue their work by enumerating these hypermaps, dividing them into S-orbits, showing that they are all reflexible, and determining their automorphism groups $\mathrm{Aut}\,\mathcal{H}$. Before investigating these hypermaps in detail, we point out that (as indicated in the first column of Table 2) our chosen representatives $\mathcal{H}_r$ have as their Walsh maps $W(\mathcal{H}_r)$ the five bipartite rotary maps $\mathcal{M}_r$ ($r = 1, \ldots, 5$) in Table 1; only $\mathcal{M}_0$, which is not bipartite, does not arise in this way.

11. The hypermap $\mathcal{H}_1$

The first case in Corn and Singerman's list is that of a rotary hypermap $\mathcal{H}$ of type $(5,5,5)$ with $\mathrm{Aut}^+\mathcal{H} \cong C_5 = \langle c \mid c^5 = 1\rangle$. Now $\mathrm{Aut}\, C_5 \cong C_4$, generated by $c \mapsto c^2$, and this has three orbits on the Δ^+-bases for C_5 of type $(5,5,5)$, containing $\mathbf{x}(i) = (c^{i-1}, c^{-i}, c)$ for $i = 2,3,4$. Thus there are (up to isomorphism) three rotary hypermaps $\mathcal{H}(i) = \mathcal{H}(2), \mathcal{H}(3)$ and $\mathcal{H}(4)$ of type $(5,5,5)$, corresponding to the triples $\mathbf{x}(i)$. Any rotary hypermap with an abelian rotation group must be reflexible, since each abelian group admits an automorphism inverting every element; hence each $\mathcal{H}(i)$ is reflexible, with

$$\mathrm{Aut}\,\mathcal{H}(i) \cong D_5 = \langle c, b \mid c^5 = b^2 = 1,\ c^b = c^{-1}\rangle.$$

As a Δ-basis for D_5 corresponding to $\mathcal{H}(i)$ we can take $\mathbf{r}(i) = (b, bc, bc^i)$. These three hypermaps, which can be distinguished from each other by the property that $r_0r_2 = (r_0r_1)^i$, are shown in Figure 12.

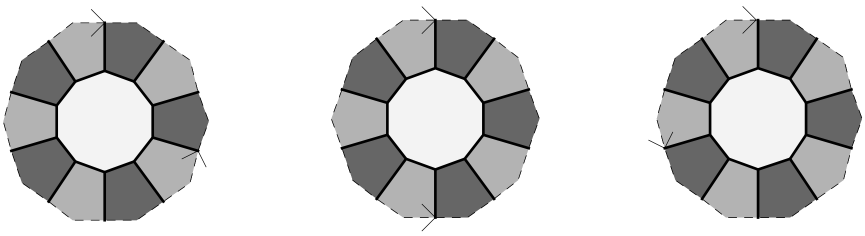

Figure 12. The hypermaps $\mathcal{H}(i)$

In each case, the sides of the decagon are identified as indicated, to give one hypervertex, one hyperedge, and one hyperface; $\mathrm{Aut}\,\mathcal{H}(i)$ is generated by a rotation through $2\pi/5$ about the centre, and a reflection in the horizontal axis. Note that $W(\mathcal{H}(3))$ is the bipartite rotary map $\mathcal{M}_1$ we have already discussed, while $\mathcal{H}(4)$ is the hypermap in Figure 9 of [4]. The automorphism of D_5 which transposes $b(= r_0)$ and $bc(= r_1)$ sends bc^i to bc^{1-i}, so $\mathcal{H}(2)^{(01)} \cong \mathcal{H}(4)$ and $\mathcal{H}(3)^{(01)} \cong \mathcal{H}(3)$. (This property of $\mathcal{H}(3)$ corresponds to the fact that $\mathcal{M}_1$, being rotary and bipartite, has an automorphism interchanging its black and white vertices.) Similarly $\mathcal{H}(3)^{(02)} \cong \mathcal{H}(2)$, so all three hypermaps are associates of each other; we have chosen $\mathcal{H}(3) = W^{-1}(\mathcal{M}_1)$ as our representative $\mathcal{H}_1$ of this S-orbit in Table 2.

12. The hypermap $\mathcal{H}_2$

The second case in [4] concerns a rotary hypermap $\mathcal{H}$ of type $(6,6,3)$ – or some permutation of this – with

$$\mathrm{Aut}^+\mathcal{H} \cong C_6 = \langle c \mid c^6 = 1\rangle.$$

Now $\mathrm{Aut}\, C_6$ (of order 2, generated by $c \mapsto c^{-1}$) has a single orbit on the Δ^+-bases for C_6 of type $(6,6,3)$, represented by $\mathbf{x} = (c, c, c^{-2})$; thus we find a single rotary hypermap $\mathcal{H}_2$ of type $(6,6,3)$, and as in case (1) since C_6 is abelian $\mathcal{H}_2$ must be reflexible, with

$$\mathrm{Aut}\,\mathcal{H}_2 \cong D_6 = \langle b, c \mid b^2 = c^6 = 1,\ c^b = c^{-1}\rangle.$$

This uniqueness implies that $\mathcal{H}_2^{(01)} \cong \mathcal{H}_2$, so $\mathcal{H}_2$ lies in an S-orbit of length 3, its associates having types $(3,6,6)$ and $(6,3,6)$. The latter is illustrated in Figure 10 of [4], while our Figure 13 shows two views of $\mathcal{H}_2$:

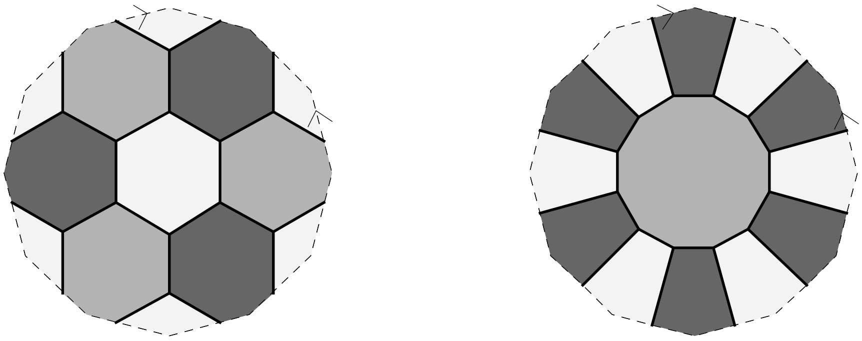

Figure 13. The hypermap $\mathcal{H}_2$

The first illustrates the fact that $W(\mathcal{H}_2) = \mathcal{M}_2$ (see Figure 1(c)), while the second shows that $\mathrm{Aut}\,\mathcal{H}_2 \cong D_6$. As a Δ-basis for D_6 corresponding to $\mathcal{H}_2$ we can take $\mathbf{r} = (b, bc^2, bc)$. Since $D_6 = \langle b, c^2\rangle \times \langle c^3\rangle \cong D_3 \times C_2$, $\mathcal{H}_2$ is a disjoint product

$$\mathcal{H}_2 = \mathcal{H}_2/\langle c^3\rangle \,\times\, \mathcal{H}_2/\langle b, c^2\rangle \cong W^{-1}(\mathcal{D}_6^{\diamond}) \times \mathcal{B}^{\hat{2}} \cong W^{-1}(\mathcal{D}_6^{\diamond})^{\hat{2}}$$

(see Figure 14).

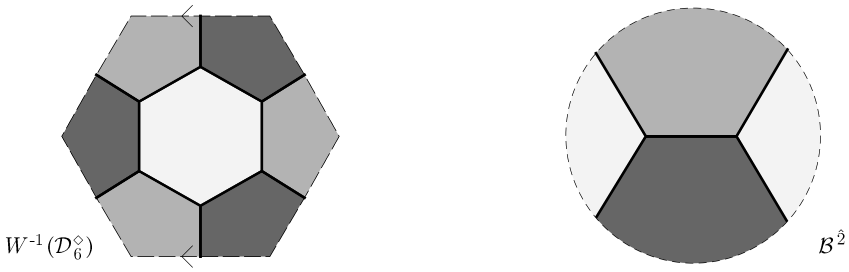

Figure 14. The hypermaps $W^{-1}(\mathcal{D}_6^{\diamond})$ and $\mathcal{B}^{\hat{2}}$

In this decomposition one can replace $\langle b, c^2\rangle$ with $\langle bc, c^2\rangle$, thus replacing $\mathcal{B}^{\hat{2}}$ with $\mathcal{B}^2$.

13. The hypermap $\mathcal{H}_3$

The third possibility in [4] is that $\mathcal{H}$ has type $(4,4,4)$ with $\mathrm{Aut}^+\mathcal{H}$ isomorphic to Q_8, the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ with

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k, \text{ etc.}$$

Now $\mathrm{Aut}\,Q_8\,(\cong S_4)$ has just one orbit on Δ^+-bases of type $(4,4,4)$, represented by $\mathbf{x} = (i, j, -k)$, so there is a unique rotary hypermap $\mathcal{H}_3$ of type $(4,4,4)$ with rotation group Q_8. By its uniqueness, $\mathcal{H}_3$ must be reflexible and S-invariant, with automorphism group $\mathrm{Aut}\,\mathcal{H}_3 = \langle Q_8, t\rangle$ where $t^2 = 1, i^t = -i$ and $j^t = -j$ (so $k^t = k$); now $i^k = -i$ and $j^k = -j$, so $\mathrm{Aut}\,\mathcal{H}_3 = \langle Q_8, u\rangle$ where $u := kt$ centralises Q_8 and satisfies $u^2 = k.k^t = k^2 = -1$; thus $\mathrm{Aut}\,\mathcal{H}_3$ is a central product

$$\mathrm{Aut}\,\mathcal{H}_3 = \mathrm{Aut}^+\mathcal{H}_3 \cdot \langle u\rangle \cong Q_8 \cdot C_4$$

of Q_8 by C_4, amalgamating the central subgroup $\{\pm 1\} = \langle u^2\rangle \cong C_2$. Being a nilpotent group with an indecomposable centre (namely $\langle u\rangle \cong C_4$), $\mathrm{Aut}\,\mathcal{H}_3$ must also be indecomposable, so $\mathcal{H}_3$ is not a disjoint product of smaller hypermaps.

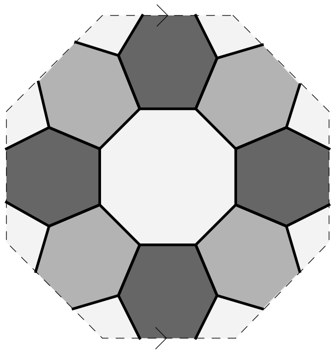

Figure 15. The hypermap $\mathcal{H}_3$

By comparing Figures 15 and 1(d), we see that $W(\mathcal{H}_3) \cong \mathcal{M}_3$. The six elements of order 4 in $\mathrm{Aut}^+\mathcal{H}_3$ are the quarter-turns fixing the centres of the two hypervertices, hyperedges and hyperfaces respectively, so the element -1 rotates each of these through a half-turn, while u (which is fixed-point-free) transposes each of these three pairs. The quotient hypermap $\overline{\mathcal{H}}_3 = \mathcal{H}_3/\langle -1\rangle$ is the unique rotary hypermap $\mathcal{D}$ of type $(2,2,2)$ with $\mathrm{Aut}\,\mathcal{D} \cong (C_2)^3$, shown in Figure 16; thus $\mathcal{H}_3$ is a 2-sheeted covering of $\mathcal{D}$, branched over its two hypervertices, two hyperedges and two hyperfaces.

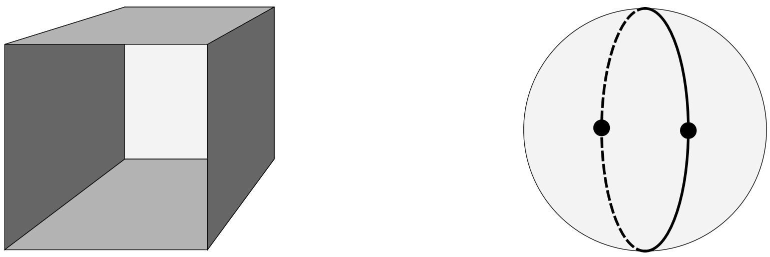

Figure 16. The hypermap $\mathcal{D}$

Comparison of Figures 15 and 1(f) shows that if we remove the face-labelling of $\mathcal{H}_3$ then the underlying trivalent map is just $\mathcal{M}_5$. This is reflexible, corresponding to the fact that $\mathcal{H}_3$ (alone among the rotary hypermaps of genus 2) is S-invariant; in general, a reflexible hypermap will give rise in this way to a vertex-transitive (but not necessarily reflexible) trivalent map on the same surface.

14. The hypermap $\mathcal{H}_4$

In the next case, $\mathcal{H}$ has type $(4,4,3)$ – or some permutation of this – with $\mathrm{Aut}^+\mathcal{H}$ isomorphic to the binary dihedral group

$$\hat{D}_3 = \langle a,b,c \mid a^2 = b^2 = c^3 = abc\rangle = \langle b,c \mid c^6 = 1,\ b^2 = c^3,\ c^b = c^{-1}\rangle;$$

this group (denoted by $\langle 2,2,3\rangle$ in §6.5 of [5]) has a central involution c^3 with $\hat{D}_3/\langle c^3\rangle \cong D_3$; apart from the powers of c, the six elements bc^i all have order 4.

Up to automorphisms (of which there are 12), $\hat{D}_3$ has a unique Δ^+-basis of type $(4,4,3)$, represented by $\mathbf{x} = (b, bc, c^2)$. Hence there is a unique rotary hypermap $\mathcal{H}_4$ of type $(4,4,3)$ with $\mathrm{Aut}^+\mathcal{H}_4 \cong \hat{D}_3$, and as in the case of $\mathcal{H}_3$ this must be reflexible, with $\mathcal{H}_4^{(01)} \cong \mathcal{H}_4$ and $\mathrm{Aut}\,\mathcal{H}_4 = \langle \mathrm{Aut}^+\mathcal{H}_4, t\rangle$ where

$$t^2 = 1,\ b^t = b^{-1} \quad \text{and} \quad (bc)^t = (bc)^{-1} = bc^4$$

(so $c^t = c$). Thus $\mathrm{Aut}\,\mathcal{H}_4$ has a normal subgroup

$$\langle c,t\rangle \cong C_6 \times C_2 \cong C_3 \times V_4$$

of index 2, complemented by

$$\langle u := bt\rangle \cong C_2,$$

where u acts by inverting $\langle c^2\rangle \cong C_3$ and by transposing the direct factors $\langle t\rangle$ and $\langle c^3 t\rangle$ of $V_4 \cong C_2 \times C_2$ (but commuting with c^3). This shows that $\mathrm{Aut}\,\mathcal{H}_4 \cong \mathrm{Aut}^{\hat{0}}\mathcal{M}_4$, and indeed by comparing Figures 17(a) and 1(e) we see that $W(\mathcal{H}_4) \cong \mathcal{M}_4$.

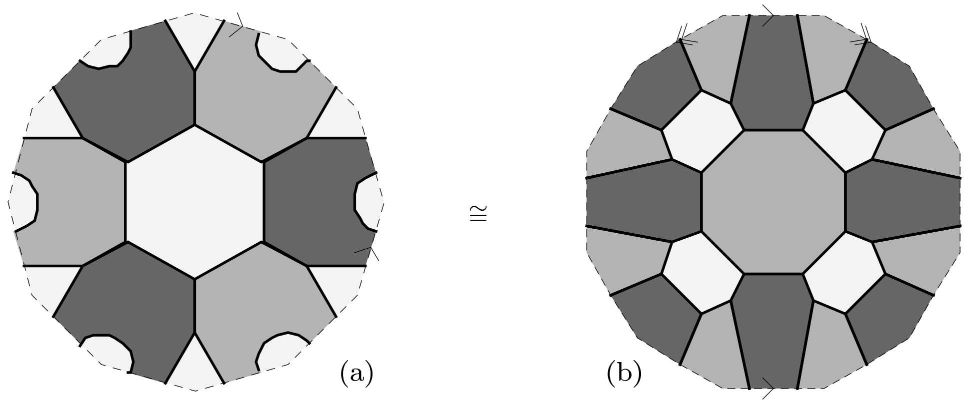

Figure 17. The hypermap $\mathcal{H}_4$

Figure 17(b), showing another view of $\mathcal{H}_4$, is based on Figure 12 of [4] which shows the hypermap $\mathcal{H}_4^{(012)}$ of type $(3,4,4)$.

Table 3 gives the cycle-structures of the non-identity elements $g \in \mathrm{Aut}^+\mathcal{H}_4$ on the hypervertices, hyperedges and hyperfaces.

g	$o(g)$	$\#g$	hypervertices	hyperedges	hyperfaces
c^3	2	1	1^3	1^3	2^2
$c^{\pm 2}$	3	2	3^1	3^1	1^4
bc^i	4	6	$1^1 2^1$	$1^1 2^1$	4^1
$c^{\pm 1}$	6	2	3^1	3^1	2^2

Table 3

For example, the unique involution c^3 in $\mathrm{Aut}^+\mathcal{H}_4$ (a half-turn of Figure 17(b) about its centre) leaves all hypervertices and hyperedges invariant, while permuting the hyperfaces in two cycles of length 2. Notice that c, of order 6, is fixed-point-free, so it is not a rotation of $\mathcal{H}_4$ about any point.

As a Δ-basis for $\mathrm{Aut}\,\mathcal{H}_4$ we can take $\mathbf{r} = (u, cu, t)$; the automorphism $c \mapsto c^{-1}$, $t \mapsto t$, $u \mapsto cu$ of $\mathrm{Aut}\,\mathcal{H}_4$ sends this to (cu, u, t), confirming that $\mathcal{H}_4 \cong \mathcal{H}_4^{(01)}$, so that $\mathcal{H}_4$ lies in an S-orbit of length 3.

As in case (3), $\mathrm{Aut}\,\mathcal{H}_4$ is indecomposable, so $\mathcal{H}_4$ is not a disjoint product. However, since $\mathrm{Aut}\,\mathcal{H}_4$ has $\langle c^3 \rangle$ as a normal subgroup, $\mathcal{H}_4$ is a double covering of the rotary map $\mathcal{H}_4/\langle c^3 \rangle \cong \mathcal{D}_6^{\ominus}$ shown in Figure 6, branched at its three vertices and three edges.

15. The hypermap $\mathcal{H}_5$

In the final case, $\mathcal{H}$ has type $(3,3,4)$ and rotation group isomorphic to the binary tetrahedral group

$$\langle 2,3,3 \rangle = \langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle;$$

this is a central extension of the tetrahedral group $(2,3,3) \cong A_4$ by $\langle abc \rangle \cong C_2$, and can be identified with $SL_2(3)$, where a, b and c correspond to $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right), -\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $-\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, so that $abc = -I$. It can also be regarded as a split extension of a normal subgroup $\langle a, a^b \rangle \cong Q_8$ by $\langle -b \rangle \cong C_3$. In any Δ^+-basis $\mathbf{x} = (x_0, x_1, x_2)$ of type $(3,3,4)$ for $\mathrm{Aut}^+\mathcal{H}$, the generators x_0 and x_1 of order 3 must not be conjugate (otherwise x_2, being outside the unique Sylow 2-subgroup $SL_2(3)' \cong Q_8$, could not have order 4); it follows easily that $\mathrm{Aut}\,SL_2(3)$ ($\cong PGL_2(3) \cong S_4$) has a unique orbit on such Δ^+-bases, represented by $\mathbf{x} = (-b, -c, -a) = (b^4, c^4, a^3) = (\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2\\2&2\end{smallmatrix}\right))$. In this case we therefore find a unique rotary hypermap $\mathcal{H}_5$ of type $(3,3,4)$ with rotation group $SL_2(3)$; as in the case of $\mathcal{H}_4$ it must be reflexible, with $\mathcal{H}_5^{(01)} \cong \mathcal{H}_5$, and with $\mathrm{Aut}\,\mathcal{H}_5$ a split extension of $\mathrm{Aut}^+\mathcal{H}_5$ by $\langle t \rangle$ where $t^2 = 1$, $b^t = b^{-1}$, $c^t = c^{-1}$. By putting $t = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$ we can identify $\mathrm{Aut}\,\mathcal{H}_5$ with $GL_2(3)$, a corresponding Δ-basis being $\mathbf{r} = (-tc, -bt, t) = (\left(\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right))$. The automorphism $g \mapsto (g^{-1})^T$ (where T denotes transpose) sends this Δ-basis to $(-bt, -tc, t)$, confirming that $\mathcal{H}_5^{(01)} \cong \mathcal{H}_5$, so that $\mathcal{H}_5$ lies in an S-orbit of length 3.

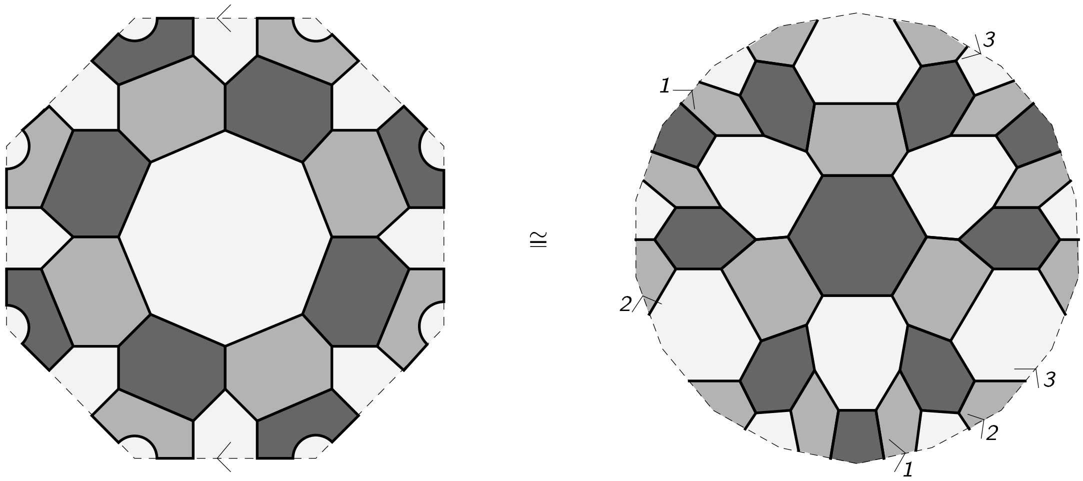

Figure 18. The hypermap $\mathcal{H}_5$

Figure 18 shows two views of $\mathcal{H}_5$, the first (based on Figure 13 of [4]) showing that $W(\mathcal{H}_5) \cong \mathcal{M}_5$ (see Figure 1(f)). Table 4 gives the cycle-structures of the non-identity elements $g \in \mathrm{Aut}^+\mathcal{H}_5$ on the hypervertices, hyperedges and hyperfaces.

$o(g)$	$\#g$	hypervertices	hyperedges	hyperfaces
2	1	2^4	2^4	1^6
3	8	$1^2 3^2$	$1^2 3^2$	3^2
4	6	4^2	4^2	$1^2 2^2$
6	8	$2^1 6^1$	$2^1 6^1$	3^2

Table 4

Since $\mathrm{Aut}\,\mathcal{H}_5$ is indecomposable, $\mathcal{H}_5$ is not a disjoint product. However, since $\langle -I \rangle$ is normal in $GL_2(3)$, with $GL_2(3)/\langle -I \rangle \cong PGL_2(3) \cong S_4$, $\mathcal{H}_5$ is a double covering of the reflexible spherical S_4-hypermap $\mathcal{H}_5/\langle -I \rangle \cong \mathcal{T}^{(12)}$ of type $(3, 3, 2)$ – where $\mathcal{T}$ is the tetrahedron – branched over its six hyperfaces.

The hypermaps $\mathcal{H}_1, \ldots, \mathcal{H}_5$, together with their associates, account for the 13 hypermaps in Table 2, so we have now described all 43 rotary hypermaps of genus 2. Notice that they are all reflexible.

16. Reflexible hypermaps of characteristic −1

In §8.8 of [5], Coxeter and Moser show that "No regular (i.e. reflexible) map can be drawn on a non-orientable surface of characteristic −1". Their argument is that the orientable double covering of such a map would be rotary map of genus 2 with an orientation-reversing fixed-point-free automorphism of order 2; however, inspection shows that none of their list of rotary maps of genus 2 has such an automorphism. We can now extend this argument from maps to hypermaps.

Theorem. *There is no reflexible hypermap of characteristic* -1.

Proof. Let $\mathcal{H}$ be a reflexible hypermap of characteristic $\chi(\mathcal{H}) = -1$, corresponding to a subgroup $H \leq \Delta$. If $H \leq \Delta^+$ then $\mathcal{H}$, being orientable and without boundary, would have even characteristic. Thus $H \not\leq \Delta^+$, so $\mathcal{H}$ has an orientable double covering $\mathcal{H}^+ \cong \mathcal{H} \times \mathcal{B}^+$, corresponding to the subgroup $H^+ = H \cap \Delta^+ \leq \Delta$ (see Figure 19).

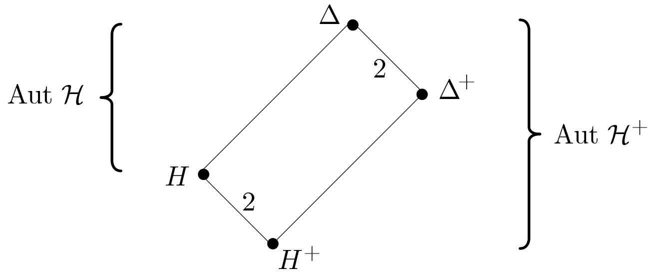

Figure 19. The subgroups H and H^+ of Δ

By hypothesis, H is normal in Δ, so H^+ is normal in Δ^+ (in fact, normal in Δ) and hence $\mathcal{H}^+$ is a rotary hypermap; having characteristic $2\chi(\mathcal{H}) = -2$, it has genus 2 and must therefore be an associate of one of the maps $\mathcal{M}_0, \ldots, \mathcal{M}_5$ or one of the hypermaps $\mathcal{H}_1, \ldots, \mathcal{H}_5$ described earlier. Furthermore,

$$\begin{aligned} \mathrm{Aut}\,\mathcal{H}^+ &\cong \Delta/H^+ \\ &\cong (\Delta^+/H^+) \times (H/H^+) \\ &\cong \mathrm{Aut}^+\mathcal{H}^+ \times C_2. \end{aligned}$$

with $\mathrm{Aut}^+\mathcal{H}^+ \cong \Delta^+/H^+ \cong \Delta/H$, so that the rotation group $\mathrm{Aut}^+\mathcal{H}^+$ is an epimorphic image of Δ.

If $\mathcal{H}^+$ is an associate of $\mathcal{M}_0, \ldots, \mathcal{M}_3$ or of $\mathcal{H}_1, \ldots, \mathcal{H}_5$ then (by its description earlier in this paper) $\mathrm{Aut}^+\mathcal{H}^+$ is not generated by involutions, so it cannot be an image of Δ: the only non-trivial case is $\mathcal{M}_3$, where the involutions are a^4 and $a^{2i}c$, generating a subgroup of index 2 in $\mathrm{Aut}^+\mathcal{M}_3$. In the cases $\mathcal{M}_4$ and $\mathcal{M}_5$, $\mathrm{Aut}^+\mathcal{H}^+$ *is* an image of Δ, but now (by inspection) the centre of $\mathrm{Aut}\,\mathcal{H}^+$ (of order 2) is contained in $\mathrm{Aut}^+\mathcal{H}^+$, so $\mathrm{Aut}\,\mathcal{H}^+ \not\cong \mathrm{Aut}^+\mathcal{H}^+ \times C_2$.

References

[1] Brahana, H. R.: *Regular maps and their groups.* Amer. J. Math. **49** (1927), 268–284.

[2] Breda d'Azevedo. A. J.; Jones, G. A.: *Double coverings and reflexible abelian hypermaps.* Preprint.

[3] Cori, R.; Machì, A.: *Maps, hypermaps and their automorphisms: a survey I, II, III.* Expositiones Math. **10** (1992), 403–427, 429–447, 449–467.

[4] Corn, D.; Singerman, D.: *Regular hypermaps.* Europ. J. Comb. **9** (1988), 337–351.

[5] Coxeter, H. S. M.; Moser, W. O. J.: *Generators and Relations for Discrete Groups.* Springer-Verlag, 4th ed. Berlin/Heidelberg/New York 1972.

[6] Izquierdo, M.; Singerman, D.: *Hypermaps on surfaces with boundary.* Europ. J. Comb. 15 **2** (1994), 159–172.

[7] Machì, A.: *On the complexity of a hypermap.* Discr. Math. **42** (1982), 221–226.

[8] Threlfall, W.: *Gruppenbilder.* Abh. sächs. Akad. Wiss. Math.-phys. Kl. **41** (1932), 1–59.

[9] Walsh, T. R. S.: *Hypermaps versus bipartite maps.* J. Combinatorial Theory, Ser. B **18** (1975), 155–163.

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