Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 43 (2002), No. 1, 121-133.

On the Chiral Archimedean Solids

Dedicated to Prof. Dr. O. Krötenheerdt

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Abstract. We discuss a unified way to derive the convex semiregular polyhedra from the Platonic solids. Based on this we prove that, among the Archimedean solids, Cubus simus (i.e., the snub cube) and Dodecaedron simum (the snub dodecahedron) can be characterized by the following property: it is impossible to construct an edge from the given diameter of the circumsphere by ruler and compass.

Keywords: Archimedean solids, enantiomorphism, Platonic solids, regular polyhedra, ruler-and-compass constructions, semiregular polyhedra, snub cube, snub dodecahedron

1. Introduction

Following Pappus of Alexandria, Archimedes was the first person who described those 13 *semiregular convex polyhedra* which are named after him: *the Archimedean solids*. Two representatives from this family have various remarkable properties, and therefore the Swiss pedagogicians A. Wyss and P. Adam called them "Sonderlinge" (eccentrics), cf. [25] and [1]. Other usual names are *Cubus simus* and *Dodecaedron simum*, due to J. Kepler [17], or *snub cube* and *snub dodecahedron*, respectively.

Immediately one can see the following property (characterizing these two polyhedra among all Archimedean solids): their symmetry group contains only proper motions. Therefore these two polyhedra are the only Archimedean solids having no plane of symmetry and having no center of symmetry. So each of them occurs in two *chiral* (or *enantiomorphic*) forms, both having different orientation.

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It is our aim to emphasize another interesting difference between these two polyhedra and the other 11 Archimedean solids. This difference (described below) is not so obvious, and in the pertinent literature it was sometimes only mentioned, without proof or deeper discussions, see, e.g., [20].

The *regular convex polyhedra* (or *Platonic solids*) and the Archimedean solids share an elementary property: all their vertices lie on a sphere, their *circumsphere*. In his dialogue "Timaeos", Plato emphasized this property (among the regular solids) only for the tetrahedron, which (also later) was called by him "pyramid". On the other hand, the existence of this circumsphere was emphasized by Euclid in Book XIII of his "Elements" as a common property of all five Platonic solids. In this book Euclid discusses the construction of these polyhedra, and it is very probable that Book XIII and also Book X go back to a scription of the mathematician Theaetetos (-415? to -369), who was a friend of Plato, see [18], pp. 32–35. To show the way of Euclid's approach, we give a typical citation (cf. C. Thaer [23]):

> "To erect a cube and to circumscribe to it, like to the pyramid, a ball; further on to show that the squared diameter of the ball yields three times the squared edge of the cube..."

So the main aim is not to construct the cube as a polyhedron bounded by six "equal" squares, but (as also confirmed by the continuation of the text) to construct an edge of the cube from the prescribed diameter of the circumsphere. And it is clear that Euclid means a construction by ruler and compass. Furthermore, the *ratio* of the diameter d of the circumsphere and the edge-length e occurs here, where later on mostly the relation $d^2 : e^2$ was studied. In modern notation, for the Platonic solids these relations can be described by the following table.

Name	symbol	$d^2 : e^2$	$e : d$
tetrahedron	$\{3,3\}$	$\frac{3}{2}$	$0,816477$
cube	$\{4,3\}$	3	$0,577350$
octahedron	$\{3,4\}$	2	$0,707107$
dodecahedron	$\{5,3\}$	$\frac{3}{2}\left(3+\sqrt{5}\right)$	$0,356822$
icosahedron	$\{3,5\}$	$\frac{1}{2}\left(5+\sqrt{5}\right)$	$0,525731$

The so-called *Schläfli-symbol* $\{p,q\}$ expresses that q regular p-gons meet at each vertex, and the numerical ratio $e : d$ is needed for building models of correspondingly equal size.

2. The Archimedean solids

As for the Platonic solids, one can pose the analogous task for the Archimedean ones: given the diameter d (as a line segment), one has to construct the edge-length e. And exactly with respect to this task, Cubus simus and Dodecaedron simum are exceptional cases in this family of semiregular polyhedra. Namely, we have the following

Theorem. *Among the Archimedean solids, Cubus simus and Dodecaedron simum are characterized by the following property: It is impossible to construct an edge from the given diameter of the circumsphere by ruler and compass.*

In the following we will give a complete proof of that theorem.

In the literature one can find various symbols and notations which are used for describing the Archimedean solids. We will follow L. Fejes Tóth [7] and describe them by the symbol $(n_1, \ldots, n_k)$, $3 \leq k \leq 5$. This means that exactly in this cyclic order (but up to orientation) a regular n_1-gon, n_2-gon, ..., and an n_k-gon meet at *every* vertex. One can use this symbol also for the Platonic solids, with the additional advantage that exchanges (which are possible in the case of Schläfli symbols) are excluded. Namely, we get $\{3,3\} = (3,3,3), \{3,4\} = (3,3,3,3), \{3,5\} = (3,3,3,3,3), \{4,3\} = (4,4,4)$, and $\{5,3\} = (5,5,5)$. (Of course, in this notation the hint is lost that these polyhedra occur in pairs of duals.) In such a way Cubus simus and Dodecaedron simum are denoted by $(3,3,3,3,4)$ and $(3,3,3,3,5)$, respectively.

Most of the names used for Archimedean solids go back to J. Kepler. Except for some cases, they are not usefully chosen. Namely, they mostly describe how the respective polyhedron can be derived from the Platonic solid $\{p, q\}$. But in this case it can also be suitably derived from $\{q, p\}$. Thus, by Kepler's notation one can see only in an incomplete manner that the Archimedean solids can be ordered in three sequences: the polyhedra that can be derived from $\{3,3\}$, those coming from $\{4,3\}$ and $\{3,4\}$, respectively (the "silvern sequence"), and those derived from $\{5,3\}$ and $\{3,5\}$, respectively (the "golden sequence").

Altogether, there are six such procedures to derive Archimedean solids from regular ones. The most impressive representation of these ways is given by the anaglyphes in the book [7] of L. Fejes Tóth.

On the one hand, these procedures can be interpreted as truncations of the Platonic solids. On the other hand, one can suitably enclose a regular p-gon or $2p$-gon into each facet of $\{p, q\}$ and then obtain the Archimedean solid as convex hull of these polygons. For the first five derviations, Fig. 1 shows the positions of the corresponding p-gons and $2p$-gons enclosed in the facets of $\{3, q\}, q \in \{3,4,5\}$, and $\{4,3\}$. In the case of $\{5,3\}$ the approach is analogous.

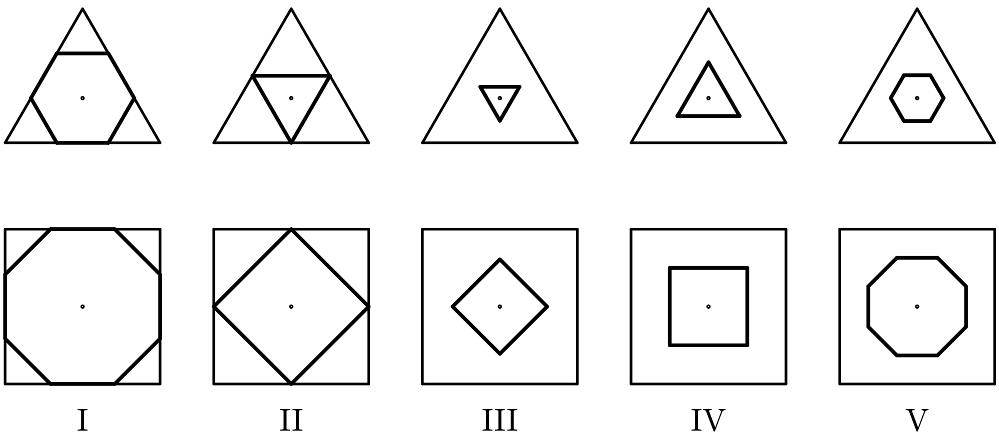

Figure 1

All the semiregular polyhedra obtained by the approaches I, II, and III should have the name "truncated $\{p, q\}$", since only vertices of regular polyhedra are truncated. (Due to Kepler, only the polyhedra obtained by the first approach have such names.) For the approaches IV and V, also edges have to be suitably truncated.

In the following table one can see which approach yields the Archimedean solid $(n_1, n_2, \ldots, n_k)$

as a truncation of $\{p, q\}$. In addition, the exact values for $d^2 : e^2$ and rounded off values for $e : d$ are given.

The application of way II to the regular tetrahedron $\{3, 3\}$ yields the Platonic octahedron $\{3, 4\}$. To obtain disjoint sequences of truncations, in the literature very often only $(3, 6, 6)$ is discussed as a truncation of $\{3, 3\}$ (see, e.g., [21]). Going this way, one ignores the following fact:

For an Archimedean solid, the ratio $d^2 : e^2$ is rational if and only if this polyhedron can be obtained from $\{3, 3\}$ by one of the basic constructions described above.

	$\{3,3\}$	$\{4,3\}$	$\{3,4\}$	$\{5,3\}$	$\{3,5\}$	$d^2 : e^2$	$e : d$
$(3,6,6)$	I,III					$\frac{11}{2}$	$0,426402$
$(4,6,6)$	V	III	I			10	$0,316228$
$(3,8,8)$		I	III			$7+4\sqrt{2}$	$0,281085$
$(4,6,8)$		V	V			$13+6\sqrt{2}$	$0,215740$
$(3,4,4,4)$		IV	IV			$5+2\sqrt{2}$	$0,357407$
$(3,4,3,4)$	IV	II	II			4	$0,500000$
$(5,6,6)$				III	I	$\frac{29}{2}+\frac{9}{2}\sqrt{5}$	$0,201774$
$(3,10,10)$				I	III	$\frac{37}{2}+\frac{5}{2}\sqrt{5}$	$0,168382$
$(4,6,10)$				V	V	$31+12\sqrt{5}$	$0,131496$
$(3,4,5,4)$				IV	IV	$11+4\sqrt{5}$	$0,223919$
$(3,5,3,5)$				II	II	$6+2\sqrt{5}$	$0,309017$

For the "silvern sequence", $d^2 : e^2$ belongs to $\mathbb{Q}(\sqrt{2})$, for the "golden sequence" to $\mathbb{Q}(\sqrt{5})$. Therefore each of the 11 Archimedean solids discussed up to now allows a ruler-and-compass construction of its edge from the given diameter.

As already mentioned, there are two additional Archimedean solids not having this property and, furthermore, also having no plane or center of symmetry. In a series of papers, F. Hohenberg investigated these polyhedra with respect to metrical and projective properties, see [8]–[14] and, for related results, [19] and [22].

3. On Cubus simus

The sixth method to construct an Archimedean solid yields the so-called *snub polyhedra*. Again, regular p-gons are situated in the facets of $\{p, q\}$, but now these p-gons are *rotated* in a special way (cf. Fig. 2).

In such a manner, the *Platonic icosahedron* is derived from $\{3, 3\}, \{4, 3\}$ as well as $\{3, 4\}$ yield *Cubus simus*, and $\{5, 3\}$ as well as $\{3, 5\}$ yield *Dodecaedron simum*. In the literature one can find relatively exact values for $d : e$ (or values coinciding with it up to a factor), see [24], or more rough estimates of $d : e$ are given, cf. [4]. But one cannot find explanations how

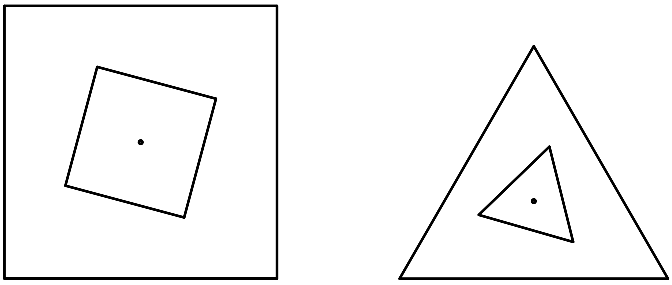

Figure 2

these estimates are obtained. This shows that a mathematical problem, clearly formulated in Euclid's Book XIII, was no longer in the center of interests.

The easiest approach to the (geometrical) properties of Cubus simus is obtained if this polyhedron is generated from a cube W. All the relations, that we derive first, are well-known. For example, one can find them collected in the paper [9] of F. Hohenberg, and they are shortly mentioned by H. S. M. Coxeter [5], too. Since only the *ratio* $d : e$ is considered, the size of the cube W is unimportant. In accordance with the mentioned authors, we choose $W = \{x(\xi_1, \xi_2, \xi_3) : |\xi_i| \leq 1,\ i = 1, 2, 3\}$. In Fig. 3, some of the squares enclosed in the facets of W are shown, and only those vertices are marked which are used in the following.

The determination of x_1 also yields the position of the respective square with midpoint $e_1(1, 0, 0)$ in the plane $\xi_1 = 0$. The corresponding squares in the other facets of W are derived from this one by suitable rotations around the axes through the midpoints of opposite facets of W. So we have

$$x_1(1, v, w),\ \ x_2(1, -w, v),\ \ x_3(v, w, 1)\,,$$

where $1 > v > w$ can be assumed (this determines the "twist").

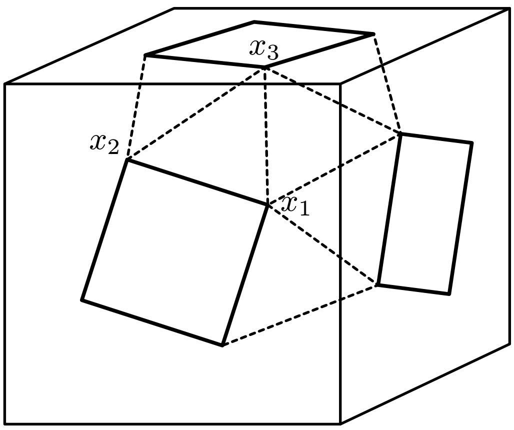

Figure 3

Since the convex hull of these six squares is a Cubus simus of edge length e, the conditions

$$\begin{aligned} e^2 &= ||x_1 - x_2||^2 = (v + w)^2 + (w - v)^2 = 2(v^2 + w^2)\,, \\ e^2 &= ||x_1 - x_3||^2 = (1 - v)^2 + (v - w)^2 + (w - 1)^2 \\ &= 2(v^2 + w^2 - vw - v - w + 1)\,, \end{aligned}$$

$$\begin{aligned} e^2 &= ||x_2 - x_3||^2 = (1-v)^2 + 4w^2 + (v-1)^2 \\ &= 2(v^2 + w^2 + w^2 - 2v + 1) \end{aligned}$$

have to be satisfied. Subtraction yields the equations

$$vw + v + w - 1 = 0 \quad \text{or} \quad v(1+w) = (1-w)$$

and

$$vw + w^2 + w - v = 0 \quad \text{or} \quad v(1-w) = w(1+w)\,.$$

By multiplication we obtain $v^2(1-w^2) = w(1-w^2)$, where $w \neq 1$ gives $v^2 = w$. Furthermore, $vw + v + w - 1 = 0$ and $e^2 = 2(v^2 + w^2)$ yield

$$\begin{aligned} &v^3 + v^2 + v - 1 = 0\,, \\ &e^2 = 2(v^4 + v^2)\,. \end{aligned}$$

Writing the cubic equation for v in the form $v^3 = -v^2 - v + 1$, one can see that by this relation each polynomial in v can be transformed into a polynomial of at most second degree. For example, one has $v^4 = v(-v^2 - v + 1) = -v^3 - v^2 + v = v^2 + v - 1 - v^2 + v = 2v - 1$, i.e.,

$$e^2 = 2(v^2 + 2v - 1)$$

and

$$e^4 = 8(-v^2 - 3v + 2)\,, \quad e^6 = 32(v^2 + 5v - 3)\,.$$

From the latter three relations and the equation $e^6 + a_2 e^4 + a_1 e^2 + a_0 = 0$ one gets

$$e^6 + 12e^4 + 32e^2 - 32 = 0\,,$$

which was already presented by F. Hohenberg [9] (without proof). From this it is easy to obtain a cubic equation for $z = d^2 : e^2$. If r is the radius of the circumsphere of the Cubus simus which is inscribed to W, then

$$r^2 = 1 + \frac{1}{2}e^2\,,$$

i.e., one obtains

$$z = \frac{d^2}{e^2} = \frac{4r^2}{e^2} = \frac{4}{e^2} + 2 \quad \text{or} \quad \frac{1}{e^2} = \frac{z-2}{4}\,.$$

Thus it is easy to get for z the equation

$$z^3 - 10z^2 + 22z - 14 = 0\,,$$

having only one real solution. The value z can be estimated by

$$7{,}22226252 < z < 7{,}22226253\,,$$

i.e., the table from above can be completed for $(3, 3, 3, 3, 4)$ by:

$$VI, \quad d^2 : e^2 = 7{,}22226252\ldots, \quad e : d = 0{,}372103\,.$$

(Remark: In [24] one can find for $d : e$ the relatively exact value $2,6874267475$, whereas the calculations of M. Brückner, cf. [4], p. 139, yield the value $2,58922$, in which only the integer part is correct.)

With the help of the equation derived for z one can easily show that an edge of Cubus simus cannot be constructed by ruler and compass from the given diameter. The polynomial $p(z) := z^3 - 10z^2 + 22z - 14$ has integer coefficients. If such a polynomial over the field $\mathbb{Q}$ of rational numbers is reducible, then (due to a Lemma of C. F. Gauß) it can be decomposed into a product of two polynomials each of which has also integer coefficients. In our case a decomposition of the form

$$p(z) = (z - a)(z^2 + b_1 z + b_0)\,, \quad \{a, b_1, b_0\} \in \mathbb{Z}\,,$$

should exist, with an integer root a. But as already mentioned, $p(z)$ has only a real root, which is not an integer. Therefore $p(z)$ is irreducible over $\mathbb{Q}$. The roots of an irreducible polynomial of third degree with coefficients from $\mathbb{Q}$ cannot belong to a quadratic field extension over $\mathbb{Q}$, i.e., they are not constructible. A particularly simple proof of this statement (which is a subcase of a more general theorem) is due to E. Landau. Regarding the bibliographic sources we refer to [2]; a representation of the proof is given in [3].

4. The construction from the regular polyhedra $\{3, q\}$

Starting from the Platonic dodecahedron, it is not so easy to derive the properties of Dodecaedron simum that are interesting for our considerations. In general it is possible to derive such properties of $(3, 3, 3, 3, q)$ from $\{3, q\}$ ($q \in \{3, 4, 5\}$), but for $q = 5$ this is even the most effective way. The corresponding approach for $q = 4$ was given above since it is descriptive, can be found at several places in the literature, and shows how to get an equation satisfied by the edge-length e.

The common construction of the polyhedra $(3, 3, 3, 3, q)$ from $\{3, q\}$ was described in a detailed manner by P. Huybers and H. S. M. Coxeter (cf. [16]). In the following we will use indeed the basic idea from [16], but our approach is a strongly modified version of the way presented there.

We start with a polyhedron $\{3, q\}$ which is circumscribed about a sphere of radius 1 and whose center is chosen as the origin of a Cartesian coordinate system. The vertices of $\{3, q\}$ be denoted by y_i, and only two neighbouring facets should have the vertices y_1, y_2, y_3 and y_2, y_3, y_4, respectively. The radius ρ of the circumsphere satisfies $||y_i||^2 = \rho^2$, and we set $\langle y_2, y_3\rangle =: \delta_0$ and $\langle y_1, y_4\rangle =: \delta_1$. It is easy to get the following well-known relations.

q	ρ^2	δ_0	δ_1
3	9	-3	-3
4	3	0	-3
5	$3(5 - 2\sqrt{5})$	$3(-2 + \sqrt{5})$	$3(2 - \sqrt{5})$

(To obtain the given values for $q = 5$, one uses the fact that the vertices of the icosahedron occur in quadruples each of which is lying in a plane, where these three planes are perpendicular to each other. Due to $\tau := \frac{1}{2}(1+\sqrt{5})$ one has with $y_2 = \left(\frac{e}{2}, 0, \tau\frac{e}{2}\right)$, $y_3 = \left(-\frac{e}{2}, 0, \tau\frac{e}{2}\right)$, $y_1 =$

$\left(0, \tau\frac{e}{2}, \frac{e}{2}\right)$, and $y_4 = \left(0, -\tau\frac{e}{2}, \frac{e}{2}\right)$ four suitable vertices of an icosahedron with edge-length e. To get an icosahedron with insphere of radius 1, one has to choose $e = \sqrt{3}(3 - \sqrt{5})$, cf. [15], p. 40 in Part I and p. 116 in Part II.)

Hence the construction of $(3, 3, 3, 3, q)$ from $\{3, q\}$ is relatively easy. This is based on a projectively invariant relation. Namely, the lines carrying the sides of the triangles embedded into the facets of $\{3, q\}$ also contain a vertex of $\{3, q\}$. This was already noticed by Cauchy when he constructed the icosahedron from the tetrahedron. Regarding the proof of this property and respective historical remarks we refer to [16].

The triangles $y_1y_2y_3$ and $y_2y_3y_4$ contained in the boundary of $\{3, q\}$ can be suitably rotated around the line through y_2 and y_3 such that the planar configuration depicted below is obtained.

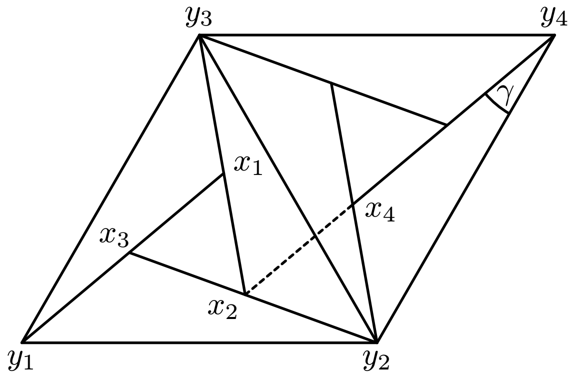

Figure 4

Therefore the position and size of the triangle $x_1x_2x_3$, which lies in the plane spanned by y_1, y_2, and y_3, only depend on one parameter. In [16], the size of the angle γ is chosen as this parameter. With $\gamma' = \frac{\pi}{3} - \gamma$ one gets the barycentric coordinates of the points x_i with respect to the equilateral triangle $y_1y_2y_3$ by cyclic exchange from $\frac{4}{3}(\sin^2\gamma', \sin\gamma' \sin\gamma, \sin^2\gamma)$. (In [16] the factor $\frac{4}{3}$ is not printed.)

Not depending on the angle γ (and on the corresponding unnecessary metrical restriction) we set affinely invariant

$$\begin{aligned} x_1 &= s(y_1 + ty_2 + t^2y_3)\,, \\ x_2 &= s(ty_1 + t^2y_2 + y_3)\,, \\ x_3 &= s(t^2y_1 + y_2 + ty_3) \end{aligned}$$

with $s = \frac{1}{1+t+t^2}$ and $t > 0$, $t \neq 1$. Then the points x_1, x_2, x_3 form an equilateral triangle, and they surely lie in the interior of $y_1y_2y_3$. One has for instance

$$x_2 - tx_1 = \frac{1}{1+t+t^2}(1-t^3)y_3 = (1-t)y_3\,,$$

i.e., y_3 belongs (as demanded) to the line trough x_1 and x_3. Analogously one can verify the other collinearities.

With respect to the triangle $y_3y_2y_4$, the point x_4 has the same position as x_2 regarding $y_1y_2y_3$. Hence

$$x_4 = s(ty_3 + t^2y_2 + y_4)\,.$$

Now only the condition $||x_1 - x_2|| = ||x_2 - x_4||$ has to be satisfied by suitable choice of t (i.e., since all the points x_i have equal norm, it suffices to ensure the condition $\langle x_1, x_2\rangle = \langle x_2, x_4\rangle$). Due to $t \neq 0$, this yields the cubic equation

$$t^3 - t^2 - t + \frac{\delta_1 - \delta_0}{\rho^2 - \delta_0} = 0\,.$$

For each $q \in \{3, 4, 5\}$ this equation has exactly one positive solution. (This holds since the "twist" was determined.) To determine only the vertex coordinates of $\{3, 3, 3, 3, q\}$, we would be ready. To get an equation yielding $z := \frac{d^2}{e^2}$, our way is analogous to that from the former section. Since the centroid c of the equilateral triangle $x_1x_2x_3$ has norm 1, we obtain

$$r^2 = 1 + \frac{1}{3}e^2 \quad \text{or} \quad \frac{1}{e^2} = \frac{1}{4}\left(\frac{4r^2}{e^2} - \frac{4}{3}\right) = \frac{1}{4}\left(z - \frac{4}{3}\right)\,.$$

Therefore we try to get an equation for e^2 (or for another quantity which is simply connected with e^2).

Fortunately, for $e^2 = ||x_1 - x_2||^2$ the convenient representation

$$e^2 = 2(\rho^2 - \delta_0)s(t-1)^2$$

is obtained. Therefore

$$\frac{2(\rho^2 - \delta_0)}{e^2} =: u = \frac{t^2 + t + 1}{(t-1)^2} = 3\frac{t^2}{(t-1)^2} - 3\frac{t}{t-1} + 1\,.$$

Setting $\frac{t}{t-1} =: \bar{t}$ $(\notin [0,1])$, we get from the cubic equation in t the relation

$$\bar{t}^3 = 3\bar{t}^2 - (1 + 2\alpha)\bar{t} + \alpha\,,$$

where $\alpha := \frac{\delta_0 - \delta_1}{\rho^2 - \delta_1}$. From this

$$\bar{t}^4 = (8 - 2\alpha)\bar{t}^2 - (3 + 5\alpha)\bar{t} + 3\alpha$$

is derived. In this relation all powers of u are representable as polynomials in $\bar{t}$ of at most second degree. We obtain

$$\begin{aligned} u &= 3\bar{t}^2 - 3\bar{t} + 1 \\ u^2 &= (33 - 18\alpha)\bar{t}^2 - (15 + 9\alpha)\bar{t} + 1 + 9\alpha \\ u^3 &= (441 - 513\alpha + 108\alpha^2)\bar{t}^2 - (171 + 108\alpha - 216\alpha^2)\bar{t} \\ &\quad +(1 + 162\alpha - 135\alpha^2)\,. \end{aligned}$$

Now for all $\bar{t}$ the relation

$$u^3 + a_2(\alpha)u^2 + a_1(\alpha)u + a_0(\alpha) = 0$$

has to be satisfied, and this yields

$$u^3 + 3(4\alpha - 5)u^2 + 3(4\alpha - 3)(3\alpha - 2)u - (3\alpha - 2)^2 = 0\,.$$

From now on, we consider the two interesting cases $q = 4$ and $q = 5$ separately. For $q = 4$, the relations $u = \frac{6}{e^2}$ and $\alpha = \frac{1}{2}$ are obtained, and the above equation gets the form (in e)

$$e^6 - 36e^4 + 1296e^2 - 864 = 0\,.$$

Obviously, this is not the same equation as in the former section. Namely, there the 4-gonal facets of $\{3,3,3,3,4\}$ had distance 1 from the midpoint of the polyhedron, whereas here the triangular facets have this property. With the equation $\frac{1}{e^2} = \frac{1}{4}(z - \frac{4}{3})$ the relation above is transformed into the cubic equation $z^3 - 10z^2 + 22z - 14 = 0$.

In the remaining case $q = 5$ we get $\alpha = \frac{2}{1+\sqrt{5}} =: \frac{1}{\tau}$ and $u = \frac{1}{e^2} \cdot \frac{12}{(\frac{3}{2}+\frac{1}{2}\sqrt{5})^2} = \frac{1}{e^2} \cdot \frac{12}{\tau^4}$. As one should have expected, in these cases all the relations are influenced by the "golden ratio" $\tau := \frac{1+\sqrt{5}}{2}$. Since $\tau^2 = \tau + 1$, all powers of τ can be presented as polynomials of first degree, where Fibonacci numbers occur as coefficients. We omit the equation for e. For $z := \frac{d^2}{e^2}$ we use $u = \frac{1}{\tau^4}(3z - 4)$ and obtain

$$(3z-4)^3 + 3(4-5\tau)\tau^3(3z-4)^2 + 3(4-3\tau)(3-2\tau)\tau^6(3z-4) - (3-2\tau)^2\tau^{10} = 0\,.$$

Reduction of the coefficients yields

$$(3z-4)^3 - 3(7\tau+6)(3z-4)^2 + 3(\tau+2)(3z-4) - (3\tau+2) = 0$$

or

$$z^3 - (7\tau+10)z^2 + (19\tau+22)z - (13\tau+14) = 0\,.$$

So one can see that *the equations for Cubus simus and Dodecaedron simum are derived in a unified manner, namely with* $\tau = 0$ *and* $\tau = \frac{1}{2}(1+\sqrt{5})$*, respectively.*

For Dodecaedron simum we obtain roughly

$$z^3 - 21{,}32623792 \cdot z^2 + 52{,}74264579 \cdot z - 35{,}03444185 = 0\,,$$

yielding the estimates

$$18{,}5905391 < z < 18{,}5905392\,.$$

(It should be noticed that in the book [24] for $\sqrt{z}$ the surprisingly exact value 4,3116747491 is given, whereas [4], p. 139, for $\frac{1}{2}\sqrt{z}$ presents the value 2,7654, from which only the integer part is correct.) Thus we complete the table from Section 2 by

$$\{3,3,3,3,5\}: \quad VI, \quad d^2 : e^2 = 18{,}5905391, \quad e : d = 0{,}2319284\,.$$

5. On the construction of Dodecaedron simum

For $\{3,3,3,3,5\}$ the equation

$$z^3 - (7\tau+10)z^2 + (19\tau+22)z - (13\tau+14) = 0\,,$$

yielding the relation $z := \frac{d^2}{e^2}$, has coefficients from $\mathbb{Q}(\sqrt{5})$. Therefore the simple criterion, which was used in the case of Cubus simus, cannot be applied. But the equation above has a special form which allows to use a more general criterion. It is obvious that

$$z^3 - 3z^2 + 3z - 1 - (7\tau+7)z^2 + (19\tau+19)z - (13\tau+13) = 0$$

or

$$(z-1)^3-(\tau+1)(7z^2-19z+13)=0\,.$$

Since $z=1$ is not a solution, this can be rewritten in the form

$$\frac{1}{\tau+1}=\frac{7z^2-19z+13}{(z-1)^3}=\frac{7(z-1)^2-5(z-1)+1}{(z-1)^3}\,.$$

Setting $\frac{1}{z-1}=:x$ and due to $(\tau+1)^{-1}=2-\tau$ we obtain for x the cubic equation

$$x^3-5x^2+7x=\frac{1}{2}\left(3-\sqrt{5}\right)\,.$$

Since z rationally depends on x, the edge-length e can be constructed from d by ruler and compass if and only if x belongs to a quadratic field extension over $\mathbb{Q}$. But generalizing the respective criterion used above, we have: If a number x satisfies an equation of n-th degree with coefficients from $\mathbb{Q}$ which is irreducible over $\mathbb{Q}$ and n is not a power of 2, then x does not belong to a quadratic field extension over $\mathbb{Q}$ (cf. [2], p. 71).

Of course it is easy to transform the above equation in x (in which only one coefficient is not from $\mathbb{Q}$) into an equation having only integer coefficients. For example, we can rewrite it in the form

$$p(x):=-(x^3-5x^2+7x)^2+3(x^3-5x^2+7x)-1=0\,.$$

But this equation of sixth degree is irreducible over $\mathbb{Q}$. This is easy to see by the following arguments: If a decomposition $p(x)=p_1(x)p_2(x)$ into polynomials p_i of at least first degree and with coefficients from $\mathbb{Q}$ would be possible, then the known Lemma of Gauß would imply a decomposition into polynomials with coefficients from $\mathbb{Z}$. Now let us restrict the polynomials onto $x\in\mathbb{Z}$. The decomposition should remain, if we pass on (from integers) to congruence classes modulo 3. First we have

$$-(x^3-5x^2+7x)^2+3(x^3-5x^2+7x)-1\equiv-(x^3+x^2+x)^2-1 \pmod 3\ .$$

But since, due to P. de Fermat, $x^3\equiv x \pmod 3$, we get furthermore

$$-(x^3+x^2+x)^2-1\equiv-(5x^2+4x)-1\equiv x^2-x-1 \pmod 3\ .$$

Since the latter polynomial has no root in $\mathbb{Z}/3\mathbb{Z}$, it is obviously irreducible over this field. This is verified by setting in the (only possible) values $1,0,-1$. This contradiction shows that $p(x)$ is irreducible in $\mathbb{Q}[x]$. The roots of $p(x)$ satisfy an equation of sixth degree which is irreducible over $\mathbb{Q}$, they cannot belong to a corresponding quadratic field extension. Thus, also in the case of Dodecaedron simum the edge-length cannot be constructed by ruler and compass from the given diameter of the circumsphere.

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Received November 23, 2000