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Maximal Facet-to-Facet Snakes of Unit Cubes

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Abstract. Let $\mathcal{C} = \langle C_1, C_2, \ldots, C_n \rangle$ be a finite sequence of unit cubes in the d-dimensional space. The sequence $\mathcal{C}$ is called a facet-to-facet snake if $C_i \cap C_{i+1}$ is a common facet of C_i and C_{i+1}, $1 \leq i \leq n-1$, and $\dim(C_i \cap C_j) \leq \max\{-1, d+i-j\}$, $1 \leq i < j \leq n$. A facet-to-facet snake of unit cubes is called maximal if it is not a proper subset of another facet-to-facet snake of unit cubes. In this paper we prove that the minimum number of d-dimensional unit cubes which can form a maximal facet-to-facet snake is $8d-1$ for all $d \geq 3$.

1. Introduction

A finite sequence $\mathcal{C} = \langle C_1, C_2, \ldots, C_n \rangle$ of pairwise nonoverlapping congruent convex bodies in the d-dimensional space where $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ is called a snake. If the snake $\mathcal{C}$ is not a proper subset of another snake of convex bodies congruent to the members of $\mathcal{C}$ then we say that the snake is maximal. Now, the problem is to determine the minimum number of convex bodies congruent to the members of $\mathcal{C}$ which can form a maximal snake. It was proved in [1] that the minimum number of congruent circular discs which can form a maximal snake is 10.

In this paper we consider a variant of this "min-max" problem which might be interesting in information theory as well. Let $\mathcal{C} = \langle C_1, C_2, \ldots, C_n \rangle$ be a finite sequence of d-dimensional unit cubes. The sequence $\mathcal{C}$ is called a facet-to-facet snake if $C_i \cap C_{i+1}$ is a common facet of

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C_i and C_{i+1}, $1 \leq i \leq n-1$, and $\dim(C_i \cap C_j) \leq \max\{-1, d+i-j\}$, $1 \leq i < j \leq n$ (by convention, $\dim(C_i \cap C_j) = -1$ if and only if $C_i \cap C_j = \emptyset$). A facet-to-facet snake of unit cubes is called maximal if it is not a proper subset of another facet-to-facet snake of unit cubes. Answering a question of H. Harborth (see [2]) it was proved in [3] and [4] that the minimum number of unit squares which can form a maximal facet-to-facet snake is 19 (see Figure 1).

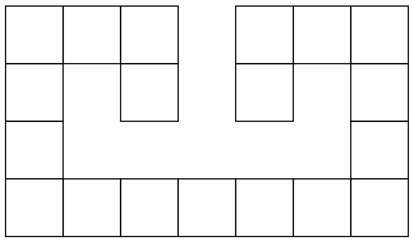

Figure 1.

H. Harborth and C. Thürmann found essentially different examples of 3-dimensional maximal facet-to facet snakes of 23 unit cubes (see Figure 2).

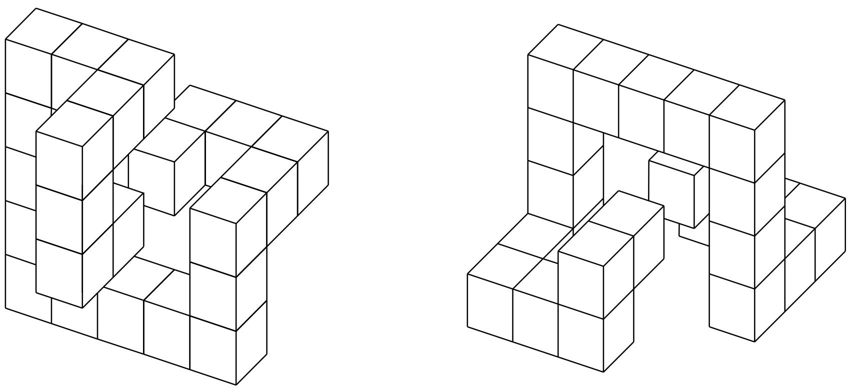

Figure 2.

Generalizing these constructions we show that there exist d-dimensional maximal facet-to-facet snakes of $8d-1$ unit cubes for all $d \geq 3$. We also show that $8d-1$ is the smallest possible number of unit cubes which can form a maximal facet-to-facet snake for all $d \geq 3$. The following theorem summarizes our results.

Theorem 1. *The minimum number of d-dimensional unit cubes which can form a maximal facet-to-facet snake is $8d-1$ for all $d \geq 3$.*

We note that the problem of determining the exact number of non-congruent d-dimensional maximal facet-to-facet snakes of $8d-1$ unit cubes remains open.

2. Constructions

In this section we show that there exist maximal facet-to-facet snakes in the d-dimensional space consisting of $8d-1$ unit cubes for all $d \geq 3$. The simplest way to describe these snakes is to list the coordinates of the centres c_i of the cubes C_i, $1 \le i \le 8d-1$, in a Cartesian coordinate system whose axes are parallel to the sides of the cubes. Let $e_1, e_2, \dots, e_n$ denote the coordinate unit vectors of such a coordinate system. The centre of the first cube is

$$c_1 = -e_1.$$

For $1 \le i \le d-1$,

$$\begin{aligned} c_{4i-2} &= -2e_i, \\ c_{4i-1} &= -2e_i - e_{i+1}, \\ c_{4i} &= -2e_i - 2e_{i+1}, \\ c_{4i+1} &= -e_i - 2e_{i+1}. \end{aligned}$$

The centres of the next four cubes are

$$\begin{aligned} c_{4d-2} &= -2e_d, \\ c_{4d-1} &= e_2 - 2e_d, \\ c_{4d} &= 2e_2 - 2e_d, \\ c_{4d+1} &= 2e_2 - e_d. \end{aligned}$$

For $2 \le i \le d-1$,

$$\begin{aligned} c_{4d+4i-6} &= 2e_i, \\ c_{4d+4i-5} &= 2e_i + e_{i+1}, \\ c_{4d+4i-4} &= 2e_i + 2e_{i+1}, \\ c_{4d+4i-3} &= e_i + 2e_{i+1}. \end{aligned}$$

Finally, the centres of the last six cubes are

$$\begin{aligned} c_{8d-6} &= 2e_d, \\ c_{8d-5} &= e_1 + 2e_d, \\ c_{8d-4} &= 2e_1 + 2e_d, \\ c_{8d-3} &= 2e_1 + e_d, \\ c_{8d-2} &= 2e_1, \\ c_{8d-1} &= e_1. \end{aligned}$$

To prove that the above cubes indeed form a facet-to-facet snake it is enough to observe that:

(1) For any two consecutive cubes there is exactly one coordinate in which their centres differ. The difference in this coordinate is one, i.e. the dimension of the intersection of the cubes is $d-1$.

(2) If the difference between the indices of two cubes is two then either there is exactly one coordinate or there are exactly two coordinates in which their centres differ. In the first case the difference in the coordinate is two, i.e. the cubes are disjoint. In the second case the difference in both coordinates is one, i.e. the dimension of the intersection of the cubes is $d-2$.

(3) If the difference between the indices of two cubes is at least three then there is a coordinate in which their centres differ by at least two, i.e. the cubes are disjoint.

We can continue the snake at C_1 neither parallel to the axis of direction e_1 because of the presence of C_2 and C_{8d-1}, nor parallel to the axis of direction e_i because of the presence of C_{4i-2} and $C_{4d-4i+6}$, $2 \le i \le d$. Similarly, we can continue the snake at C_{8d-1} neither parallel to the axis of direction e_1 because of the presence of C_1 and C_{8d-2}, nor parallel to the axis of direction e_i because of the presence of C_{4i-2} and $C_{4d-4i+6}$, $2 \le i \le d$. Therefore the above snake is maximal.

We note that the above construction coincides with the construction given on the left hand side of Figure 2 for $d = 3$. We also note that one can generalize the construction given on the right hand side of Figure 2 as well for all $d \ge 3$.

3. Proof of Theorem 1

In what follows facet-to-facet snakes of d-dimensional unit cubes will be briefly called snakes. Consider a snake $\mathcal{C} = \langle C_1, C_2, \ldots, C_n \rangle$. Let $e_1, \ldots, e_d$ denote the coordinate unit vectors of a Cartesian coordinate system whose axes are parallel to the edges of the cubes in $\mathcal{C}$. With the snake $\mathcal{C}$ we can associate a sequence $V = \langle v_1, \ldots, v_{n-1} \rangle$ of unit vectors parallel to the coordinate axes so that $C_{i+1} = v_i + C_i$, $i = 1, 2, \ldots, n-1$. Thus $|\mathcal{C}| = |V| + 1$ holds. We mention a simple property of $\mathcal{C}$ and V.

Proposition 1. *For $1 \le i < j \le n$ either $C_i \cap C_j = \emptyset$ or $\dim(C_i \cap C_j) = d + i - j$. In addition, in the latter case the vectors $v_i, v_{i+1}, \ldots, v_{j-1}$ are mutually orthogonal.*

Proof. If $C_i \cap C_j \ne \emptyset$ then $\dim(C_i \cap C_j) \le d + i - j$ by definition. The projections of C_i and C_j on the coordinate axes are also not disjoint and $\dim(C_i \cap C_j)$ is equal to the number k of axes where the projections of C_i and C_j coincide. If the projections of C_i and C_j do not coincide on a coordinate axis then at least one of the vectors $v_i, v_{i+1}, \ldots, v_{j-1}$ is parallel to this axis from which $d - k \le j - i$. Together with the previous inequality this implies that $\dim(C_i \cap C_j) = d + i - j$ and the vectors $v_i, v_{i+1}, \ldots, v_{j-1}$ are mutually orthogonal. □

Corollary 1. *For $1 \le i < j < k \le n$ the inequality $\dim(C_i \cap C_k) \le \dim(C_i \cap C_j)$ holds with equality if and only if both $C_i \cap C_k$ and $C_i \cap C_j$ are empty.*

Our strategy for proving that any maximal snake consists of at least $8d - 1$ cubes will be the following. Consider a maximal snake $\mathcal{C}$. With this snake we associate the subsequences $V_1, \ldots, V_d$ of V consisting of vectors parallel to $e_1, \ldots, e_d$, respectively. We will show that there are at most five axes such that the corresponding subsequences V_i consist of at most seven elements. Then the proof will be completed by a rather technical case-by-case analysis based on the number and the structure of the subsequences V_i consisting of at most seven elements.

First we introduce the concept of *blocking.* If $\mathcal{C} = \langle C_1, C_2, \ldots, C_n \rangle$ is a maximal snake then for each $e = \pm e_m$, $m = 1, \ldots, d$ there exists a cube C_i in $\mathcal{C}$ which intersects $e + C_1$ in a face of dimension at least $\max\{d - i + 1, 0\}$. In this case we will say that C_1 is blocked by C_i from direction e. Project $C_i, C_1, e + C_1$ onto the coordinate axis of direction e. Since

$(e+C_1)\cap C_i \neq \emptyset$ the same holds for their projections as well. There are three different cases, (1) the projections of $2e+C_1$ and C_i coincide, (2) the projections of $e+C_1$ and C_i coincide, (3) the projections of C_1 and C_i coincide. Since the projections of $e+C_1$ and C_1 on the other coordinate axes coincide therefore the projections of C_1 and C_i on the other coordinate axes intersect each other. Thus in the second and third cases $C_1 \cap C_i \neq \emptyset$ which implies that $i \leq d+1$ in these cases. Observe that the third case cannot occur because in this case $\dim((e+C_1)\cap C_i)+1 = \dim(C_1 \cap C_i) = d-i+1$, i.e. C_1 is not blocked by C_i from direction e, a contradiction. The second case may occur of course. Similar things hold for C_n as well.

The above discussion easily implies that $C_1 \cap C_n = \emptyset$ and no cube in $\mathcal{C}$ intersects both C_1 and C_n.

Since $C_1 \cap C_n = \emptyset$ therefore there exists at least one coordinate axis where the projections of C_1 and C_n are disjoint. The axes with this property will be called *primary axes* while the axes where the projections of C_1 and C_n are not disjoint will be called *secondary axes*. As we have already mentioned, with the snake $\mathcal{C}$ we can associate the subsequences $V_1, \dots, V_d$ of V consisting of vectors parallel to $e_1, \dots, e_d$, respectively. Instead of the vectors of these subsequences we will also use the sign $+$ when the vector is identical with the corresponding coordinate unit vector and the sign $-$ otherwise.

We distinguish four types P1–P4 of subsequences associated with the primary axes. Consider the projections of the centres of the cubes in $\mathcal{C}$ on a primary axis. For the sake of simplicity assume that the direction of this axis is e_1. Let A and B denote the centres of the projections of C_1 and C_n, respectively. Without loss of generality we may assume that $\overrightarrow{AB} = te_1$ where $t \geq 2$ is the distance between A and B. Let D, C, E, F be points on the axis of direction e_1 such that $\overrightarrow{DC} = \overrightarrow{CA} = \overrightarrow{BE} = \overrightarrow{EF} = e_1$. The cube C_1 is blocked from $-e_1$ by a cube in $\mathcal{C}$ and the projection of the centre of this cube is C or D. Similarly, the cube C_n is blocked from e_1 by a cube in $\mathcal{C}$ and the projection of the centre of this cube is E or F.

Type P1. The projection of $\mathcal{C}$ goes through both D and F. Then $|V_1| \geq t+8$ with equality if and only if $V_1 = \langle -,-,+,+,\dots,+,+,-,- \rangle$ where the number of the $+$ signs is $t+4$.

Type P2. The projection of $\mathcal{C}$ goes through F and avoids D. The projection of the centre of the cube in $\mathcal{C}$ which blocks C_1 from $-e_1$ must be C. Therefore this cube intersects C_1. This implies that the first element of V_1 is $-$ and before the first vector of V_1 there cannot be two identical vectors in V. Now $|V_1| \geq t+6$ with equality if and only if $V_1 = \langle -,+,+,\dots,+,+,-,- \rangle$ where the number of the $+$ signs is $t+3$.

Type P3. The projection of $\mathcal{C}$ goes through D and avoids F. The projection of the centre of the cube in $\mathcal{C}$ which blocks C_n from e_1 must be E. Therefore this cube intersects C_n. This implies that the last element of V_1 is $-$ and after the last vector of V_1 there cannot be two identical vectors in V. Now $|V_1| \geq t+6$ with equality if and only if $V_1 = \langle -,-,+,+,\dots,+,+,- \rangle$ where the number of the $+$ signs is $t+3$.

Type P4. The projection of $\mathcal{C}$ avoids both F and D. Here both the first and the last elements of V_1 are $-$. Now $|V_1| \geq t+4$ with equality if and only if $V_1 = \langle -,+,+,\dots,+,+,- \rangle$ where the number of the $+$ signs is $t+2$.

We also distinguish six types S1–S6 of the subsequences associated with the secondary axes. Consider the projections of the centres of the cubes in $\mathcal{C}$ on a secondary axis. For the sake of simplicity assume again that the direction of this axes is e_1. Let A and B denote the centres of the projections of C_1 and C_n, respectively.

In the first two types S1, S2 the points A and B coincide. Without loss of generality we may assume that the first element of V_1 is $-e_1$. Let D, C, E, F be points on the axis of direction e_1 such that $\overrightarrow{DC} = \overrightarrow{CA} = \overrightarrow{BE} = \overrightarrow{EF} = e_1$. The cube C_1 is blocked from $-e_1$ by a cube in $\mathcal{C}$ and the projection of the centre of this cube is C or D. Similarly, the cube C_n is blocked from e_1 by a cube in $\mathcal{C}$ and the projection of the centre of this cube is E or F. Obviously, the cube in $\mathcal{C}$ blocking C_1 from e_1 does not intersect C_1 hence the projection of $\mathcal{C}$ cannot avoid F.

Type S1. The projection of $\mathcal{C}$ goes through both D and F. Then $|V_1| \geq 8$ with equality if and only if $V_1 = \langle -, -, +, +, +, +, -, - \rangle$.

Type S2. The projection of $\mathcal{C}$ avoids D and goes through F. In this case the cubes in $\mathcal{C}$ blocking C_1 and C_n from $-e_1$ intersect C_1 and C_n, respectively. This implies that the first two and the last two elements of V_1 are $-, +$ and it cannot be two identical vectors in V before the first and after the last vector of V_1. Thus $|V_1| \geq 8$ with equality if and only if $V_1 = \langle -, +, +, +, -, -, -, + \rangle$.

In the remaining four types S3–S6 the points A and B do not coincide. Without loss of generality we may assume that $\overrightarrow{AB} = e_1$. Let D, C, E, F be points on the axis of direction e_1 such that $\overrightarrow{DC} = \overrightarrow{CA} = \overrightarrow{BE} = \overrightarrow{EF} = e_1$. The cube C_1 is blocked from $-e_1$ by a cube in $\mathcal{C}$ and the projection of the centre of this cube is C or D. Similarly, the cube C_n is blocked from e_1 by a cube in $\mathcal{C}$ and the projection of the centre of this cube is E or F.

Type S3. The projection of $\mathcal{C}$ goes through both D and F. Then $|V_1| \geq 9$ with equality if and only if $V_1 = \langle -, -, +, +, +, +, +, -, - \rangle$.

Type S4. The projection of $\mathcal{C}$ goes through F and avoids D. Here the first two elements of V_1 are $-, +$ and it cannot be two identical vectors in V before the first vector of V_1. Now $|V_1| \geq 7$ with equality if and only if $V_1 = \langle -, +, +, +, +, -, - \rangle$.

Type S5. The projection of $\mathcal{C}$ goes through D and avoids F. Here the last two elements of V_1 are $+, -$ and it cannot be two identical vectors in V after the last vector of V_1. Now $|V_1| \geq 7$ with equality if and only if $V_1 = \langle -, -, +, +, +, +, - \rangle$.

Type S6. The projection of $\mathcal{C}$ avoids both F and D. The first two elements of V_1 are $-, +$ while the last two elements are $+, -$, and there cannot be two identical vectors in V before the first and after the last vector of V_1. Now $|V_1| \geq 5$ with equality if and only if $V_1 = \langle -, +, +, +, - \rangle$. Here we mention that if $|V_1| \neq 5$ then $|V_1| \geq 7$ with equality if and only if V_1 is $\langle -, +, -, +, +, +, - \rangle$, $\langle -, +, +, -, +, +, - \rangle$, or $\langle -, +, +, +, -, +, - \rangle$.

The following simple observation will be used frequently in the proof.

Lemma 1. *For the vectors of V, if $v_i = -v_j$ for some $1 \leq i < j \leq n-1$ then one can find two indices $i < k < l < j$ such that $v_k = v_l$.*

Proof. It is enough to prove the lemma when $v_m \perp v_i$ for all $i < m < j$. First we show that there exist two indices $i < k' < l' < j$ such that $v_{k'} \parallel v_{l'}$. If this is not true then $v_{k'} \perp v_{l'}$ for all $i < k' < l' < j$ which implies that $\dim(C_i \cap C_j) = d + i - j$. Thus $\dim(C_i \cap C_{j+1}) = d + i + 1 - j$, a contradiction. Now, if $v_{k'} = v_{l'}$ then we are done. On the other hand, if $v_{k'} = -v_{l'}$ then we can repeat the above argument with $i = k'$ and $j = l'$. □

Corollary 2. *The first two vectors cannot be opposite in all* $V_1, \ldots, V_d$.

The next three lemmas will show that there are at most five axes such that the corresponding subsequences V_i consist of at most seven elements.

Lemma 2. *If there is a subsequence* V_i *corresponding to a primary axis which consists of at most seven elements then the other subsequences corresponding to the primary axes consist of at least nine elements.*

Proof. If we have only one primary axis then there is nothing to prove. Without loss of generality we may assume that $i = 1$ and V_2 also corresponds to a primary axis. Now $V_1 = \langle -, +, +, \ldots, +, +, - \rangle$ where the number of the $+$ signs is 4 or 5. Obviously, if the first element of V_2 is $+$ then $|V_2| \geq 10$ and we are done. Thus we may assume that the first element of V_2 is $-$. Let t_1 and t_2 be the distances between the projections of C_1 and C_n on the axes of direction e_1 and e_2, respectively.

If $t_2 \geq 3$ then consider the projection of the snake on the plane of e_1 and e_2 (see Figure 3).

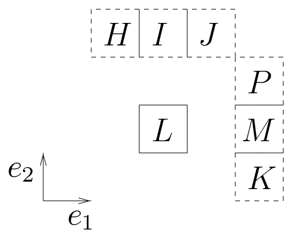

Figure 3.

Using the notations of Figure 3 the projection of C_1 is the square L. The cube C_1 is blocked from e_1 and from e_2 by cubes C_i and C_j in $\mathcal{C}$, respectively. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J, since the first element of V_1 and V_2 is $-$ and thus both $C_1 \cap C_i$ and $C_1 \cap C_j$ are empty. Then $j < i$ because of the structure of V_1. This implies that $V_2 \neq \langle -, +, +, \ldots, +, +, - \rangle$ from which $|V_2| \geq t_2 + 6 \geq 9$.

If $t_2 = 2$ and $t_1 = 3$ then consider the projection of the snake on the plane of e_1 and e_2 (see Figure 4).

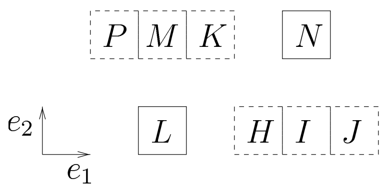

Figure 4.

Using the notations of Figure 4 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_2 by a cube C_i and the cube C_n is blocked from $-e_2$ by a cube

C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. Then $i < j$ because of the structure of V_1. This implies that $|V_2| \geq 10$ with equality if and only if $V_2 = \langle -, +, +, +, -, -, +, +, +, - \rangle$.

Finally, if $t_2 = 2$ and $t_1 = 2$ then consider the projection of the snake on the plane of e_1 and e_2 (see Figure 5).

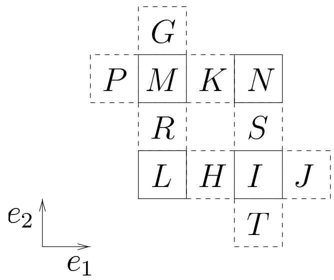

Figure 5.

Using the notations of Figure 5 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_2 by a cube C_i and the cube C_n is blocked from $-e_2$ by a cube C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. If $i < j$ then using a similar argument as before we conclude that $|V_2| \geq 10$. On the other hand, if $i > j$ then consider a cube C_k which blocks C_1 from e_1 and a cube C_l which blocks C_n from $-e_1$. The projection of C_k is S, I, or T while the projection of C_l is G, M, or R. Then $i < k$ and $l < j$ because of the structure of V_1. Together with $i > j$ these imply that $l < k$ from which $|V_2| \geq 10$ with equality if and only if $V_2 = \langle -, +, +, -, +, +, -, +, +, - \rangle$. □

To formulate the next two lemmas we need a definition. A secondary axis will be called a bad secondary axis if the subsequence corresponding to this axis consists of at most seven elements. Recall that there are only six possibilities for bad secondary axes:

$$\langle -, +, +, +, +, -, - \rangle,$$
$$\langle -, -, +, +, +, +, - \rangle,$$
$$\langle -, +, +, +, - \rangle,$$
$$\langle -, +, -, +, +, +, - \rangle,$$
$$\langle -, +, +, -, +, +, - \rangle,$$
$$\langle -, +, +, +, -, +, - \rangle.$$

Lemma 3. *There are at most two bad secondary axes of the forms*

$$\langle -, +, +, +, +, -, - \rangle,$$
$$\langle -, +, +, +, - \rangle,$$
$$\langle -, +, +, -, +, +, - \rangle,$$
$$\langle -, +, +, +, -, +, - \rangle.$$

In addition, if there are exactly two bad secondary axes of the above forms then the first element of each subsequence associated with a primary axis is +.

Lemma 4. *There are at most two bad secondary axes of the forms*

$$\langle -,-,+,+,+,+,- \rangle,$$
$$\langle -,+,+,+,- \rangle,$$
$$\langle -,+,+,-,+,+,- \rangle,$$
$$\langle -,+,-,+,+,+,- \rangle.$$

In addition, if there are exactly two bad secondary axes of the above forms then the last element of each subsequence associated with a primary axis is $+$.

By symmetry, it is enough to prove Lemma 3 only.

Proof of Lemma 3. If we have at most one secondary axis of the form described in the lemma then there is nothing to prove. Thus we may assume, without loss of generality, that the axes of direction e_1 and e_2 are bad secondary axes of the forms described in the lemma. We may also assume that the first vector of V_1 is before the first vector of V_2 in the sequence V. Consider the projection of the snake on the plane of e_1 and e_2 (see Figure 6).

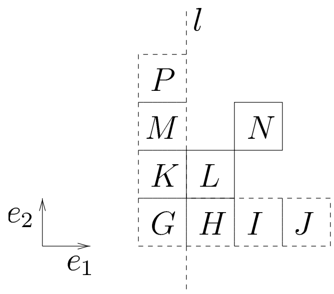

Figure 6.

Using the notations of Figure 6 the projections of C_1 and C_n are L and N, respectively. The cube C_n is blocked from $-e_1$ and $-e_2$ by a cube C_i and a cube C_j, respectively. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. Any cube in $\mathcal{C}$ blocking C_1 from the direction $-e_2$ must intersect C_1 because of the structure of V_2. This implies that it cannot be two parallel vectors in V before the first vector of V_2. Now there exists a cube in $\mathcal{C}$ whose projection on the plane of e_1 and e_2 is the square G. Among these cubes let C_k be that one whose index is minimal. Then the first vector in V_2 is v_{k-1}. Observe that $k < j$ and the projections of the cubes in $\mathcal{C}$ whose indices are greater than j are on the right of the line l separating the squares K and L because of the structure of V_1. Moreover, none of the vectors $v_k, \dots, v_{j-1}$ belongs to V_2 because of the structure of V_2. This immediately yields $i < k$. Since C_n is blocked by C_i therefore the projections of C_i and C_n intersect each other on the axes of direction different from e_1, especially on every primary axis. Since there do not exist two parallel vectors in V before the first vector of V_2, therefore the projections of C_1 and C_i intersect each other on every axis. Thus the distance between the projections of the centres of C_1 and C_n on the primary axes is exactly two and the first element of each subsequence associated with the primary axes is $+$. In addition, the first element of V_2 is after the first element of the subsequences associated with the primary axes in V. Let

v_s denote the first element in V_1. Obviously $s \leq i-1$. Furthermore $s < i-1$ otherwise $C_{i-1} \cap C_n \neq \emptyset$, a contradiction. The projections of C_s and C_n on the secondary axes intersect each other. Indeed, this is trivial for the axis of direction e_1 while on the other secondary axes the projections of C_1 and C_i intersect the projection of C_n and thus the projections of the cubes in $\mathcal{C}$ between C_1 and C_i also intersect the projection of C_n since there do not exist two parallel vectors in V before v_{i-1}. Combining this with the fact that $C_s \cap C_n = \emptyset$ we obtain that there is a primary axis on which the projections of C_s and C_n are disjoint. The index of the first vector in the subsequence associated with such a primary axis is greater than s because the first element of the subsequence associated with any primary axis is +. Without loss of generality we may assume that V_3 is that subsequence whose first vector is the last one in V among the first vectors of the subsequences associated with the primary axes. It is clear that V_3 is independent from the choice of V_1 and V_2, i.e. from the two bad secondary axes chosen at the beginning. Furthermore, the first element of V_3 is between the first elements of V_1 and V_2 in V. This implies that a third bad secondary axis of the forms described in the lemma different from V_1 and V_2 cannot occur. □

By Lemma 3 and Lemma 4 there are at most four bad secondary axes. We distinguish five different cases with respect to the number of the bad secondary axes.

Case 1. There are four bad secondary axes. Then the subsequences associated with these axes consist of seven elements and both the first and the last element in the subsequences associated with the primary axes are +. This implies that the subsequences associated with the primary axes consist of at least 14 elements from which $|V| \geq 4 \cdot 7 + 14 + (d-5) \cdot 8 = 8d + 2$ follows.

Case 2. There are three bad secondary axes. Then at least two of the subsequences associated with these axes consist of seven elements. In the subsequences associated with the primary axes the first or the last element is +. This implies that the subsequences associated with the primary axes consist of at least 10 elements. If there is a five-element subsequence associated with a bad secondary axis then in the subsequences associated with the primary axes both the first and the last element are + from which $|V| \geq 5+7+7+14+(d-4) \cdot 8 = 8d+1$ follows. On the other hand, if the subsequences associated with the bad secondary axes consist of seven elements then $|V| \geq 7+7+7+10+(d-4) \cdot 8 = 8d-1$.

Case 3. There are two bad secondary axes. We may assume that V_1 and V_2 are associated with these axes. We may also assume, without loss of generality, that $|V_1| \leq |V_2|$.

Case 3.1. $|V_1| = |V_2| = 5$. Then both the first and the last element in the subsequences associated with the primary axes are + from which $|V| \geq 5+5+14+(d-3) \cdot 8 = 8d$.

Case 3.2. $|V_1| = 5$ and $|V_2| = 7$. Then the first or the last element in the subsequences associated with the primary axes is +. This implies that $|V| \geq 5+7+10+(d-3) \cdot 8 = 8d-2$.

Case 3.3. $|V_1| = |V_2| = 7$ and the two bad secondary axes belong to the same group among the two groups introduced in Lemma 3 and Lemma 4. Then the first or the last element in the subsequences associated with the primary axes is +. This implies that $|V| \geq 7+7+10+(d-3) \cdot 8 = 8d$.

Case 3.4. $|V_1| = |V_2| = 7$ and the two bad secondary axes belong to different groups among the two groups introduced in Lemma 3 and Lemma 4. Without loss of generality we may assume that V_1 is $\langle -,+,+,+,+,-,- \rangle$ or $\langle -,+,+,+,-,+,- \rangle$ while V_2 is $\langle -,-,+,+,+,+,- \rangle$ or $\langle -,+,-,+,+,+,- \rangle$. If the subsequences associated with the primary axes consist of at least 8 elements then we are done. Therefore assume that there is a subsequence, say V_3, associated with a primary axis such that $|V_3| \le 7$. Then V_3 is $\langle -,+,+,+,+,- \rangle$ or $\langle -,+,+,+,+,+,- \rangle$.

Case 3.4.1. There is one more primary axis besides the axis of direction e_3. Without loss of generality we may assume that this axis is of direction e_4. Then, by Lemma 2, $|V_4| \ge 9$. If $|V_3| = 7$ then we are done. Otherwise $V_3 = \langle -,+,+,+,+,- \rangle$. Consider the projection of the snake on the plane of e_1 and e_3 (see Figure 7).

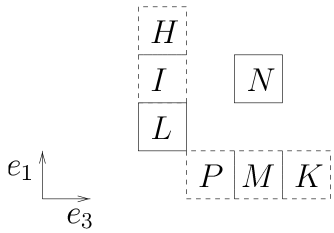

Figure 7.

Using the notations of Figure 7 the projections of C_1 and C_n are L and N, respectively. The cube C_n is blocked from $-e_1$ and $-e_3$ by a cube C_i and a cube C_j, respectively. The projection of C_i is P, M, or K while the projection of C_j is H, I, or L. Then $j < i$ because of the structure of V_3. Moreover, the projection of C_j is L because of the structure of V_1. The projections of C_j and C_n on the axis of direction e_4 intersect each other since C_j blocks C_n from $-e_3$. On the other hand, the projections of C_1 and C_n on the axis of direction e_4 are disjoint since the axis of direction e_4 is a primary axis. These observations imply that C_1 and C_j are different cubes. The vector v_{j-1} is before the first element of V_1 in V because of the structure of V_1. This implies that there are no two parallel vectors in V before v_{j-1}. Therefore the first element in V_4 is $+$ from which V_4 is of type P1 or P3. Moreover, the distance between the projections of the centres of C_1 and C_n is two. Thus $|V_4| \ge 10$ since $|V_4|$ is an even number and $V_4 \ne \langle -,-,+,+,+,+,+,- \rangle$. Summing up the vectors of the subsequences we obtain that $|V| \ge 7 + 7 + 6 + 10 + (d-4) \cdot 8 = 8d - 2$.

Case 3.4.2. There is no primary axis besides the axis of direction e_3. Consider the projection of the snake on the plane of e_1 and e_3 (see Figure 8). Here Figures 8a and 8b correspond to the cases where $|V_3| = 6$ and $|V_3| = 7$, respectively.

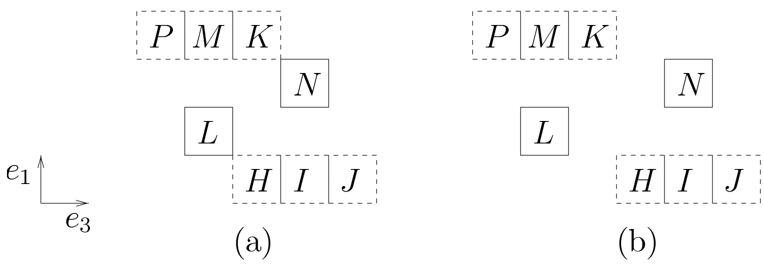

Figure 8.

Using the notations of Figure 8 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_1 by a cube C_i and the cube C_n is blocked from $-e_1$ by a cube C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. It is clear that $j < i$ because of the structure of V_1. If $|V_3| = 7$ then this is impossible because of the structure of V_3. Therefore $|V_3| = 6$ and the projections of C_i and C_j are K and H, respectively. Now C_1 and C_j are disjoint therefore there is an axis, say the axis of direction e_k, $k \neq 1, 3$, on which the projections of C_1 and C_j are also disjoint. The projections of C_j and C_n on the axis of direction e_k are not disjoint since C_n is blocked by C_j, therefore the projection of C_1 and C_n on this axis cannot coincide. Thus the axis of direction e_k is a secondary axis of type different from S1 or S2. The projections of cubes blocking C_n from e_k and $-e_k$ intersects the projection of C_n on the plane of e_1 and e_3. Therefore these cubes are after C_j in $\mathcal{C}$ from which $k \neq 2$ follows taking the structure of V_2 into account. Repeating the above argument with C_n and C_i instead of C_1 and C_j, respectively, we again find an axis, say the axis of direction e_l where $l \neq 1, 2, 3$, on which the projections of C_n and C_i are disjoint. This axis is again not of type S1 or S2. If $k \neq l$ then $|V| \geq 7 + 7 + 6 + 9 + 9 + (d - 5) \cdot 8 = 8d - 2$. On the other hand, if $k = l$ then the projections of C_i, C_1, C_n, C_j are in this order on the axis of direction e_l and $j < i$. This implies that $|V_k| \geq 11$ from which $|V| \geq 7+7+6+11+(d-4)\cdot 8 = 8d-1$.

Case 4. There is only one bad secondary axis. We may assume that V_1 is associated with this axis.

Case 4.1. $|V_1| = 7$. If there is a V_k consisting of at least 9 elements then we are done. Therefore assume that $|V_r| \leq 8$ for all $1 \leq r \leq d$. If the sequences associated with the primary axes consist of at least 7 elements then we are again done. Thus we assume that there is a sequence, say V_2, associated with a primary axis which consists of 6 elements. By Lemma 2 this is the only primary axis. Consider the projection of the snake on the plane of e_1 and e_2 (see Figure 9).

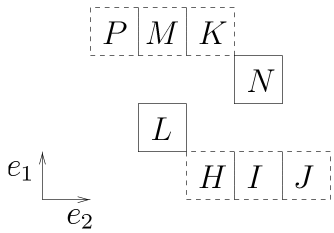

Figure 9.

Using the notations of Figure 9 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_1 by a cube C_i and the cube C_n is blocked from $-e_1$ by a cube C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. Then $j < i$ because of the structure of V_1 and the projections of C_i and C_j are K and H, respectively. Now C_1 and C_j are disjoint therefore there is an axis, say the axis of direction e_k, on which the projections of C_1 and C_j are also disjoint. It is clear that $k \neq 1, 2$. The projections of C_j and C_n on the axis of direction e_k are not disjoint, therefore the projections of C_1 and C_n on this axis cannot coincide. Thus the axis of direction e_k is a secondary axis of type different from S1 or S2. This implies that $|V_k|$ is an odd number and thus $|V_k| \geq 9$, a contradiction.

Case 4.2. $|V_1| = 5$. Then $V_1 = \langle -, +, +, +, - \rangle$.

Case 4.2.1. There is a primary axis such that the subsequence, say V_2, associated with this axis consists of at most 7 elements. Consider the projection of the snake on the plane of e_1 and e_2 (see Figure 10). Here Figures 10a and 10b correspond to the cases where $|V_2| = 6$ and $|V_2| = 7$, respectively.

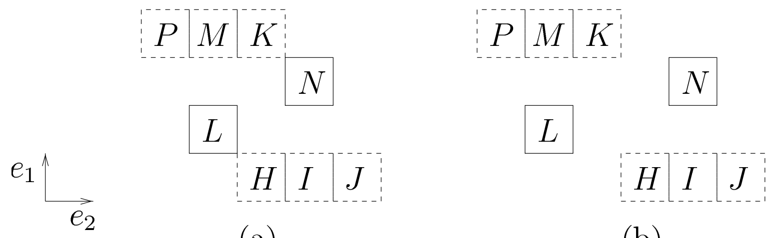

Figure 10.

Using the notations of Figure 10 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_1 by a cube C_i and the cube C_n is blocked from $-e_1$ by a cube C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. It is clear that $j < i$ because of the structure of V_1. If $|V_2| = 7$ then this is impossible because of the structure of V_2. Therefore $|V_2| = 6$ and the projections of C_i and C_j are K and H, respectively.

If there is one more primary axis then the subsequence, say V_l, associated with this axis starts and ends with $+$. Indeed, let C_k be a cube in $\mathcal{C}$ which blocks C_n from $-e_2$. The projections of C_k and C_n on the axis of direction e_2 are disjoint since the last element of V_2 is $-$. Then $k < j$ because of the structure of V_2. Thus the projection of C_k on the plane of e_1 and e_2 is L because of the structure of V_1. The projections of C_k and C_n on the axis of direction e_l intersect each other since C_k blocks C_n. On the other hand, the projections of C_1 and C_n on the axis of direction e_l are disjoint since the axis of direction e_l is a primary axis. These observations imply that C_1 and C_k are different cubes. The vector v_{k-1} is before the first element of V_1 in V because of the structures of V_1. This implies that there are no two parallel vectors in V before v_{k-1}. Therefore the first element in V_l is $+$. Repeating the above argument with a cube of $\mathcal{C}$ which blocks C_1 from e_2 we obtain that V_l also ends with $+$ from which V_l is of type P1 and thus $|V_l| \geq 14$ with equality if and only if V_l is $\langle +, -, -, -, +, +, +, +, +, +, -, -, -, + \rangle$ or $\langle +, +, +, +, -, -, -, -, -, -, +, +, +, + \rangle$. Thus $|V| \geq 5 + 6 + 14 + (d-3) \cdot 8 = 8d + 1$.

Therefore assume that the only primary axis is the axis of direction e_2. Now there is an axis, say the axis of direction e_3, on which the projections of C_1 and C_j are also disjoint since C_1 and C_j are disjoint. The projections of C_j and C_n on the axis of direction e_3 are not disjoint therefore the projection of C_1 and C_n on this axis cannot coincide. Thus the axis of direction e_3 is a secondary axis of type different from S1 or S2. Assume that $|V_3| = 9$ otherwise $|V| \geq 5 + 6 + 11 + (d-3) \cdot 8 = 8d - 2$ and we are done. Let C_m be a cube in $\mathcal{C}$ which blocks C_n from $-e_3$. The projections of C_m and C_n on the plane of e_1 and e_2 are not disjoint which implies that $m > j$. The projections of C_m and C_n on the axis of direction e_3 are not disjoint since $|V_3| = 9$. Thus $V_3 = \langle -, +, +, +, +, -, -, -, + \rangle$.

Similar argument for C_i and C_n shows that there exists a secondary axis, say the axis of direction e_4, such that $V_4 = \langle +, -, -, -, +, +, +, +, - \rangle$. Assume that $|V_r| = 8$ for all $5 \le r \le d$ otherwise $|V| = 5 + 6 + 9 + 9 + 9 + (d - 5) \cdot 8 = 8d - 2$ and we are done. This implies that the axes of directions $e_5, \dots, e_d$ are of types S1 or S2. The first two vectors in V_2 are opposite hence, by Lemma 3, between these two vectors there exist two identical vectors in V. Therefore there exists a subsequence, say V_5, whose first two elements are identical and are before the second vector of V_2 in V. Also, the first two vectors of V_5 are before v_j in V. In fact, the first three vectors of V_5 are before v_j in V since the projections of C_j and C_n on the axis of direction e_5 are not disjoint. The cube blocking C_n from $-e_5$ is after C_j in $\mathcal{C}$ since the projection of this cube intersects the projection of C_n on the plane of e_1 and e_2. But this is impossible since there are at least three vectors of V_5 before v_j in V and $V_5 = \langle -, -, +, +, +, +, -, - \rangle$.

Case 4.2.2. The subsequences associated with the primary axes consist of at least 8 elements. Assume that $|V_r| = 8$ for all $2 \le r \le d$ otherwise $|V| = 5 + 9 + (d - 2) \cdot 8 = 8d - 2$ and we are done. This implies among others that the secondary axes are of types S1 or S2. Let V_2 be a subsequence associated with a primary axis of type P2, P3, or P4. Then either the first four elements of V_2 are $-, +, +, +$ or the last four elements of V_2 are $+, +, +, -$. By symmetry, we may assume that the first four elements of V_2 are $-, +, +, +$. Consider the projection of the snake on the plane of e_1 and e_2 (see Figure 11). Here Figures 11a and 11b correspond to the cases where the distance between the projections of the centers of C_1 and C_n on the axis of direction e_2 is two and four, respectively (note that this distance cannot be an odd number).

Using the notations of Figure 11 the projections of C_1 and C_n are L and N, respectively. The cube C_1 is blocked from e_1 by a cube C_i and the cube C_n is blocked from $-e_1$ by a cube C_j. The projection of C_i is P, M, or K while the projection of C_j is H, I, or J. It is clear that $j < i$ because of the structure of V_1.

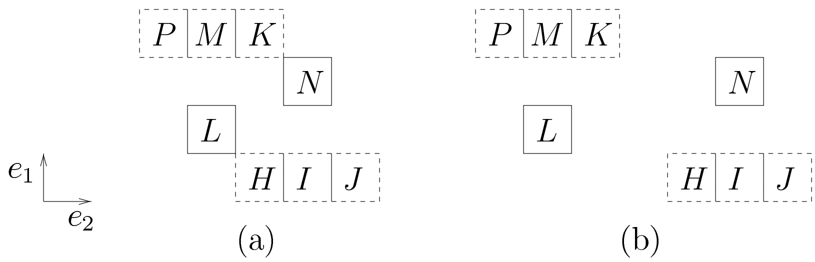

Figure 11.

The situation on Figure 11b cannot occur because of the structure of V_2. Now, the projections of C_i and C_j are K and H, respectively. Let C_k be a cube in $\mathcal{C}$ which blocks C_n from $-e_2$. The projections of C_k and C_n on the axis of direction e_2 are disjoint since the last element of V_2 is $-$. Thus the projection of C_k on the plane of e_1 and e_2 is L taking the structures of V_1 and V_2 into account. This implies, as in Case 4.2.1, that the first elements of the subsequences associated with the primary axes different from the axis of direction e_2 are $+$ which is impossible since the number of elements of these subsequences is eight.

Therefore the only primary axis is the axis of direction e_2. The first two vectors in V_2 are opposite hence, by Lemma 3, between these two vectors there exist two identical vectors in V. Therefore there exists a subsequence, say V_3, whose first two elements are identical and are before the second vector of V_2 in V. Then $V_3 = \langle -, -, +, +, +, +, -, - \rangle$ and the first two vectors of V_3 are before v_j in V. In fact, the first three vectors of V_3 are before v_j in V since the projections of C_j and C_n on the axis of direction e_3 are not disjoint. The cube blocking C_n from $-e_3$ is after C_j in $\mathcal{C}$ since the projection of this cube intersect the projection of C_n on the plane of e_1 and e_2. But this is impossible since there are at least three vectors of V_3 before v_j in V.

Case 5. There is no bad secondary axis. Then, by Lemma 2, all subsequences associated with the axes consist of at least eight elements with at most one exception in which the number of elements is at least six. Thus $|V| \geq 8d - 2$.

This completes the proof. □

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