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Geodesic CR-Lightlike Submanifolds

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Abstract. Geodesic (totally geodesic, D-geodesic, D'-geodesic and mixed geodesic) CR-lightlike submanifolds in indefinite Kaehler manifold are investigated. Some necessary and sufficient conditions on totally geodesic, D-geodesic, D'-geodesic and mixed geodesic CR-lightlike submanifolds are obtained. We find geometric properties of CR-lightlike submanifolds of an indefinite Kaehler manifold.

1. Introduction

The general theory of a lightlike submanifold has been developed by Kupeli [10] and Bejancu-Duggal [8]. In [9], the authors constructed the principal vector bundles to a lightlike submanifold in semi-Riemann manifold and obtained Gauss-Weingarten formulae as well as other properties of this submanifold.

The study of the geometry of CR-submanifolds in a Kaehler manifold was initiated by Bejancu and has been developed by [2], [4], [5], [6] and others.

In this paper, CR-lightlike submanifolds of indefinite Kaehler manifolds which were defined in [8] are considered. In particular, we study geodesic CR-lightlike submanifolds in indefinite Kaehler manifolds. Some characterizations of totally geodesic, D-geodesic, D'-geodesic and mixed geodesic CR-lightlike submanifolds in indefinite Kaehler manifolds are given.

2. Preliminaries

Let $(\overline{M},\overline{g})$ be a real $(m+n)$-dimensional semi-Riemann manifold, $m, n > 1$ and $\overline{g}$ be a semi-Riemann metric on $\overline{M}$. We denote by q the constant index of $\overline{g}$ and we suppose that $\overline{M}$ is not Riemann manifold.

Let M be a lightlike submanifold of dimension m of $\overline{M}$. In this case there exists a smooth distribution on M, named a radical distribution such that $N_p = TM_p \cap TM_p^{\perp}$, $\forall p \in M$. If the rank of $RadTM$ is $r > 0$, M is called an r-lightlike submanifold of $\overline{M}$. Then, there are four cases: I. $0 < r < \min\{m, n\}$; II. $1 < r = n < m$; III. $1 < r = m < n$; IV. $1 < r = m = n$. In the first case the submanifold is called an r-lightlike submanifold, in the second a coisotropic submanifold, in the third an isotropic submanifold and in the fourth a totally lightlike submanifold.

Let M be an r-lightlike submanifold of $\overline{M}$. We consider the complementary distribution $S(TM)$ of $Rad(TM)$ on TM which is called a screen distribution. Then, we have the direct orthogonal sum

$$TM = RadTM \perp S(TM). \tag{2.1}$$

As $S(TM)$ is a nondegenerate vector subbundle of $T\overline{M} \mid_M$, we put

$$T\overline{M} \mid_M = S(TM) \perp S(TM)^{\perp}, \tag{2.2}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\overline{M} \mid_M$. Moreover, $S(TM)$, $S(TM)^{\perp}$ are non-degenerate, we have the following orthogonal direct decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}. \tag{2.3}$$

Theorem 2.1. [9] *Let $\left(M, g, S(TM), S(TM^{\perp})\right)$ be an r-lightlike submanifold of a semi-Riemannian manifold $\left(\overline{M}, \overline{g}\right)$. Then, there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and the basis of $\Gamma\left(ltr(TM) \mid_U\right)$ consists of smooth sections $\{N_1, \ldots, N_r\}$ of $S(TM^{\perp})^{\perp} \mid_U$ such that*

$$\overline{g}\left(N_i, \xi_j\right) = \delta_{ij}, \overline{g}\left(N_i, N_j\right) = 0,\ i, j = 0, 1 \ldots, r$$

where $\{\xi_1, \ldots, \xi_r\}$ is a basis of $\Gamma\left(RadTM\right) \mid_U$.

We consider the vector bundle

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.4}$$

Thus

$$T\overline{M} = TM \oplus tr(TM) = S(TM) \perp S(TM^{\perp}) \perp (Rad(TM) \oplus ltr(TM). \tag{2.5}$$

Now, let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\, \forall X, Y \in \Gamma\left(TM\right)$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V,\ \forall X \in \Gamma\left(TM\right) \text{ and } V \in \Gamma\left(tr(TM)\right).$$

Using the projectors $L : tr(TM) \longrightarrow ltr(TM)$, $S : tr(TM) \longrightarrow S(TM^{\perp})$, from [9], we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \tag{2.6}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N) \tag{2.7}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W) \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma\left(S(TM^\perp)\right)$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X,N) \in \Gamma\left(S(TM^\perp)\right)$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Denote by P the projection morphism of TM to the screen distribution, we consider the decomposition

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY) \tag{2.9}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*^t}_X \xi \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$. Then we have the following equations

$$\overline{g}\left(h^l(X,PY),\xi\right) = g\left(A^*_\xi X, PY\right), \ \overline{g}\left(h^*(X,PY),N\right) = g\left(A_N X, PY\right), \tag{2.11}$$

$$g\left(A^*_\xi PX, PY\right) = g\left(PX, A^*_\xi PY\right), \ A^*_\xi \xi = 0. \tag{2.12}$$

Let $\left(\overline{M}, \overline{J}, \overline{g}\right)$ be a real $2m$-dimensional indefinite almost Hermitian manifold and M be a real n-dimensional submanifold of $\overline{M}$.

Definition 2.1. [8] *A submanifold M of an indefinite almost Hermitian manifold $\overline{M}$ is said to be a CR-lightlike submanifold if the following two conditions are fulfilled:*

i) $\overline{J}(Rad(TM))$ *is a distribution on M such that*

$$Rad(TM) \cap \overline{J} Rad(TM) = \{0\}.$$

ii) *There exist vector bundles $S(TM)$, $S(TM^\perp)$, $ltr(TM)$, D_0 and D' over M such that*

$$S(TM) = \left\{\overline{J}(RadTM) \oplus D'\right\} \perp D_0, \ \overline{J} D_0 = D_0, \ \overline{J} D' = L_1 \perp L_2,$$

where D_0 is a nondegenerate distribution on M and L_1, L_2 are vector bundles of $ltr(TM)$ and $S(TM^\perp)$, respectively.

From the definition of CR-lightlike submanifold, we have

$$TM = D \oplus D'$$

where

$$D = RadTM \perp \overline{J} RadTM \perp D_0.$$

We denote by S and Q the projections on D and D', respectively. Then we have

$$\overline{J} X = fX + \omega X \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$, where $fX = \overline{J} SX$ and $\omega X = \overline{J} QX$. On the other hand, we set

$$\overline{J} V = BV + CV \tag{2.14}$$

for any $V \in \Gamma(tr(TM))$, where $BV \in \Gamma(TM)$ and $CV \in \Gamma(tr(TM))$. Unless otherwise stated, M_1 and M_2 are supposed to as $\overline{J} L_1$ and $\overline{J} L_2$, respectively.

3. Geodesic CR-lightlike submanifolds

Definition 3.1. *A CR-lightlike submanifold of an indefinite almost Hermitian manifold is called mixed geodesic CR-lightlike submanifold if its second fundamental form h satisfies*

$$h(X,U)=0$$

for any $X\in\Gamma(D)$ and $U\in\Gamma(D')$.

Definition 3.2. *A CR-lightlike submanifold of an indefinite almost Hermitian manifold is called D-geodesic CR-lightlike submanifold if its second fundamental form h satisfies*

$$h(X,Y)=0$$

for any $X,Y\in\Gamma(D)$.

Definition 3.3. *A CR-lightlike submanifold of an indefinite almost Hermitian manifold is called D'-geodesic CR-lightlike submanifold if its second fundamental form h satisfies*

$$h(U,V)=0$$

for any $U,V\in\Gamma(D')$.

Theorem 3.1. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then, M is totally geodesic if and only if*

$$\left(L_\xi\overline{g}\right)(X,Y)=0$$

and

$$\left(L_W\overline{g}\right)(X,Y)=0$$

for any $X,Y\in\Gamma\left(TM\right),\xi\in\Gamma\left(Rad(TM)\right)$ and $W\in\Gamma\left(S(TM^{\perp})\right)$.

Proof. We note that to show M is totally geodesic, it suffices to show that

$$h(X,Y)=0$$

for any $X,Y\in\Gamma\left(TM\right)$. On the other hand, by the definition of lightlike submanifolds $h(X,Y)=0$ if and only if

$$\overline{g}\left(h(X,Y),\xi\right)=0$$

and

$$\overline{g}\left(h(X,Y),W\right)=0.$$

From (2.6) and definition of Lie derivative we have

$$\begin{aligned}
\overline{g}\left(h(X,Y),\xi\right) &= \overline{g}\left(\overline{\nabla}_XY,\xi\right)\\
&= X\overline{g}\left(Y,\xi\right)-\overline{g}\left(Y,\overline{\nabla}_X\xi\right)\\
&= -\overline{g}\left(Y,[X,\xi]\right)-\overline{g}\left(Y,\overline{\nabla}_\xi X\right)\\
&= -\overline{g}\left(Y,[X,\xi]\right)-\xi\overline{g}\left(Y,X\right)+\overline{g}\left(\overline{\nabla}_\xi Y,X\right)\\
&= -\overline{g}\left(Y,[X,\xi]\right)-\xi\overline{g}\left(Y,X\right)+\overline{g}\left(X,[\xi,Y]\right)+\overline{g}\left(\overline{\nabla}_Y\xi,X\right)\\
&= -\left(L_\xi\overline{g}\right)(X,Y)+\overline{g}\left(\overline{\nabla}_Y\xi,X\right)\\
&= -\left(L_\xi\overline{g}\right)(X,Y)-\overline{g}\left(\xi,\overline{\nabla}_YX\right)
\end{aligned}$$

or

$$2\overline{g}\left(h(X,Y),\xi\right)=-\left(L_{\xi}\overline{g}\right)(X,Y). \tag{3.1}$$

In a similar way we obtain

$$\begin{aligned}
\overline{g}\left(h(X,Y),W\right) &= \overline{g}\left(\overline{\nabla}_X Y, W\right)\\
&= X\overline{g}\left(Y,W\right)-\overline{g}\left(Y,\overline{\nabla}_X W\right)\\
&= -\overline{g}\left(Y,[X,W]\right)-\overline{g}\left(Y,\overline{\nabla}_W X\right)\\
&= -\overline{g}\left(Y,[X,W]\right)-W\overline{g}\left(Y,X\right)+\overline{g}\left(\overline{\nabla}_W Y,X\right)\\
&= -\overline{g}\left(Y,[X,W]\right)-W\overline{g}\left(Y,X\right)+\overline{g}\left(X,[W,Y]\right)+\overline{g}\left(\overline{\nabla}_Y W,X\right)\\
&= -\left(L_W\overline{g}\right)(X,Y)+\overline{g}\left(\overline{\nabla}_Y W,X\right)\\
&= -\left(L_W\overline{g}\right)(X,Y)-\overline{g}\left(W,\overline{\nabla}_Y X\right)
\end{aligned}$$

or

$$2\overline{g}\left(h(X,Y),W\right)=-\left(L_W\overline{g}\right)(X,Y) \tag{3.2}$$

for any $W\in\Gamma\left(S(TM^{\perp})\right)$. Thus, from (3.1) and (3.2), the proof is complete.

It is obvious that from the proof of the theorem, the assertion of the theorem is true for any lightlike submanifold of a semi-Riemann manifold.

Lemma 3.2. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then*

$$\overline{g}\left(h\left(X,Y\right),W\right)=\overline{g}\left(A_W X,Y\right)$$

for any $X\in\Gamma(D), Y\in\Gamma(D')$ and $W\in\Gamma\left(S(TM^{\perp})\right)$.

Proof. For any $X\in\Gamma(D), Y\in\Gamma(D')$ and $W\in\Gamma\left(S(TM^{\perp})\right)$ we have

$$\begin{aligned}
\overline{g}\left(h\left(X,Y\right),W\right) &= \overline{g}\left(\overline{\nabla}_X Y,W\right)\\
&= -\overline{g}\left(Y,\overline{\nabla}_X W\right).
\end{aligned}$$

From (2.8) it follows

$$\begin{aligned}
\overline{g}\left(h\left(X,Y\right),W\right) &= -\overline{g}\left(Y,-A_W X+\nabla^s_X W+D^l(X,W)\right)\\
&= \overline{g}\left(Y,A_W X\right).
\end{aligned}$$

Theorem 3.3. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is mixed geodesic if and only if*

$$A^*_{\xi}X\in\Gamma\left(D_0\perp\overline{J}L_1\right)$$

and

$$A_W X\in\Gamma\left(D_0\perp RadTM\perp\overline{J}L_1\right)$$

for any $X\in\Gamma(D)$, $\xi\in\Gamma\left(Rad(TM)\right)$ and $W\in\Gamma\left(S(TM^{\perp})\right)$.

Proof. By the definition of CR-lightlike submanifolds, M is mixed geodesic if and only if

$$\overline{g}\left(h(X,Y),\xi\right)=0$$

and

$$\overline{g}\left(h(X,Y),W\right)=0$$

for any $X\in\Gamma(D)$ and $Y\in\Gamma(D')$ and $W\in\Gamma(S(TM^{\perp}))$. Thus, from (2.6) and (2.10) we get

$$\begin{aligned}\overline{g}\left(h(X,Y),\xi\right) &= \overline{g}\left(h^{l}(X,Y),\xi\right)\\ &= \overline{g}\left(\overline{\nabla}_{X}Y,\xi\right)\\ &= -\overline{g}\left(Y,\overline{\nabla}_{X}\xi\right)\\ &= -\overline{g}\left(Y,\nabla_{X}\xi\right)\end{aligned}$$

or

$$\overline{g}\left(h(X,Y),\xi\right)=\overline{g}\left(Y,A^{*}_{\xi}X\right). \tag{3.3}$$

Thus assertion of theorem follows from (3.3) and Lemma 3.2.

Theorem 3.4. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is D-geodesic if and only if*

$$\overline{g}\left(Y,A_{W}X\right)=\overline{g}\left(D^{l}(X,W),Y\right)$$

and

$$\nabla^{*}_{X}\overline{J}\xi\notin\Gamma\left(D_{0}\perp\overline{J}L_{1}\right),\ \ A^{*}_{\xi}Y\notin\Gamma\left(\overline{J}L_{1}\right)$$

for any $X,Y\in\Gamma\left(D\right)$, $\xi,\ \xi'\in\Gamma\left(Rad(TM)\right)$ and $W\in\Gamma\left(S(TM^{\perp})\right)$.

Proof. By the definition of lightlike submanifolds and Definition 3.2, M is D-geodesic if and only if

$$\overline{g}\left(h^{l}\left(X,Y\right),\xi\right)=0$$

and

$$\overline{g}\left(h^{s}\left(X,Y\right),W\right)=0$$

for any $X,Y\in\Gamma\left(D\right)$, $\xi,\xi'\in\Gamma\left(Rad(TM)\right)$ and $W\in\Gamma\left(S(TM^{\perp})\right)$. Thus we have

$$\begin{aligned}\overline{g}\left(h^{s}\left(X,Y\right),W\right) &= \overline{g}\left(\overline{\nabla}_{X}Y,W\right)\\ &= -\overline{g}\left(Y,\overline{\nabla}_{X}W\right)\\ &= -\overline{g}\left(Y,-A_{W}X+\nabla^{s}_{X}W+D^{l}(X,W)\right)\\ &= -\overline{g}\left(Y,-A_{W}X\right)-\overline{g}\left(Y,D^{l}(X,W)\right)\end{aligned}$$

or

$$\overline{g}\left(h^{s}\left(X,Y\right),W\right)=\overline{g}\left(Y,A_{W}X\right)-\overline{g}\left(Y,D^{l}(X,W)\right). \tag{3.4}$$

In a similar way we get

$$\begin{aligned}\overline{g}\left(h^{l}\left(X,Y\right),\xi\right) &= \overline{g}\left(\overline{\nabla}_{X}Y,\xi\right)\\ &= -\overline{g}\left(\overline{J}Y,\overline{\nabla}_{X}\overline{J}\xi\right)\\ &= -\overline{g}\left(\overline{J}Y,\nabla_{X}\overline{J}\xi+h\left(X,\overline{J}\xi\right)\right)\\ &= -\overline{g}\left(\overline{J}Y,\nabla_{X}\overline{J}\xi\right)-\overline{g}\left(\overline{J}Y,h\left(X,\overline{J}\xi\right)\right)\end{aligned}$$

or from (2.9) we obtain

$$\overline{g}\left(h^l\left(X,Y\right),\xi\right) = -\overline{g}\left(\overline{J}Y, \nabla^*_X \overline{J}\xi\right) - \overline{g}\left(\overline{J}Y, h\left(X, \overline{J}\xi\right)\right). \tag{3.5}$$

Since $Y \in \Gamma(D)$ for the second expression in the right side of equation (3.5), we have $Y \in \Gamma(Rad(TM))$, $Y \in \Gamma\left(\overline{J}Rad(TM)\right)$ or $Y \in \Gamma(D_0)$. If $Y \in \Gamma(Rad(TM))$, we get

$$\overline{g}\left(h^l\left(X,\overline{J}\xi\right),\overline{J}Y\right) = 0$$

and if $Y \in \Gamma(D_0)$ then we obtain

$$\overline{g}\left(h^l\left(X,\overline{J}\xi\right),\overline{J}Y\right) = 0.$$

If $Y \in \Gamma\left(\overline{J}Rad(TM)\right)$ then we put $Y = \overline{J}\xi'$. Hence we derive

$$-\overline{g}\left(h^l\left(X,\overline{J}\xi\right),\xi'\right) = -\overline{g}\left(A^*_{\xi'}X, \overline{J}\xi\right). \tag{3.6}$$

Thus, from (3.5) follows

$$\overline{g}\left(h^l\left(X,Y\right),\xi\right) = -\overline{g}\left(Y, \nabla^*_X \overline{J}\xi\right) - \overline{g}\left(A^*_{\xi'}X, \overline{J}\xi\right).$$

Hence if $\nabla^*_X \overline{J}\xi \notin \Gamma(D_0 \perp M_1)$ and $A^*_\xi X \notin \Gamma(M_1)$ we get $h^l(X,Y) = 0$.

Conversely, if $h^l(X,Y) = 0$ then, for any $Y \in \Gamma(\overline{J}RadTM)$, since

$$g(h^l(X,Y),\xi) = g(A^*_\xi X, Y) = 0,$$

we have $A^*_\xi X \notin \Gamma(M_1)$ and for any $Y \in \Gamma(D_0 \perp RadTM)$, we obtain

$$\begin{aligned} \overline{g}(h^l(X,Y),\xi) &= \overline{g}(\overline{\nabla}_X Y, \xi) \\ &= \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}\xi) \\ &= -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}\xi) \\ &= -\overline{g}(\overline{J}Y, \nabla_X \overline{J}\xi) \\ &= -\overline{g}(\overline{J}Y, \nabla^*_X \overline{J}\xi). \end{aligned}$$

Since M is D-geodesic, we derive $\nabla^*_X \overline{J}\xi \notin \Gamma(D_0 \perp M_1)$.

Theorem 3.5. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is D'-geodesic if and only if $A_W Z$ and $A^*_\xi Z$ have no components in $M_2 \perp \overline{J}RadTM$ for any $Z \in \Gamma(D')$, $\xi \in \Gamma(RadTM)$ and $W \in \Gamma\left(S(TM^\perp)\right)$.*

Proof. From (2.6) we have

$$\begin{aligned} \overline{g}\left(h(Z,V),W\right) &= \overline{g}\left(\overline{\nabla}_Z V, W\right) \\ &= -\overline{g}\left(V, \overline{\nabla}_Z W\right) \end{aligned}$$

for any $Z, V \in \Gamma(D')$, or

$$\overline{g}(h(Z,V),W) = \overline{g}(A_W Z, V). \tag{3.7}$$

On the other hand we get

$$\begin{aligned}\overline{g}(h(Z,V),\xi) &= \overline{g}\left(\overline{\nabla}_Z V, \xi\right)\\ &= -\overline{g}\left(V, \overline{\nabla}_Z \xi\right)\end{aligned}$$

or

$$\overline{g}(h(Z,V),\xi) = \overline{g}\left(A^*_\xi Z, V\right). \tag{3.8}$$

Then our assertion follows from (3.7) and (3.8).

Corollary 3.6. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is D'-geodesic if and only if*

i) *$A_W Z$ has no component in $M_2 \perp \overline{J}RadTM$,*

ii) *$A_{\overline{J}V} Z$ has no component in M_1, for any $Z, V \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$.*

Proof. From (2.6)

$$\begin{aligned}\overline{g}(h(Z,V),\xi) &= \overline{g}\left(\overline{\nabla}_Z V, \xi\right)\\ &= \overline{g}\left(\overline{J}\overline{\nabla}_Z V, \overline{J}\xi\right)\\ &= \overline{g}\left(\overline{\nabla}_Z \overline{J}V, \overline{J}\xi\right)\\ &= -\overline{g}\left(A_{\overline{J}V} Z, \overline{J}\xi\right)\end{aligned}$$

for any $Z, V \in \Gamma(D')$. Then our assertion follows from Theorem 3.4.

Lemma 3.7. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If the distribution D is integrable then the following assertions hold:*

i) $\overline{g}\left(D^l\left(\overline{J}X, W\right), Y\right) = \overline{g}\left(D^l(X,W), \overline{J}Y\right) \Longleftrightarrow \overline{g}\left(A_W \overline{J}X, Y\right) = \overline{g}\left(A_W X, \overline{J}Y\right)$,

ii) $\overline{g}\left(D^l\left(\overline{J}X, W\right), \xi\right) = \overline{g}\left(A_W X, \overline{J}\xi\right)$,

iii) $\overline{g}\left(D^l(X,W), \xi\right) = \overline{g}\left(A_W \overline{J}X, \overline{J}\xi\right)$.

Proof. From (2.8) we have

$$\begin{aligned}\overline{g}\left(D^l\left(\overline{J}X, W\right), Y\right) &= \overline{g}\left(\overline{\nabla}_{\overline{J}X} W - \nabla^s_{\overline{J}X} W + A_W \overline{J}X, Y\right)\\ &= \overline{g}\left(\overline{\nabla}_{\overline{J}X} W + A_W \overline{J}X, Y\right)\\ &= -\overline{g}\left(W, \overline{\nabla}_{\overline{J}X} Y\right) + \overline{g}\left(A_W \overline{J}X, Y\right)\\ &= -\overline{g}\left(W, \nabla_{\overline{J}X} Y + h\left(\overline{J}X, Y\right)\right) + \overline{g}\left(A_W \overline{J}X, Y\right)\\ &= -\overline{g}\left(W, h\left(\overline{J}X, Y\right)\right) + \overline{g}\left(A_W \overline{J}X, Y\right).\end{aligned}$$

Then, taking in account that D is integrable, we obtain

$$\begin{aligned}
\overline{g}\left(D^l\left(\overline{J}X,W\right),Y\right) &= -\overline{g}\left(W,h\left(\overline{J}X,Y\right)\right)+\overline{g}\left(A_W\overline{J}X,Y\right)\\
&= -\overline{g}\left(W,h\left(X,\overline{J}Y\right)\right)+\overline{g}\left(A_W\overline{J}X,Y\right)\\
&= -\overline{g}\left(W,\overline{\nabla}_X\overline{J}Y\right)+\overline{g}\left(A_W\overline{J}X,Y\right)\\
&= \overline{g}\left(\overline{\nabla}_XW,\overline{J}Y\right)+\overline{g}\left(A_W\overline{J}X,Y\right)\\
&= -\overline{g}\left(A_WX,\overline{J}Y\right)+\overline{g}\left(D^l\left(X,W\right),\overline{J}Y\right)+\overline{g}\left(A_W\overline{J}X,Y\right)
\end{aligned}$$

or

$$\overline{g}\left(D^l\left(\overline{J}X,W\right),Y\right)-\overline{g}\left(D^l\left(X,W\right),\overline{J}Y\right)=\overline{g}\left(A_W\overline{J}X,Y\right)-\overline{g}\left(A_WX,\overline{J}Y\right).$$

This is proof of (i). Substituting $Y=\xi,\ Y=\overline{J}\xi$ in (i) we obtain (ii) and (iii).

Lemma 3.8. *Let M be a mixed geodesic CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then we have*

$$A^*_\xi X\in\Gamma\left(\overline{J}Rad(TM)\right)$$

for any $X\in\Gamma\left(D'\right)$.

Proof. Using the Kaehler character of $\overline{M}$ and (2.6),

$$\begin{aligned}
h\left(\overline{J}\xi,X\right) &= \overline{\nabla}_X\overline{J}\xi-\nabla_X\overline{J}\xi\\
&= \overline{J}\,\overline{\nabla}_X\xi-\nabla_X\overline{J}\xi\\
&= \overline{J}\nabla_X\xi+\overline{J}h(X,\xi)-\nabla_X\overline{J}\xi
\end{aligned}$$

for any $X\in\Gamma(D')$, $Y\in\Gamma(D')$. Since M is mixed geodesic, we have

$$\overline{J}\nabla_X\xi=\nabla_X\overline{J}\xi.$$

From (2.9) and (2.10) we derive

$$-\overline{J}A^*_\xi X+\overline{J}{\nabla^*_X}^t\xi=\nabla^*_X\overline{J}\xi+h^*(X,\overline{J}\xi)$$

or from (2.13)

$$-fA^*_\xi X-\omega A^*_\xi X+\overline{J}{\nabla^*_X}^t\xi=\nabla^*_X\overline{J}\xi+h^*(X,\overline{J}\xi).$$

Thus

$$\omega A^*_\xi X=0$$

or

$$A^*_\xi X\in\Gamma\left(\overline{J}Rad(TM)\perp D_0\right).$$

If $A^*_\xi X\in\Gamma\left(D_0\right)$ then since D_0 is nondegenerate, it must be

$$\overline{g}\left(A^*_\xi X,Z_0\right)\neq 0.$$

But from (2.6) and (2.10) we get

$$\begin{aligned} \overline{g}\left(A^*_\xi X, Z_0\right) &= \overline{g}\left(-\nabla_X \xi + {\nabla^*_X}^t \xi, Z_0\right) \\ &= \overline{g}\left(-\nabla_X \xi, Z_0\right) \\ &= \overline{g}\left(-\overline{\nabla}_X \xi, Z_0\right) \\ &= \overline{g}\left(\xi, \overline{\nabla}_X Z_0\right) \\ &= \overline{g}\left(\xi, \nabla_X Z_0 + h(X, Z_0)\right) \\ &= 0. \end{aligned}$$

Thus $A^*_\xi X \notin \Gamma(D_0)$.

From (2.10) and Lemma 3.2, we have the following lemma.

Lemma 3.9. *Let M be a mixed geodesic CR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then*

$$\overline{g}\left(h^l(X,Y), \xi\right) = 0$$

for any $X \in \Gamma(D'), Y \in \Gamma(M_2)$ and $\xi \in \Gamma(Rad(TM))$.

By the definition of CR-lightlike submanifolds and from (2.11), (2.12) we have the following corollaries.

Corollary 3.10. *Let $\overline{M}$ be an indefinite almost complex manifold and M be a mixed CR-lightlike submanifold of $\overline{M}$. Then*

$$A^*_\xi X \in \Gamma\left(\overline{J}Rad(TM) \perp M_2\right)$$

for any $X \in \Gamma(D')$.

Corollary 3.11. *Let $\overline{M}$ be an indefinite almost complex manifold and M be a mixed CR-lightlike submanifold of $\overline{M}$. Then*

$$A^*_\xi X \in \Gamma(D_0 \perp M_1)$$

for any $X \in \Gamma(D)$.

Corollary 3.12. *Let $\overline{M}$ be an indefinite almost complex manifold and M be a CR-lightlike submanifold of $\overline{M}$. If $h^*(X,Y) = 0$ then we have:*

a) *$A_N X$ has no component in $\overline{J}Rad(TM) \perp M_2$*

b) *$A_N Y$ has no component $D_0 \perp M_1$*

for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$.

Corollary 3.13. *Let $\overline{M}$ be an indefinite almost complex manifold and M be a mixed CR-lightlike submanifold of $\overline{M}$. Then:*

a) *$A_W X$ has no component in $\overline{J}RadTM \perp M_2$*

b) *$A_W Y$ has no component in $D_0 \perp M_1$*

for any $X \in \Gamma(D), Y \in \Gamma(D')$.

Lemma 3.14. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then*

$$\bar{g}\left(A_W \overline{J}X, Y\right) = \bar{g}\left(A_W Y, \overline{J}X\right) - \bar{g}\left(\overline{J}X, D^l(Y, W)\right)$$

for any $X \in \Gamma(D), Y \in \Gamma(D')$ and $W \in \Gamma\left(S(TM^\perp)\right)$.

Proof. From (2.8),

$$\begin{aligned}
\bar{g}\left(A_W \overline{J}X, Y\right) &= \bar{g}\left(h\left(\overline{J}X, Y\right), W\right) \\
&= \bar{g}\left(\overline{\nabla}_Y \overline{J}X, W\right) \\
&= -\bar{g}\left(\overline{J}X, \overline{\nabla}_Y W\right) \\
&= \bar{g}\left(\overline{J}X, A_W Y\right) - \bar{g}\left(D^l(Y, W), \overline{J}X\right)
\end{aligned}$$

for $X \in \Gamma(D), Y \in \Gamma(D')$ and $W \in \Gamma\left(S(TM^\perp)\right)$.

Theorem 3.15. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a mixed geodesic CR-lightlike submanifold of $\overline{M}$. Then*

$$A_V X \in \Gamma(D)$$

for any $X \in \Gamma(D)$, $V \in \Gamma(L_1 \perp L_2)$.

Proof. Since M is mixed geodesic, $h(X, Y) = 0$ for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$. From (2.6) we have

$$0 = \overline{\nabla}_X Y - \nabla_X Y$$

Since D' is anti-invariant there exists $V \in \Gamma(L_1 \perp L_2)$ such that $\overline{J}V = Y$. Thus, from (2.8), (2.13) and (2.14) we get

$$\begin{aligned}
0 &= \overline{\nabla}_X \overline{J}V - \nabla_X Y \\
&= \overline{J}\,\overline{\nabla}_X V - \nabla_X Y \\
&= \overline{J}(-A_V X + \nabla^t_X V) - \nabla_X Y \\
&= -\overline{J}A_V X + \overline{J}\nabla^t_X V - \nabla_X Y \\
&= -f A_V X - \omega A_V X + B\nabla^t_X V + C\nabla^t_X V - \nabla_X Y.
\end{aligned}$$

Hence

$$\omega A_V X = C\nabla^t_X V$$

or

$$A_V X \in \Gamma(D).$$

Corollary 3.16. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a mixed geodesic CR-lightlike submanifold of $\overline{M}$. Then M is a mixed geodesic CR-lightlike submanifold if and only if*

$$A_V X \in \Gamma(D)$$

and $L_1 \perp L_2$ is parallel with respect to D (that is, ${\nabla^t}_X V \in \Gamma(L_1 \perp L_2)$ for any $V \in \Gamma(L_1 \perp L_2)$ and $X \in \Gamma(D)$).

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