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Drinfeld-Anderson Shtukas and Uniformization of *A*-Motives via Sato Grassmannians

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Abstract. In this paper we continue to investigate the algebro-geometric structure of Drinfeld-Anderson motives introduced in [28] and [29]. In the first part we construct shtukas related to Drinfeld-Anderson motives. The main result of the second part is uniformization Theorem 3.4.2.

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1. Introduction

Drinfeld-Anderson motives are “toy models” of hypothetical twisted (noncommutative) motives in positive characteristic. They are a direct generalization of Drinfeld modules [12] and Anderson t-motives [4]. In [29] we showed how these motives are related to the multicomponent KP hierarchy. There are however many open questions and this paper is devoted to two of them. First of all, one would like to have an algebro-geometric definition of Drinfeld-Anderson A-motives valid over an arbitrary $\mathbb{F}_q$-scheme. We were able to give such a definition earlier ([28, 1.6], [29, 6.2]). In the first part of this paper we go further and define Drinfeld-Anderson shtukas. Then the purity of A-motives is the property of a quasi-periodic propagation of associated shtukas. In the second chapter we consider another important question concerning the uniformization of A-motives. As Anderson showed [4, §2] not all motives of this kind are uniformizable. However it is possible to uniformize formally (or rigid-analytically) trivial motives. The main result of the second part of this paper is the uniformization of such motives via Sato Grassmannians (Theorem 3.4.2). This uniformization may be considered as an analogue of the Krichever map ([1], [11, Sect. 6]) and also as a period morphism [30]. Of course, it is also possible to uniformize A-motives via $\mathfrak{p}$-adic symmetric domains. In fact, our result implies that $\mathfrak{p}$-adic symmetric domains associated to formally trivial Drinfeld-Anderson motives may be embedded into multicomponent Sato Grassmannians. It should give a generalization of Genestier's results ([15], [16]). We hope to describe these embeddings in a sequel.

When this paper was mostly finished, the author discovered that similar ideas concerning the uniformization of A-motives were developped by Alvarez [3]. It seems also that the original idea to exploit ind-algebraic structures of Drinfeld symmetric domains Ω^d is due to Genestier [16]. He described the Ω^d as generalized Deligne-Lusztig varieties embedded in ind-algebraic flag varieties (loc. cit.).

2. Shtukas related to Drinfeld-Anderson sheaves

2.1. Torsion-free (bi)shtukas

Let X be a geometrically irreducible (possibly singular) complete curve over $\mathbb{F}_q$ and S an arbitrary $\mathbb{F}_q$-scheme. We denote by Fr_S the Frobenius morphism of S and by ${}^{\tau}\mathcal{E} = (\mathrm{Id}_X \times \mathrm{Fr}_S)^*\mathcal{E}$ the Frobenius pull-back of a sheaf $\mathcal{E}$ on $X \times_{\mathbb{F}_q} S$.

Definition 2.1.1. *A left (resp. right) torsion-free Drinfeld-Anderson shtuka of rank r and τ-rank n with a zero $\alpha : S \to X$ and a pole $\beta : S \to X$ over S is a diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{s_\alpha} & \mathcal{E} \\ & {\scriptstyle i_\beta}\searrow & \\ & & {}^{\tau}\mathcal{E} \end{array} \qquad \textit{resp.} \qquad \begin{array}{ccc} \mathcal{E} & \xrightarrow{j_\beta} & \mathcal{G} \\ & \nearrow{\scriptstyle t_\alpha} & \\ {}^{\tau}\mathcal{E} & & \end{array} \tag{2.1.1}$$

of torsion-free sheaves over $X \times S$ of rank r such that cokernels of s_α and i_β (resp. of t_α and j_β) are direct images of locally free $\mathcal{O}_S$-modules of rank n under the morphisms $\Gamma_\alpha : S \to X \times S$ and $\Gamma_\beta : S \to X \times S$ induced by the graphs of α and β.

Remarks 2.1.2. 1) The necessity of torsion-free sheaves (and not only vector bundles) for the algebro-geometric classification of Krichever (Drinfeld) modules was underlined by Mumford [26].
2) The τ-rank appears in the definitions of t-motives [4] and Drinfeld-Anderson motives [29]. Drinfeld shtukas have τ-rank 1 [14] but $\mathcal{D}$-elliptic sheaves [24] and $\mathcal{D}$-shtukas [21] have τ-rank d (where d^2 is the rank of a division $\mathcal{O}_X$-algebra $\mathcal{D}$).

We say that a torsion-free shtuka is *separated* if its zero and pole are disjoint. It is easy to see that a separated right (resp. left) shtuka may be completed to a *bishtuka*, that is, to a "bicartesian square":

$$\begin{array}{ccccc} & \mathcal{E} & \xrightarrow{j} & \mathcal{G} & \\ {\scriptstyle t^{\mathrm{c}}}\nearrow & & \nearrow{\scriptstyle t} & & \\ \mathcal{G}^{\mathrm{c}} & \xrightarrow{j^{\mathrm{c}}} & {}^{\tau}\mathcal{E} & & \end{array} \qquad \text{resp.} \qquad \begin{array}{ccccc} \mathcal{F} & \xrightarrow{s} & \mathcal{E} & & \\ & {\scriptstyle i}\searrow & & \searrow{\scriptstyle i^{\mathrm{c}}} & \\ & & {}^{\tau}\mathcal{E} & \xrightarrow{s^{\mathrm{c}}} & \mathcal{F}^{\mathrm{c}} \end{array} \tag{2.1.2}$$

([14], [21, I.1], [23]). This is a natural functorial construction and the stacks of left and right shtukas are naturally equivalent outside of the diagonal $\Delta_X = X \times X$ (loc. cit.). We say that a left (resp. right) shtuka with the same zero and pole

$$\mathcal{E} \xleftarrow{t^c} \mathcal{G}^c \xrightarrow{j^c} {}^\tau\mathcal{E} \quad \text{resp.} \quad {}^\tau\mathcal{E} \xrightarrow{i^c} \mathcal{F}^c \xleftarrow{s^c} \mathcal{E} \tag{2.1.3}$$

is *conjugate* to a right (resp. left) shtuka (2.1.1) and we fix such a conjugation. One can also consider a *dual shtuka* of a left (resp. right) shtuka (2.1.1) by taking dual sheaves and morphisms. The zeros and poles of a shtuka and its dual are interchanged.

2.2. Shifts and propagations

There is an one more procedure to pass from a left to a right shtuka and vice versa. Namely, let ${}^r_n\mathrm{Sht}$ (resp. Sht^r_n) denote the moduli stacks of left (resp. right) shtukas of rank r and τ-rank n. Consider maps $\mathrm{Fr}_0 : {}^r_n\mathrm{Sht} \to \mathrm{Sht}^r_n$ and $\mathrm{Fr}_\infty : \mathrm{Sht}^r_n \to {}^r_n\mathrm{Sht}$ such that:

$$\mathrm{Fr}_0 : \begin{array}{ccc} \mathcal{F} & \xrightarrow{s} & \mathcal{E} \\ & {\scriptstyle i}\searrow & \\ & & {}^\tau\mathcal{E} \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{F} & & \\ & \searrow{\scriptstyle i} & \\ {}^\tau\mathcal{F} & \xrightarrow{{}^\tau s} & {}^\tau\mathcal{E} \end{array} \qquad \text{and} \quad \mathrm{Fr}_\infty : \begin{array}{ccc} \mathcal{E} & \xrightarrow{j} & \mathcal{G} \\ & \nearrow{\scriptstyle t} & \\ {}^\tau\mathcal{E} & & \end{array} \quad \mapsto \quad \begin{array}{ccc} & & \mathcal{G} \\ & {\scriptstyle t}\nearrow & \\ {}^\tau\mathcal{E} & \xrightarrow{{}^\tau j} & {}^\tau\mathcal{G} \end{array}$$

then, obviously, $\mathrm{Fr}_0 \circ \mathrm{Fr}_\infty = \mathrm{Fr}_S$ and $\mathrm{Fr}_\infty \circ \mathrm{Fr}_0 = \mathrm{Fr}_S$. Combining it with the conjugation we obtain a left (resp. right) shtuka

$${}^\tau\mathcal{F} \xleftarrow{i_1} \mathcal{F}_1 \xrightarrow{s_1} \mathcal{F} \quad \text{resp.} \quad \mathcal{G} \xrightarrow{j_1} \mathcal{G}_1 \xleftarrow{t_1} {}^\tau\mathcal{G}.$$

Such a procedure will be called a *left* (resp. *right*) *1-shift*. A left-shifted (resp. right-shifted) shtuka has the zero $\alpha \circ \mathrm{Fr}_S$ (resp. α) and the pole β (resp. $\beta \circ \mathrm{Fr}_S$). Continuing such shifts indefinitely we obtain a *propagated* left (resp. right) shtuka, that is a diagram

$$\begin{array}{ccccccc} & \mathcal{F}_2 & \xrightarrow{s_2} & \mathcal{F}_1 & \xrightarrow{s_1} & \mathcal{F} & \xrightarrow{s} \mathcal{E} \\ \ldots & {\scriptstyle i_2}\searrow & & {\scriptstyle i_1}\searrow & & {\scriptstyle i}\searrow & \\ & & {}^\tau\mathcal{F}_1 & \longrightarrow & {}^\tau\mathcal{F} & \longrightarrow & {}^\tau\mathcal{E} \end{array} \qquad \text{resp.} \qquad \begin{array}{ccccccc} \mathcal{E} & \xrightarrow{j} & \mathcal{G} & \xrightarrow{j_1} & \mathcal{G}_1 & \xrightarrow{j_2} & \mathcal{G}_2 \\ & \nearrow{\scriptstyle t} & & \nearrow{\scriptstyle t_1} & & \nearrow{\scriptstyle t_2} & \ldots \\ {}^\tau\mathcal{E} & \longrightarrow & {}^\tau\mathcal{G} & \longrightarrow & {}^\tau\mathcal{G}_1 & & \end{array}$$

2.3. Relatively pure shtukas

Let $\mathcal{I}$ be an invertible sheaf on X and denote $\widetilde{\mathcal{I}} = \mathcal{I} \boxtimes \mathcal{O}_S$ the corresponding sheaf on $X \times S$. Then

$${}^\tau\mathcal{E} \otimes \widetilde{\mathcal{I}} \xleftarrow{i} \mathcal{F} \otimes \widetilde{\mathcal{I}} \xrightarrow{s} \mathcal{E} \otimes \widetilde{\mathcal{I}} \quad \text{resp.} \quad \mathcal{E} \otimes \widetilde{\mathcal{I}} \xrightarrow{j} \mathcal{G} \otimes \widetilde{\mathcal{I}} \xleftarrow{t} {}^\tau\mathcal{E} \otimes \widetilde{\mathcal{I}} \tag{2.3.1}$$

are also shtukas with the same zero and pole as (2.1.1) [14, constr. 5].

Definition 2.3.1. *A left (resp. right) torsion-free shtuka* (2.1.1) *is called relatively pure of weight $w = \deg \mathcal{I}/k$ with respect to $\widetilde{\mathcal{I}}$ if the k-shifted shtuka*

$$\begin{array}{ccc} \mathcal{F}_k & \xrightarrow{s_k} & \mathcal{F}_{k-1} \\ & {\scriptstyle i_k}\searrow & \\ & & {}^\tau\mathcal{F}_{k-1} \end{array} \qquad \textit{resp.} \qquad \begin{array}{ccc} \mathcal{G}_{k-1} & \xrightarrow{j_k} & \mathcal{G}_k \\ & \nearrow{\scriptstyle t_k} & \\ {}^\tau\mathcal{G}_{k-1} & & \end{array} \tag{2.3.3}$$

is isomorphic to left (resp. right) shtuka (2.3.1).

When $\mathcal{I} = \mathcal{O}_X(\infty)$ this is the usual condition of purity (with weight $1/d$) for shtukas associated to Drinfeld modules [14, §1].

2.4. Drinfeld-Anderson sheaves

Let ∞ be a smooth closed point on X and denote $A = \mathrm{H}^0(X \setminus \infty, \mathcal{O}_X)$.

Definition 2.4.1. ([29, 6.2]) *A Drinfeld-Anderson sheaf of pole ∞, of rank r and of τ-rank n over an A-scheme S, consists of the following commutative diagram:*

$$\begin{array}{ccccccccc} \dots & \overset{j}{\hookrightarrow} & \mathcal{E}_{i-1} & \overset{j}{\hookrightarrow} & \mathcal{E}_i & \overset{j}{\hookrightarrow} & \mathcal{E}_{i+1} & \overset{j}{\hookrightarrow} & \dots \\ & t\nearrow & & t\nearrow & & t\nearrow & & t\nearrow & \\ \dots & \overset{\tau j}{\hookrightarrow} & {}^\tau\mathcal{E}_{i-1} & \overset{\tau j}{\hookrightarrow} & {}^\tau\mathcal{E}_i & \overset{\tau j}{\hookrightarrow} & {}^\tau\mathcal{E}_{i+1} & \overset{\tau j}{\hookrightarrow} & \dots \end{array} \tag{2.4.1}$$

such that any left i-truncation

$$\begin{array}{ccccccc} \mathcal{E}_i & \overset{j}{\hookrightarrow} & \mathcal{E}_{i+1} & \overset{j}{\hookrightarrow} & \mathcal{E}_{i+2} & \dots \\ & t\nearrow & & t\nearrow & & \\ {}^\tau\mathcal{E}_i & \overset{\tau j}{\hookrightarrow} & {}^\tau\mathcal{E}_{i+1} & \dots & & \end{array}$$

is a propagated right torsion-free shtuka of pole ∞ and of zero $\alpha : S \to \operatorname{Spec} A$. A Drinfeld-Anderson sheaf (2.4.1) *is pure of weight $w = u/v$ if any right shtuka*

$$\mathcal{E}_i \xrightarrow{j} \mathcal{E}_{i+1} \xleftarrow{t} {}^\tau\mathcal{E}_i$$

is relatively pure of weight w with respect to $\mathcal{O}_X(u\infty) \boxtimes \mathcal{O}_S$. In other words, it means that

$$\mathcal{E}_{i+v \deg \infty} \simeq \mathcal{E}_i(\{u\infty\} \times S)$$

for any integer i.

3. Uniformization of Drinfeld-Anderson motives

3.1. Associated bundles on twisted projective line

Let L be a perfect field over $\mathbb{F}_q$ equipped with a $\mathbb{F}_q$-morphism $\alpha_L : A \to L$ and $L[\tau]$ the twisted polynomial ring with the commutation rule $\tau a = a^q \tau$. Denote $L(\tau)$ the quotient skew-field of $L[\tau]$, $L[\![\tau]\!]$ the ring of skew power series and $L(\!(\tau)\!)$ the skew-field of Laurent series [24, Sect. 3].

Let $\mathbb{P}_{L(\tau)}$ denote the projective line over $L(\tau)$ (cf. [31, Ch. VII] for the general definition). Vector bundles over $\mathbb{P}_{L(\tau)}$ may be defined using the following well-known description (due to Grothendieck) of vector bundles on a smooth curve X. Any closed point P on X gives a "covering" of X by $X - P$ and the infinitesimal disc D_P at P. Then any vector bundle is given by an isomorphism of restrictions of the trivial bundles to the infinitesimal punctured disc D_P^*, that is, by an automorphism of the trivial bundle on D_P^* [20, 1.4]. Hence, by definition, a *vector bundle of rank* n over $\mathbb{P}_{L(\tau)}$ is a pair (M, V_∞) where M is a free $L[\tau]$-module of rank n, V_∞ is a free $L[\![\tau^{-1}]\!]$-submodule of $L(\!(\tau^{-1})\!) \otimes_{L[\tau]} M$ such that the induced map

$$L(\!(\tau^{-1})\!) \bigotimes_{L[\![\tau^{-1}]\!]} V_\infty \to L(\!(\tau^{-1})\!) \widehat{\bigotimes_{L[\tau]}} M \tag{3.1.1}$$

is an isomorphism [24, 3.13].

For a Drinfeld-Anderson sheaf (2.4.1) we denote $M_i = \mathrm{H}^0(X \setminus \infty, \mathcal{E}_i)$ and $V_{i,\infty} = \mathrm{H}^0(\mathrm{Spec}(\mathcal{O}_\infty \widehat{\otimes} L), \mathcal{E}_i)$. It is easy to see that $V_{i,\infty}$ is a free $L[\![\tau^{-1}]\!]$-module of rank n [24, 3.11]. The following result may be proved in the same manner as [24, 3.17].

Proposition 3.1.1. *The functor associating the pair*

$$(M = \mathrm{H}^0(X \setminus \infty, \mathcal{E}_0), V_\infty = \mathrm{H}^0(\mathrm{Spec}(\mathcal{O}_\infty \widehat{\otimes} L), \mathcal{E}_0))$$

to (2.4.1) *defines an equivalence between the category of Drinfeld-Anderson sheaves over* L *and the full subcategory of the category of vector bundles over* $\mathbb{P}^1_{L(\tau)}$ *such that*

(i) *A acts on $M/\tau M$ via α_L,*

(ii) *M is finitely generated as an $A \otimes_{\mathbb{F}_q} L$-module and*

(iii) *V_∞ is finitely generated as $\mathcal{O}_\infty \widehat{\otimes}_{\mathbb{F}_q} L$-module.*

If the Drinfeld-Anderson sheaf (2.4.1) *is pure of weight* $w = u/v$ *then, in addition,*

(iv) $\tau^{-v \deg(\infty)} V_\infty = \varpi_\infty^u V_\infty$

where ϖ_∞ is an uniformizer of the completed local ring $\mathcal{O}_\infty$.

3.2. Formally trivial motives

When we work over a perfect field L the notion of Drinfeld-Anderson motive is somewhat more general. First of all, the action of A on $M/\tau M$ may be not diagonal and, as Anderson showed [4, §2], a lattice V_∞ verifying (3.1.1) does not always exist. However, it is always possible to uniformize rigid-analytically trivial t-motives (loc. cit.). In this section we generalize Anderson's results. In the following definition we don't fix X and ∞ but just suppose that A is a Dedekind domain with the constant field $\mathbb{F}_q$.

Definition 3.2.1. [29, 5.1] *A Drinfeld-Anderson A-motif M of rank r and τ-rank n is a left $(A \otimes_{\mathbb{F}_q} L[\tau])$-module verifying the following conditions:*

(i) *M is a free $L[\tau]$-module of rank n;*

(ii) *M is a torsion-free $(A \otimes_{\mathbb{F}_q} L)$-module of rank r;*

(iii) *$(a - \alpha_L(a))$ acts nilpotently on $M/\tau M$ for any $a \in A$.*

A morphism of Drinfeld-Anderson motives is an $(A \otimes_{\mathbb{F}_q} L[\tau])$-linear map.

Furthermore, M is said to be formally trivial if there exists a lattice V_∞ in $L(\!(\tau^{-1})\!) \otimes_{L[\tau]} M$ verifying (3.1.1). *A formally trivial Drinfeld-Anderson motive is pure of weight $w = u/v$ if condition* (iv) *of Proposition* 3.1.1 *is satisfied.*

If there exist a complete curve X over $\mathbb{F}_q$, a smooth closed point ∞ such that $A = \mathrm{H}^0(X \setminus \infty, \mathcal{O}_X)$ and a torsion-free sheaf $\mathcal{E}_0$ as in Proposition 3.1.1 then (M, V_∞) is said to be of geometric origin. In this case $(X, \infty, \mathcal{E}_0)$ will be called the underlying geometric triple of a formally trivial motive (M, V_∞).

Such formally trivial A-motives are analogues of Anderson's rigid-analytically trivial t-motives (loc. cit.).

3.3. Multicomponent Sato Grassmannians and Schur pairs

First of all, recall one of the "classical" definitions of the Sato Grassmannian ([25], [32], [33]). For the moment, let L be a field of characteristic zero and let

$$L[\![t]\!](\!(\partial^{-1})\!) = L[\![t]\!](\!((d/dt)^{-1})\!)$$

denote the ring of microdifferential operators in one variable, i.e. the ring of Laurent series in a formal symbol $\partial^{-1} = (d/dt)^{-1}$ with the commutation rule

$$\left(\frac{d}{dt}\right)^{-1} \cdot a = \sum_{k=0}^{\infty} (-1)^k \frac{d^k a}{dt^k} \cdot \left(\frac{d}{dt}\right)^{-k-1}$$

for any $a \in L[\![t]\!]$ (cf. [26, p. 140]). Consider the subring $L(\!(\partial^{-1})\!) \subset L[\![t]\!](\!(\partial^{-1})\!)$ of microdifferential operators with constant coefficients. We say that a map of vector spaces is *Fredholmian* if it has both finite kernel and cokernel. The index of a Fredholm map γ is defined by:

$$\mathrm{ind}\, \gamma = \dim_L \mathrm{Ker}\, \gamma - \dim_L \mathrm{Coker}\, \gamma.$$

For any natural integer n the set

$$\begin{aligned} \mathrm{Gr}_n = \{ & \text{ subspaces } W \subset L(\!(\partial^{-1})\!)^{\oplus n} \text{ such that the projection} \\ & \gamma_W : W \to (L(\!(\partial^{-1})\!)/\partial^{-1}L[\![\partial^{-1}]\!])^{\oplus n} \text{ is Fredholmian } \} \end{aligned}$$

is called the n-component Sato Grassmannian. In the q-twisted case we simply mimic this definition. Let L be again a perfect field over $\mathbb{F}_q$.

Definition 3.3.1. *The set*

$$\begin{aligned} q\text{-}\mathrm{Gr}_n = \{ & \textit{ subspaces } W \subset L(\!(\tau^{-1})\!)^{\oplus n} \textit{ such that the projection} \\ & \gamma_W : W \to (L(\!(\tau^{-1})\!)/\tau^{-1}L[\![\tau^{-1}]\!])^{\oplus n} \textit{ is Fredholmian } \} \end{aligned}$$

will be called the q-twisted Sato Grassmannian. The virtual dimension of W is just the index of γ_W. Denote q-$\mathrm{Gr}_n(l)$ the Grassmannian of subspaces of virtual dimension l. Then

$$q\text{-}\mathrm{Gr}_n^+(0) \stackrel{\mathrm{def}}{=} \{W \in q\text{-}\mathrm{Gr}_n(0) \mid \dim_L \mathrm{Ker}\, \gamma_W = \dim_L \mathrm{Coker}\, \gamma_W = 0\}$$

is called the big cell of q-Gr_n.

Remarks 3.3.2. 1) The ind-proalgebraic variety structure of q-Gr_n will be discussed in the next section.

2) The n-component Sato Grassmannian may be also described as the set of colattices in $L(\!(\tau^{-1})\!)^{\oplus n}$ with the respect to the Tate topology (cf. Appendix 4.2).

We would like to have a functorial correspondence between the category of Drinfeld-Anderson motives of τ-rank n and a certain category related to the n-component Sato Grassmannian. Fortunately, an apropriate construction was already found by Mulase in the differential context [25].

Definition 3.3.3. *Let R be a non-trivial commutative subring of $\mathrm{M}_n(L[\tau])$ stabilizing a subspace $W \in q\text{-}\mathrm{Gr}_n$ (that is, such that $RW \subset W$) then (R, W) will be called a q-twisted Schur pair.*

It is well-known (cf. [25]) that if for a fixed $W \in \mathrm{Gr}_n$ a Schur pair (R, W) exists then W has the algebro-geometric origin.

3.4. Admissible Schur pairs and uniformizable Drinfeld-Anderson motives

In this subsection we shall prove the anti-equivalence of the category of formally trivial Drinfeld-Anderson motives and the category of admissible Schur pairs. It describes Drinfeld-Anderson motives in terms of (co)lattices and, consequently, it generalizes uniformization results of Drinfeld and Anderson ([12, §3] and [4, §2]).

In [29] we considered non-trivial and non-degenerate commutative subrings $R \subset M_n(L[\tau])$ satisfying the following Anderson's condition:

$$\mathrm{Hom}(\mathbb{G}^n_{\mathrm{a},L}, \mathbb{G}_{\mathrm{a},L}) = \sum_{a \in R} V \circ a \tag{3.4.1}$$

for a certain finite-dimensional L-subspace $V \subset \mathrm{Hom}(\mathbb{G}^n_{\mathrm{a},L}, \mathbb{G}_{\mathrm{a},L})$ (cf. [29, Sect. 5], [4, 1.1.3]) and such that for any $D \in R$

$$\mathrm{ev}_0(D) - D_\alpha \cdot \mathrm{Id}_n \tag{3.4.2}$$

is nilpotent for a certain $D_\alpha \in L$. Here

$$\mathrm{ev}_0 : \mathrm{M}_n(L[\tau]) \to \mathrm{M}_n(L)$$

is the evaluation map at "$\tau = 0$". We suppose also that R contains $\mathbb{F}_q$ via the diagonal injection $a \mapsto \mathrm{diag}(a, \ldots, a)$. It was shown that any such ring is given by an embedding

$$\varphi : A = \mathrm{H}^0(X \setminus \infty, \mathcal{O}_X) \hookrightarrow \mathrm{M}_n(L[\tau])$$

for an appropriate curve X and a smooth closed point ∞ on it. In addition, the correspondence $a \mapsto D_\alpha$ for $D = \varphi(a)$ equips L with an A-module structure.

Definition 3.4.1. *A commutative ring R as above satisfying conditions* (3.4.1) *and* (3.4.2) *is called admissible or Drinfeld-Anderson abelian A-module* (*by analogy with Anderson's abelian t-modules* [4, 1.1]). *A morphism*

$$u : R_1 = \mathrm{Im}(\varphi_1) \to R_2 = \mathrm{Im}(\varphi_2)$$

is an element $u \in \mathrm{M}_n(L[\tau])$ such that $u\varphi_1(a) = \varphi_2(a)u$ for any $a \in A$.

Theorem 3.4.2. *The category of formally trivial Drinfeld-Anderson motives M of τ-rank n is anti-equivalent to the category of admissible Schur pairs (R, W) where R is admissible and $W \in q\text{-}\mathrm{Gr}_n$. If $(X, \infty, \mathcal{E}_M)$ is the underlying geometric triple of M and (R, W_M) is the corresponding Schur pair then*

$$\dim.\mathrm{virt}.\, W_M = h^0(\mathcal{E}_M) - h^1(\mathcal{E}_M)$$

where $h^0(\mathcal{E}_M) = \dim \mathrm{H}^0(X, \mathcal{E}_M)$ and $h^1(\mathcal{E}_M) = \dim \mathrm{H}^1(X, \mathcal{E}_M)$.

Proof. The anti-equivalence of the category of Drinfeld-Anderson motives of τ-rank n with the category of admissible commutative subrings of $\mathrm{M}_n(L[\tau])$ was proved in [29, Th. 5.3]. Namely, the $(A \otimes_{\mathbb{F}_q} L)$-module structure of a Drinfeld-Anderson motive M of rank n defines a morphism $\varphi_M : A \to \mathrm{End}_{L[\tau]} M$, that is, a commutative subring of $M_n(L[\tau])$. Then the functor $M \mapsto \mathrm{Im}(\varphi_M)$ makes these categories anti-equivalent (loc. cit.).

If, in addition, (M, V_∞) is a formally trivial Drinfeld-Anderson motive then we define

$$W \stackrel{\mathrm{def}}{=} \eta(M) \subset L((\tau^{-1}))^{\oplus n}$$

choosing the following trivialization:

$$\eta : V_\infty \xrightarrow{\sim} (\tau^{-1} L[\![\tau^{-1}]\!])^{\oplus n}.$$

It is precisely condition (3.1.1) which makes W a well-defined subspace of $L((\tau^{-1}))^{\oplus n}$. We should prove that W is a point of $q\text{-}\mathrm{Gr}_n(l)$ with $l = h^0(\mathcal{E}_M) - h^1(\mathcal{E}_M)$.

First of all, let us show that any (M, V_∞) is a Drinfeld-Anderson motive of *geometric origin*, that is, it has an underlying geometric triple $(X, \infty, \mathcal{E})$ (cf. Def. 3.2.1). It is easy to see that $X = \mathrm{Proj}\,(\mathrm{gr}\, R)$ where the gradation is taken with respect to the degree function on $R \subset \mathrm{M}_n(L[\tau])$ [29, (5.2)]. Moreover, ∞ is the unique point of X corresponding to the same degree function in such a manner that $\deg(D)/\deg(\infty)$ is the pole order of D at ∞. Finally, isomorphism (3.1.1) is exactly the gluing condition for a torsion-free sheaf $\mathcal{E}$.

Denote $U_\infty = \mathrm{Spec}(\mathcal{O}_\infty \widehat{\otimes} L)$ (resp. $U_\infty^* = \mathrm{Spec}(K_\infty \widehat{\otimes} L)$) the infinitesimal (resp. infinitesimal punctured) disc at ∞ on $X \times L$ where K_∞ is the quotient field of $\mathcal{O}_\infty$. We have

$$\begin{aligned} \mathrm{H}^0(X, \mathcal{E}) &\simeq \mathrm{H}^0(X \setminus \infty, \mathcal{E}) \cap \mathrm{H}^0(U_\infty, \mathcal{E}_{U_\infty}) \\ &\simeq W \cap V_\infty = \mathrm{Ker}\, \gamma_W \end{aligned}$$

and

$$\begin{aligned} \mathrm{H}^1(X, \mathcal{E}) &\simeq \mathrm{H}^0(U_\infty^*, \mathcal{E}_{U_\infty}) / (\mathrm{H}^0(X \setminus \infty, \mathcal{E}) + \mathrm{H}^0(U_\infty, \mathcal{E}_{U_\infty})) \\ &\simeq L((\tau^{-1}))^{\oplus n} / (W + V_\infty) \simeq \mathrm{Coker}\, \gamma_W \end{aligned}$$

where γ_W is the projection $W \to (L((\tau^{-1}))/\tau^{-1} L[\![\tau^{-1}]\!])^{\oplus n}$ (cf. [25, proof of Th. 2.7]). The following equalities finish the proof:

$$\dim.\mathrm{virt}.\, W = \dim_L \mathrm{Ker}\, \gamma_W - \dim_L \mathrm{Coker}\, \gamma_W = h^0(\mathcal{E}) - h^1(\mathcal{E}).$$

Indeed, the inverse construction $(R, W) \mapsto (M, V_\infty)$ is obvious since we can identify W with M and V_∞ with $(\tau^{-1}L[\![\tau^{-1}]\!])^{\oplus n}$ as above. □

Remark. For the sake of simplicity we suppose that morphisms in the category of Schur pairs are defined as morphisms of underlying admissible subrings. Thus, the uniformization of Drinfeld-Anderson motives arise as a by-product of the anti-equivalence between the categories of A-motives and admissible abelian A-modules restricted to the formally trivial case.

The correspondence $M \mapsto W_M$ is an analogue of the famous Krichever map ([1], [11, Sect. 6]). It is well-known that its image is the so-called algebraic part of the Sato Grassmannian. If we consider Drinfeld-Anderson motives of fixed rank r then one can give more precise description. Let $q\text{-Gr}_n^r$ denote the r-reduced n-component Sato Grassmannian (cf. Appendix 4.2 for a definition).

Corollary 3.4.3. *If M is a Drinfeld-Anderson motive of rank r and τ-rank n then*

$$W_M \in q\text{-Gr}_n^r(h^0(\mathcal{E}_M) - h^1(\mathcal{E}_M)).$$

4. Appendices

4.1. Proalgebraic structure of Sato Grassmannians

The n-component Sato Grassmannian has many different structures. From the algebro-geometric point of view it may be easily described as a proalgebraic variety [8, 4.3], [25, Sect. 1].

Let us recall that the set of L-subspaces of $L(\!(\tau^{-1})\!)^{\oplus n}$ has the following *Tate topology* [8, 2.4.1]: $U \subset L(\!(\tau^{-1})\!)^{\oplus n}$ is *open* if $\tau^{-N}L[\![\tau^{-1}]\!]^{\oplus n} \subset U$ for a sufficiently big positive integer N. Moreover, U is *bounded* if $U \subset \tau^N L[\![\tau^{-1}]\!]^{\oplus n}$ for $N \gg 0$. A subspace both open and bounded is called a *lattice*. Finally, a subspace V is *discrete* if for some open U one has $U \cap V = 0$. In this language the points of the n-component Sato Grassmannian are exactly colattices (maximal discrete subspaces) in $L(\!(\tau^{-1})\!)^{\oplus n}$. According to [8, 4.3] the set $q\text{-Gr}_n^{(V)}$ of colattices transversal to any fixed lattice V is a $\mathrm{Hom}(L(\!(\tau^{-1})\!)^{\oplus n}/V, V)$-torsor and, consequently, the projective limit of finite-dimensional spaces. The gluing of $q\text{-Gr}_n^{(V)}$ for different V defines a structure of a proalgebraic variety on $q\text{-Gr}_n$.

4.2. Ind-algebraic structure of r-reduced (loop) Grassmannians

As shown above the Sato Grassmannians are disjoint unions of strata of different virtual dimension. Another "stratification" is given by *r-reduced Sato Grassmannians:*

$$q\text{-Gr}_n^r = \{W \in q\text{-Gr}_n \mid \tau^r W \subset W\}.$$

Notice that if $W \in q\text{-Gr}_n^r$ then $W/\tau^r W$ is an nr-dimensional vector space. Any nr-tuple $w = (w_1, \ldots, w_{nr})$ of W spanning $W/\tau^r W$ defines a twisted formal loop γ_w with values in GL_{nr} [33, p. 14], that is,

$$\gamma_w \in q\text{-LGL}_{nr} := \mathrm{GL}_{nr}(L(\!(\tau^{-1})\!)).$$

This loop is uniquely defined up to an element of

$$q\text{-}\mathrm{L}^{+}\mathrm{GL}_{nr} := \mathrm{GL}_{nr}(L[\![\tau^{-1}]\!]).$$

As a consequence we can identify $q\text{-}\mathrm{Gr}_n^r$ with the *formal loop Grassmannian* $q\text{-}\mathrm{LGL}_{nr}/q\text{-}\mathrm{L}^{+}\mathrm{GL}_{nr}$. There are two well-known and equivalent ind-structures on the (twisted) loop Grassmannians: the usual one given by the pole orders and the ind-structure given by generalized Schubert varieties ([20], [34], [22, Sect. 4]).

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