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CR Singular Immersions of Complex Projective Spaces

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Abstract. Quadratically parametrized smooth maps from one complex projective space to another are constructed as projections of the Segre map of the complexification. A classification theorem relates equivalence classes of projections to congruence classes of matrix pencils. Maps from the 2-sphere to the complex projective plane, which generalize stereographic projection, and immersions of the complex projective plane in four and five complex dimensions, are considered in detail. Of particular interest are the CR singular points in the image.

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1. Introduction

It was shown by [23] that the complex projective plane $\mathbb{C}P^2$ can be embedded in $\mathbb{R}^7$. An example of such an embedding, where $\mathbb{R}^7$ is considered as a subspace of $\mathbb{C}^4$, and $\mathbb{C}P^2$ has complex homogeneous coordinates $[z_1 : z_2 : z_3]$, was given by the following parametric map:

$$[z_1 : z_2 : z_3] \mapsto \frac{1}{|z_1|^2 + |z_2|^2 + |z_3|^2}(z_2\bar{z}_3, z_3\bar{z}_1, z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

Another parametric map of a similar form embeds the complex projective line $\mathbb{C}P^1$ in $\mathbb{R}^3 \subseteq \mathbb{C}^2$:

$$[z_0 : z_1] \mapsto \frac{1}{|z_0|^2 + |z_1|^2}(2\bar{z}_0 z_1, |z_1|^2 - |z_0|^2).$$

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This may look more familiar when restricted to an affine neighborhood, $[z_0 : z_1] = (1, z) = (1, x + iy)$, so the set of complex numbers is mapped to the unit sphere:

$$z \mapsto \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right),$$

and the "point at infinity", $[0 : 1]$, is mapped to the point $(0, 0, 1) \in \mathbb{R}^3$. This is the usual form of the "stereographic projection" map.

This article will consider embeddings of $\mathbb{C}P^m$ which generalize the above examples by considering quadratic polynomials with arbitrary complex coefficients on terms $z_i\bar{z}_j$. By considering two parametric maps equivalent if one is related to another by complex linear coordinate changes of the domain and target, the classification of these maps is reduced to a problem in matrix algebra.

This project was originally motivated by the study of real submanifolds of $\mathbb{C}^n$, and in particular how the topology of a compact submanifold is related to whether any of its tangent planes contain a complex line.

For example, [9] and [4] considered real 4-manifolds immersed in $\mathbb{C}^5$ (or some other (almost) complex 5-manifold), which will generally have isolated points where the real tangent space contains a complex line. Such points are called complex jump points, complex tangents, or CR singularities; a manifold without such points will be called *totally real*. Isolated complex tangents can be assigned an integer index, which is 1 or -1 when the submanifold is in general position, and which reverses when the submanifold's orientation is switched. For compact submanifolds, the sum of these indices is then determined by a characteristic class formula. In the case where the complex projective plane $\mathbb{C}P^2$, considered only as a smooth, oriented 4-manifold, is immersed in $\mathbb{C}^5$, it cannot be totally real, and the index sum for a generic immersion is the first Pontrjagin number, $p_1\mathbb{C}P^2 = 3$. The existence of an embedding with exactly three complex tangents follows from a lemma of [9] which uses results of Gromov. One of the main results of this paper is an explicit formula defining such an embedding (Example 5.3).

The next section will set up a general construction for mapping a complex projective m-space into a complex projective n-space. Section 3 is a brief review of the topology of generically immersed real submanifolds, which will define the notion of "general position" and give a formula for the expected dimension of CR singular loci. Sections 4 and 5 will consider immersions of $\mathbb{C}P^m$ in $\mathbb{C}P^n$, in the cases where $m = 1$, $n = 2$, and $m = 2$, $n = 5$.

2. The projective geometric construction

The complex projective m-space, $\mathbb{C}P^m$, is the set of complex lines containing the origin in $\mathbb{C}^{m+1}$, so each line z will have homogeneous coordinates $[z_0 : z_1 : \ldots : z_m]$. A nonzero vector spanning the line z will be written as a column vector $\vec{z}$. A vector $\vec{z}$ can be multiplied by an invertible square matrix A with complex entries: $\vec{z} \mapsto A\vec{z}$, and this defines a group action on $\mathbb{C}P^m$. The set of nonzero complex scalar multiples $\{c \cdot A,\ c \neq 0\}$ is an element of the projective general linear group, $PGL(m + 1, \mathbb{C})$. (Usually, the equivalence class of matrices $\{c \cdot A\}$ will be abbreviated as A.)

The following map is formed by all the $(m+1)^2$ quadratic monomials $z_i w_j$ in the components of two vectors $\vec{z}$ and $\vec{w}$:

$$\begin{aligned} s : \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} &\to \mathbb{C}^{(m+1)^2} \\ (\vec{z}, \vec{w}) &\mapsto (z_0 w_0, z_0 w_1, \dots, z_0 w_m, \dots, z_m w_0, \dots, z_m w_m). \end{aligned}$$

Since it has the property that $s(\lambda \cdot \vec{z}, \mu \cdot \vec{w}) = \lambda \cdot \mu \cdot s(\vec{z}, \vec{w})$ for all $\lambda, \mu \in \mathbb{C}$, it induces a map:

$$\begin{aligned} s : \mathbb{C}P^m \times \mathbb{C}P^m &\to \mathbb{C}P^{m^2+2m} \\ (z, w) &\mapsto [z_0 w_0 : z_0 w_1 : \dots : z_0 w_m : \dots : z_m w_0 : \dots : z_m w_m], \end{aligned}$$

called the Segre map, which is a holomorphic embedding.

Define a vector space isomorphism from the space of $d \times d$ complex matrices to the space of column d^2-vectors by stacking the columns of the matrix:

$$\begin{aligned} vec : M(d, \mathbb{C}) &\to \mathbb{C}^{d^2} \\ (\vec{z}^{\,1} \cdots \vec{z}^{\,d})_{d\times d} &\mapsto \begin{pmatrix} \vec{z}^{\,1} \\ \vdots \\ \vec{z}^{\,d} \end{pmatrix}_{d^2\times 1} . \end{aligned}$$

This is the well-known "vectorization" map from matrix algebra ([14]). Denote its inverse by $k : \mathbb{C}^{d^2} \to M(d, \mathbb{C})$. The induced map $\mathbb{C}P^{d^2-1} \to \mathbb{C}P(M(d, \mathbb{C}))$ is also denoted k.

The composition of the Segre map with the isomorphism k (in the case $d = m+1$) has the following interpretation in terms of matrix multiplication:

$$(k \circ s)(\vec{z}, \vec{w}) = \vec{w} \cdot \vec{z}^T. \tag{1}$$

$\vec{z}^T$ is a row vector, the transpose of $\vec{z}$, so the RHS is a (line spanned by a) rank ≤ 1 matrix of size $(m+1) \times (m+1)$.

This construction could be considered more abstractly. The following (optional) sketch links the above notation with standard notions from geometry and multilinear algebra (see [12], [15], [5]). Let V be a finite-dimensional complex vector space, and denote by V^* the "dual" space of $\mathbb{C}$-linear functions $\phi : V \to \mathbb{C}$, and by $\mathrm{End}(V)$ the set of endomorphisms $V \to V$. Then, define a map $\mathbf{k} : V^* \otimes V \to \mathrm{End}(V)$, first by stating the following formula for tensor products: for $\vec{v}$, $\vec{w} \in V$, and $\phi \in V^*$,

$$\mathbf{k}(\phi \otimes \vec{w}) : \vec{v} \to (\phi(\vec{v})) \cdot \vec{w},$$

and then defining the map $\mathbf{k}$ for all elements of $V^* \otimes V$ by extending by $\mathbb{C}$-linearity, to get an isomorphism of vector spaces. Let $\mathbf{s}$ be the universal bilinear function $V^* \times V \to V^* \otimes V$. Then, a vector $\vec{z} \in V$ determines a dual vector ϕ, by $\phi : \vec{v} \mapsto \vec{z}^{\,T} \cdot \vec{v}$, and $(\mathbf{k} \circ \mathbf{s})(\phi, \vec{w}) = \mathbf{k}(\phi \otimes \vec{w})$ is an endomorphism taking every vector $\vec{v}$ to some multiple of $\vec{w}$, just as in equation (1).

The next ingredients in the construction are a number n such that $0 \leq 2m \leq n \leq m^2+2m$, and a $(n+1) \times (m+1)^2$ matrix P with complex entries and full rank $n+1 \leq (m+1)^2$, called the coefficient matrix. The linear transformation $\mathbb{C}^{(m+1)^2} \to \mathbb{C}^{n+1}$ (also denoted P)

induces a "projection" map $\mathbb{C}P^{m^2+2m} \to \mathbb{C}P^n$, (also denoted P) defined for all elements z except those lines in the kernel of P. Let $\mathbb{C}P^n$ have homogeneous coordinates $[Z_0 : \ldots : Z_n]$.

Formally, the composition $P \circ s$ can be written:

$$(z, w) \mapsto [P_0 : \ldots : P_n],$$

with complex coefficients $p_k^{i,j}$ (the $(n+1) \times (m+1)^2$ entries of matrix P) on each term:

$$P_k = \sum_{i,j=0}^{m} p_k^{i,j} z_i w_j.$$

$P \circ s$ will be a well-defined map $\mathbb{C}P^m \times \mathbb{C}P^m \to \mathbb{C}P^n$ if the image of s is disjoint from the kernel of P, and otherwise will be a "rational map," well-defined only at each point whose image under s is not in the kernel of P.

The previously mentioned bound $2m \leq n$ means that the dimension of the domain of $P \circ s : \mathbb{C}P^m \times \mathbb{C}P^m \to \mathbb{C}P^n$ is less than or equal to the dimension of the target. It also implies that the dimension of the image of s in $\mathbb{C}P^{m^2+2m}$ is less than the codimension $(n+1)$ of the (projective) kernel of P, so that generically, but not always, the image of s is disjoint from the kernel of P.

Example 2.1. The $m = 1$, $n = 2$ case is in the assumed dimension range. A 3×4 matrix P with rank 3 has a kernel equal to a line in $\mathbb{C}^4$, or a single point $x \in \mathbb{C}P^3$. $P \circ s : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$ is well-defined if the two-dimensional image of s misses the point x in $\mathbb{C}P^3$, and otherwise is defined on all but one point of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Although s is an embedding, the composition $P \circ s$ may not be one-to-one, and may also have singular points, where its (complex) Jacobian has rank less than $2m$.

Definition 2.2. *For a given pair m, n, two coefficient matrices P and Q are c-equivalent if there exist three invertible matrices, A, $B \in GL(m+1, \mathbb{C})$, $C \in GL(n+1, \mathbb{C})$ such that for all $(\vec{z}, \vec{w}) \in \mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$,*

$$(Q \circ s)(\vec{z}, \vec{w}) = (C \circ P \circ s)(A\vec{z}, B\vec{w}) \in \mathbb{C}^{n+1}.$$

It is easy to check c-equivalence is an equivalence relation.

Theorem 2.3. *P and Q are c-equivalent if and only if there exist A, $B \in GL(m+1, \mathbb{C})$ such that the following $(m^2 + 2m - n)$-dimensional subspaces of $M(m+1, \mathbb{C})$ are equal:*

$$k(\ker(P)) = B \cdot (k(\ker(Q))) \cdot A^T.$$

Proof. For invertible matrices A, $B \in M(m+1, \mathbb{C})$, the map

$$\vec{v} \mapsto vec(B \cdot (k(\vec{v})) \cdot A^T)$$

is a $\mathbb{C}$-linear invertible map $\mathbb{C}^{(m+1)^2} \to \mathbb{C}^{(m+1)^2}$. Its representation as a square matrix will be denoted $[A \otimes B]$.

Using equation (1),

$$\begin{aligned}(k \circ s)(A\vec{z}, B\vec{w}) &= (B \cdot \vec{w}) \cdot (A \cdot \vec{z})^T \\ &= B \cdot \vec{w} \cdot \vec{z}^T \cdot A^T \\ &= B \cdot ((k \circ s)(\vec{z}, \vec{w})) \cdot A^T \\ &= k([A \otimes B] \cdot (s(\vec{z}, \vec{w}))).\end{aligned}$$

Since k is an isomorphism,

$$s(A\vec{z}, B\vec{w}) = [A \otimes B] \cdot (s(\vec{z}, \vec{w})).$$

(This motivates the notation $[A \otimes B]$, in terms of the abstract version of the construction. For present purposes, $[A \otimes B]$ is merely a convenient label; see [14] or [5] for the connections between *vec* and tensor products.)

So, from the definition of c-equivalence,

$$(Q \circ s)(\vec{z}, \vec{w}) = (C \circ P \circ s)(A\vec{z}, B\vec{w}) = (C \circ P \circ [A \otimes B])(s(\vec{z}, \vec{w})),$$

and since the image of $s : \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} \to \mathbb{C}^{(m+1)^2}$ spans the target space, Q and P are c-equivalent if and only if there exist A, B, C so that

$$Q = C \cdot P \cdot [A \otimes B].$$

This equation says Q and $P \cdot [A \otimes B]$ are row-equivalent, and therefore also *solution-equivalent*, i.e., there exists such an invertible C if and only if $\ker(Q) = \ker(P \cdot [A \otimes B])$ (see [15], [13]). Of course, this equality of subspaces of $\mathbb{C}^{(m+1)^2}$ is equivalent to the equality of subspaces of $M(m+1, \mathbb{C})$:

$$k(\ker(Q)) = k(\ker(P \cdot [A \otimes B])).$$

Suppose $K \in k(\ker(P \cdot [A \otimes B]))$. This is equivalent to

$$\vec{0} = (P \cdot [A \otimes B])(vec(K)) = P \cdot vec(B \cdot K \cdot A^T),$$

by definition of $[A \otimes B]$, or, equivalently,

$$vec(B \cdot K \cdot A^T) \in \ker(P) \iff B \cdot K \cdot A^T \in k(\ker(P)).$$

This proves the claim of the theorem. □

It follows immediately from the Definition that if P and Q are c-equivalent coefficient matrices, then there exist automorphisms A, $B \in PGL(m+1, \mathbb{C})$, $C \in PGL(n+1, \mathbb{C})$ such that the compositions of induced maps are equal for all $(z, w) \in \mathbb{C}P^m \times \mathbb{C}P^m$ where the quantities are defined:

$$(Q \circ s)(z, w) = (C \circ P \circ s)(Az, Bw) \in \mathbb{C}P^n.$$

Geometrically, C corresponds to a linear transformation of the target $\mathbb{C}P^n$, and the maps $(z, w) \mapsto (Az, Bw)$ form a subgroup of the group of holomorphic automorphisms of the domain $\mathbb{C}P^m \times \mathbb{C}P^m$. This is the connected component containing the identity in the automorphism group, and a proper subgroup since $(z, w) \mapsto (w, z)$, for example, is holomorphic.

The converse assertion, that if there exist automorphisms such that the above induced maps are equal, then the corresponding coefficient matrices are c-equivalent, is another issue, which we will not address here.

The last map to be introduced in this section is the totally real diagonal embedding,

$$\begin{aligned} \Delta : \mathbb{C}P^m &\to \mathbb{C}P^m \times \mathbb{C}P^m \\ z &\mapsto (z, \bar{z}). \end{aligned}$$

The image of Δ is exactly the fixed point set of the involution $(z, w) \mapsto (\bar{w}, \bar{z})$, and the product space $\mathbb{C}P^m \times \mathbb{C}P^m$ could be considered the "complexification" of its real submanifold $\Delta(\mathbb{C}P^m)$. The composition $s \circ \Delta : \mathbb{C}P^m \to \mathbb{C}P^{m^2+2m}$ is a smooth embedding, but not holomorphic for $m > 0$. It has the following form:

$$z \mapsto [z_0\bar{z}_0 : z_0\bar{z}_1 : \ldots : z_0\bar{z}_m : \ldots : z_m\bar{z}_0 : \ldots : z_m\bar{z}_m].$$

[16] calls $s \circ \Delta$ the "skew-Segre" embedding, and shows how it is related to the Mannoury embedding of $\mathbb{C}P^m$ into an affine space $\mathbb{C}^{m^2+2m+1}$ (the $m = 2$ case will appear in Example 5.6).

For a projection map P, the composition $P \circ s \circ \Delta : \mathbb{C}P^m \to \mathbb{C}P^n$ is also smooth where it is defined, but not necessarily one-to-one or nonsingular. It is possible that $P \circ s \circ \Delta$ is an embedding even if $P \circ s$ is not.

Example 2.4. The stereographic projection from the introduction can be written as a map $\mathbb{C}P^1 \to \mathbb{C}P^2$,

$$[z_0 : z_1] \mapsto [|z_0|^2 + |z_1|^2 : 2\bar{z}_0 z_1 : |z_1|^2 - |z_0|^2],$$

so the image is contained in the affine neighborhood $\{[Z_0 : Z_1 : Z_2] : Z_0 \neq 0\}$. The map $s \circ \Delta$ in this case has the form

$$[z_0 : z_1] \mapsto [z_0\bar{z}_0 : z_0\bar{z}_1 : z_1\bar{z}_0 : z_1\bar{z}_1],$$

and the coefficient matrix (acting on columns) is

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}_{3\times 4}.$$

Note that $\ker(P)$ is the complex line $[0 : 1 : 0 : 0]$, and this point is in the image of s, so the composition $P \circ s : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$ is not defined at the point $x = ([1 : 0], [0 : 1])$. The singular locus in the domain is a 1-dimensional subvariety S defined by $z_1 w_0 = 0$, which is the union of two lines,

$$S = ((\mathbb{C}P^1 \times \{[0 : 1]\}) \cup (\{[1 : 0]\} \times \mathbb{C}P^1)) \setminus \{([1 : 0], [0 : 1])\}$$

(their point of intersection is x, which is not in the domain). The real diagonal $\Delta(\mathbb{C}P^1)$ does not meet x, and meets S at two points, $([0:1],[0:1])$ and $([1:0],[1:0])$. The image of S is a set of two points,

$$(P \circ s)(S) = \{[1:0:1],[1:0:-1]\} \subseteq \mathbb{C}P^2.$$

The image $(P \circ s \circ \Delta)(\mathbb{C}P^1)$ is a sphere which is contained in the affine neighborhood $\{[Z_0 : Z_1 : Z_2] : Z_0 = 1\}$, and the two points in the image of the singular locus are the "North and South Poles" of the stereographic projection where (not coincidentally) the tangent plane to the sphere is a complex line.

Example 2.5. The first example from the introduction falls in the $m = 2$, $n = 4$ case, and the coefficient matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}_{5\times 9}.$$

As in the previous example, the top row of P is the sequence of coefficients from the denominator. The kernel of P is a 4-dimensional subspace of $\mathbb{C}^9$, equal to the set of vectors of the form

$$(c_1, 0, c_2, c_3, c_1, 0, 0, c_4, -2c_1)^T$$

for any complex constants $c_1, \dots, c_4$. The kernel of P meets the image of s at exactly three points, corresponding to

$$\{([0:0:1],[0:1:0]),([0:1:0],[1:0:0]),([1:0:0],[0:0:1])\}$$

in the domain of s, so $P \circ s$ is not defined at those three points. The real diagonal $\Delta(\mathbb{C}P^2)$ does not meet any of the three points, and $P \circ s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^4$ is an embedding into the affine neighborhood $\{Z_0 = 1\}$.

There are several ways to calculate the intersection of $\ker P$ and the image of s. An easy way is to use equation (1), recalling that a matrix is in the image of $k \circ s$ if it has rank 1. In the above example, $k(\ker P)$ is the subspace of matrices of the form

$$\begin{pmatrix} c_1 & c_3 & 0 \\ 0 & c_1 & c_4 \\ c_2 & 0 & -2c_1 \end{pmatrix}.$$

A matrix of this form has rank 1 only if c_1 and two out of three of the other coefficients are 0, for example,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot (0,0,1).$$

Lemma 2.6. *The subgroup of $PGL(m+1,\mathbb{C}) \times PGL(m+1,\mathbb{C})$ that leaves invariant the set $\Delta(\mathbb{C}P^m)$ is the set of automorphisms of the form $(z,w) \mapsto (Az, \bar{A}w)$.*

Proof. $\bar{A}$ denotes the entrywise complex conjugate of the matrix A, but the above automorphisms are still holomorphic, and obviously form a subgroup. For $(z,\bar{z}) \in \Delta(\mathbb{C}P^m)$, $(Az, \bar{A}\bar{z}) = \Delta(Az)$, so this subgroup fixes the image of Δ. Conversely, if $(A,B) \in PGL(m+1,\mathbb{C}) \times PGL(m+1,\mathbb{C})$ has the property that for all $(z,\bar{z}) \in \Delta(\mathbb{C}P^m)$, $(Az, B\bar{z})$ is also in $\Delta(\mathbb{C}P^m)$, then $Az = \overline{B\bar{z}} = \bar{B}z$ for all z, so $A = \bar{B}$. $\square$

Definition 2.7. *For a given pair m, n, two coefficient matrices P and Q are r-equivalent if there exist two invertible matrices $A \in GL(m+1,\mathbb{C})$, $C \in GL(n+1,\mathbb{C})$ such that for all $(\vec{z},\vec{w}) \in \mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$,*

$$(Q \circ s)(\vec{z},\vec{w}) = (C \circ P \circ s)(A\vec{z}, \bar{A}\vec{w}).$$

This is obviously an equivalence relation, and if P and Q are r-equivalent, then they are also c-equivalent. Lemma 2.6 gives a geometric interpretation of the relationship between the two equivalences.

Theorem 2.8. *P and Q are r-equivalent if and only if there exists an invertible matrix $A \in M(m+1,\mathbb{C})$ such that the following (m^2+2m-n)-dimensional subspaces of $M(m+1,\mathbb{C})$ are equal:*

$$k(\ker(P)) = A \cdot (k(\ker(Q))) \cdot \bar{A}^T.$$

Proof. The proof of Theorem 2.3 goes through with only the obvious modifications. Since $s(A\vec{z}, \bar{A}\vec{w}) = [A \otimes \bar{A}] \cdot (s(\vec{z},\vec{w}))$, the following are equivalent: Q and P are r-equivalent; $Q = C \circ P \circ [A \otimes \bar{A}]$; $\ker(Q) = \ker(P \cdot [A \otimes \bar{A}])$; $k(\ker(Q)) = k(\ker(P \cdot [A \otimes \bar{A}]))$. Also, $K \in k(\ker(P \cdot [A \otimes \bar{A}])) \iff \bar{A} \cdot K \cdot A^T \in k(\ker(P))$. $\square$

In matrix algebra, a subspace of a space of matrices is called a "pencil," and matrices or pencils K, M satisfying $M = AK\bar{A}^T$ are "congruent" or "conjunctive." Theorems 2.3 and 2.8 were motivated by a similar construction in [7], where the real projective plane was mapped to $\mathbb{R}P^3$ by projections of the Veronese map, and the (finitely many) equivalence classes of such projections were found by classifying congruence classes of real symmetric matrix pencils.

The r-equivalence of matrices P and Q also implies the existence of automorphisms $A \in PGL(m+1,\mathbb{C})$, $C \in PGL(n+1,\mathbb{C})$ such that the compositions of induced maps are equal:

$$Q \circ s \circ \Delta = C \circ P \circ s \circ \Delta \circ A : \mathbb{C}P^m \to \mathbb{C}P^n.$$

As with c-equivalence, the A and C matrices from Definition 2.7 are complex automorphisms of the domain and range of a map $\mathbb{C}P^m \to \mathbb{C}P^n$, and r-equivalence seems to be a natural way to classify maps of the form $P \circ s \circ \Delta$.

Once again, the converse assertion, whether the equality of induced maps implies the r-equivalence of matrices, will not be treated in general. However, something even stronger can be proved in the case $m = 1$, $n = 2$. It will follow from the classification of Theorem 4.3 that if there are automorphisms A, C such that $Q \circ s \circ \Delta$ and $C \circ P \circ s \circ \Delta \circ A$, as maps from $\mathbb{C}P^1$ to $\mathbb{C}P^2$, have the same image, then P and Q are r-equivalent matrices.

If the images of $P \circ s \circ \Delta$ and $Q \circ s \circ \Delta$ are both contained in some affine neighborhood, as in the two examples from the introduction, a weaker notion of equivalence would allow real-linear transformations of the affine target space. However, such transformations could distort the interesting CR singular structure.

3. Review of complex tangents

The following general facts about real submanifolds of high codimension in $\mathbb{C}^n$ are recalled from [10], [4], which both generalize these ideas to real submanifolds of almost complex manifolds, and give further references on this subject.

Consider $0 \leq d \leq n$, and an oriented manifold M of real dimension d inside $\mathbb{C}^n$, where $\mathbb{C}^n$ can be described as a $2n$-dimensional real vector space, together with a real-linear complex structure operator J such that $J \circ J = -Id$. If the tangent space at a point $x \in M$ satisfies $T_xM \cap JT_xM = \{\vec{0}\}$, i.e., the subspace meets its rotation by J only at the origin, then T_xM is called *totally real*. This implies that the subspace T_xM contains no complex lines. The manifold M is also called *totally real at* x, and a *totally real submanifold* if it is totally real at every point.

The totally real subspaces form a dense open subset of the Grassmann manifold $G(d, 2n)$ of all real, oriented, d-dimensional subspaces in $\mathbb{C}^n$. The subspaces T which are not totally real form a subvariety of real codimension $2(n-d+1)$ in the $d(2n-d)$-dimensional space $G(d, 2n)$. More generally, $T \cap JT$ is always a complex subspace of $\mathbb{C}^n$, and the set of d-planes T such that $\dim_{\mathbb{C}} T \cap JT \geq j$ is a subvariety D_j. $D_j \setminus D_{j+1}$ is a smooth, oriented submanifold of real codimension $2j(n-d+j)$ in $G(d, 2n)$. If the Gauss map $\gamma : x \mapsto T_xM$ of a submanifold M in $\mathbb{C}^n$ misses D_j for $j > 0$, M is totally real. Otherwise, M has "CR singular" loci, indexed by j,

$$N_j = \{x \in M : \gamma(x) \in D_j\} = \{x \in M : \dim_{\mathbb{C}} T_xM \cap JT_xM \geq j\},$$

which have an "expected" codimension $2j(n-d+j)$ in M. The usual warnings about intersections apply — N_j could be empty, and need not be a submanifold of M. Example 5.4 will demonstrate an exception to the expected codimension formula.

When the dimension d is equal to $2j(n-d+j)$ for some j, the locus N_j is expected to be a set of isolated points. For the purposes of this article, M is said to be "in general position" if $d = 2j(n-d+j)$, $\gamma(M)$ meets D_j transversely in $G(d, 2n)$, and $N_{j+1} = \emptyset$. Then, the "index" of each point in N_j is the oriented intersection number, ± 1, of $\gamma(M)$ and D_j.

Example 3.1. A real n-manifold immersed in $\mathbb{C}^n$ is expected to have complex tangents along a locus N_1 of real codimension 2. A manifold with nonzero euler characteristic cannot have a totally real embedding in $\mathbb{C}^n$. The stereographic embedding from the introduction is an example of a 2-sphere embedded in $\mathbb{C}^2$ with two complex tangents.

Example 3.2. In the geometric construction of the previous section, the complex projective m-space (real dimension $d = 2m$) is mapped to a complex n-manifold with $2m \leq n$. The expected behavior is that the image will be totally real outside a locus N_1 of real codimension $2(n-2m+1)$. So, when the real dimension is less than this number, $2m < 2(n-2m+1) \iff m < \frac{1}{3}(n+1)$, the image of $\mathbb{C}P^m$ will generically be totally real in $\mathbb{C}P^n$. Otherwise, N_1 will generically be either empty, or of real dimension $2m - 2(n-2m+1) = 2(3m-n-1)$.

Example 3.3. For a real 8-manifold in general position in a complex 8-manifold, the locus of complex tangents is a (possibly empty) 6-dimensional subset N_1, and the points x where $j = \dim_{\mathbb{C}} T_x \cap JT_x = 2$ will form a subset N_2 of isolated points in N_1.

A complex automorphism of the ambient space will not change the CR singular structure of any real submanifold; the dimension of $T_xM \cap JT_xM$ remains invariant under any local biholomorphism around the point x. Also, suppose there is a holomorphic map f from one complex manifold to another, so that f is an embedding when restricted to a neighborhood of a point x in the domain, and M is a submanifold in this neighborhood which is totally real at x. Then the image $f(M)$ is a submanifold in a neighborhood of $f(x)$, and $f(M)$ is totally real at $f(x)$.

In particular, two maps $P \circ s \circ \Delta$ and $Q \circ s \circ \Delta$ with r-equivalent coefficient matrices will have images with the same number and dimension (j) of complex tangents. Given a coefficient matrix P, and a point $(z, \bar{z})$ where $P \circ s$ is nonsingular, the restriction of $P \circ s$ to a neighborhood of $(z, \bar{z})$ will be an embedding. Since $\Delta(\mathbb{C}P^m)$ is totally real in $\mathbb{C}P^m \times \mathbb{C}P^m$, its image will also be totally real at $(P \circ s \circ \Delta)(z, \bar{z}) \in \mathbb{C}P^n$. The only place a complex tangent could occur in the image of $P \circ s \circ \Delta$ would be in the singular value set of $P \circ s$. This was already observed in Example 2.4, and this phenomenon will be the crucial step in finding the exact CR singular locus of some immersions in Section 5.

4. Real spheres in $\mathbb{C}P^2$

This section will cover the $m = 1$, $n = 2$ case of the construction from Section 2, establishing the c-equivalence and r-equivalence classes of 3×4 coefficient matrices, which correspond to maps from $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^1$ to $\mathbb{C}P^2$.

Theorem 4.1. *There are two c-equivalence classes of 3×4 matrices P, characterized by the rank of $k(\ker P)$.*

Proof. Of course, by "rank($k(\ker P)$)" we mean the rank of a non-zero matrix which spans the line.

From Theorem 2.3, the c-equivalence classes are defined by classifying $k(\ker P)$, a one-dimensional subspace of $M(2, \mathbb{C})$, spanned by some nonzero matrix K, up to the following equivalence relation: given any $A, B \in GL(2, \mathbb{C})$, the following complex lines are equivalent:

$$\{c \cdot K : c \in \mathbb{C}\} \sim \{c \cdot AKB : c \in \mathbb{C}\}.$$

The first case is where K is nonsingular, in which case choosing $A = K^{-1}$, $B = Id$, shows that all such subspaces are equivalent to the subspace spanned by the identity matrix.

The second case is that K is singular, and since it spans a line, it has rank 1. It is a straightforward calculation to check that there exist A, B so that $AKB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so all P so that K has rank 1 are c-equivalent. □

To summarize, the two classes of coefficient matrices P can be distinguished in several different ways.

Case 1: The following are equivalent.

- P is c-equivalent to $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

- There exist A, B, C, so that $(C \circ P \circ s)(Az, Bw) = [z_0w_0 - z_1w_1 : z_0w_1 : z_1w_0]$.
- $k(\ker(P))$ is spanned by a rank 2 matrix, so $\ker(P)$ does not intersect the image of s, and $P \circ s$ is defined on all of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Case 2: The following are equivalent.

- P is c-equivalent to $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.
- There exist A, B, C, so that $(C \circ P \circ s)(Az, Bw) = [z_0w_0 + z_1w_1 : 2z_1w_0 : z_1w_1 - z_0w_0]$.
- $k(\ker(P))$ is spanned by a rank 1 matrix, so $\ker(P)$ intersects the image of s at one point, and $P \circ s$ is defined on all of $\mathbb{C}P^1 \times \mathbb{C}P^1$ except for one point.

In some higher-dimensional cases where $\ker(P)$ is still just a line, Theorem 4.1 easily generalizes, and we only sketch a proof.

Theorem 4.2. *For $n = (m + 1)^2 - 2$, so that the kernel of each of the coefficient matrices P, Q, is one-dimensional, P and Q are c-equivalent if and only if* $\operatorname{rank}(k(\ker(P))) = \operatorname{rank}(k(\ker(Q)))$*, so there are $m + 1$ c-equivalence classes.*

Proof. The rank of a matrix is the only invariant under the equivalence of K and AKB (see [13], §3.5). The kernel of P is spanned by some non-zero $(m+1) \times (m+1)$ matrix, which can be put into a diagonal normal form with 1 and 0 entries, according to its rank, $1, \dots, m+1$. Theorem 2.3 then establishes the c-equivalence classes. ☐

The two cases of Theorem 4.1 break down into more cases under r-equivalence, and there will be some continuous invariants.

Theorem 4.3. *The r-equivalence classes of 3×4 matrices P are characterized by the congruence class of $k(\ker(P))$, each class corresponding to exactly one of the following normal forms for basis elements of $k(\ker(P))$:*

- $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha = \cos(\theta) + i\sin(\theta)$, $0 \le \theta \le \pi$,
- $\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$, $0 \le \beta < 1$,
- $\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$,
- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. The correspondence between r-equivalence and congruence of the kernels was established in Theorem 2.8, so it is enough to find the congruence classes of the one-dimensional kernels of P. If K spans $\ker(P)$, it can be decomposed into its Hermitian and skew-Hermitian parts, $K = K_h + iK_s$, $K_h = \frac{1}{2}(K + \bar{K}^T)$, $K_s = \frac{1}{2i}(K - \bar{K}^T)$. This is somewhat arbitrary, since K spans a complex line, and the decomposition is not respected by complex scalar multiplication. However, the decomposition is respected by the congruence operation:

$(AK\bar{A}^T)_h = AK_h\bar{A}^T$, and there is a well-developed theory of simultaneous congruence for pairs of Hermitian matrices K_h, K_s.

Following [13] §4.5, there are two main cases, the first is where K_h is nonsingular. Then, there are a few possible normal forms for pairs, according to the following list.

1. $K_h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $K_s = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}$.
2. $K_h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $K_s = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}$.
3. $K_h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $K_s = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}$.
4. $K_h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $K_s = \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}$, $\alpha \in \mathbb{C}$.
5. $K_h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $K_s = \begin{pmatrix} 0 & \lambda \\ \lambda & 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.
6. $K_h = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $K_s = \begin{pmatrix} 0 & -\lambda \\ -\lambda & -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

In each of these cases, recombining the two matrices as $K = K_h + iK_s$, then scaling by a complex number c, and then possibly using another congruence transformation, will bring K to one of the normal forms claimed in the theorem.

1. $K = \begin{pmatrix} 1+i\lambda_1 & 0 \\ 0 & 1+i\lambda_2 \end{pmatrix}$ spans the same line as $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i\lambda_2}{1+i\lambda_1} \end{pmatrix}$. As λ_1, λ_2 can be any real numbers, the fraction $\frac{1+i\lambda_2}{1+i\lambda_1}$ can be 1, or any non-real complex number α. For $A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, $AK\bar{A}^T$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & x\bar{x}\alpha \end{pmatrix}$, so the entry α can be scaled to have absolute value 1, and can be any element of $S^1 \subseteq \mathbb{C}$ except -1. Then, congruence by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ transforms $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ to $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$, so α can be assumed to lie in the closed upper half-plane, and the congruence classes are parametrized by $\alpha = \cos(\theta) + i\sin(\theta)$, for $0 \le \theta < \pi$.
2. $K = \begin{pmatrix} 1+i\lambda_1 & 0 \\ 0 & -1-i\lambda_2 \end{pmatrix}$ spans the same line as $\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1+i\lambda_2}{1+i\lambda_1} \end{pmatrix}$. By the same calculations as in the previous case, the lower right entry can be scaled to any element of S^1 except $1 \in \mathbb{C}$, so this case overlaps with the previous to give $\alpha = \cos(\theta) + i\sin(\theta)$, $0 < \theta \le \pi$. By the Law of Inertia ([13]), the $\alpha = 1$ and $\alpha = -1$ cases are not equivalent. It is a straightforward calculation to check that the lines spanned by matrices with different values of α in the upper half-plane are not congruent.
3. This case is the same as case 1, since K spans the same complex line as the matrix from case 1.
4. In this case, $K = \begin{pmatrix} 0 & 1+i\alpha \\ 1+i\bar{\alpha} & 0 \end{pmatrix}$. Let $a = i\alpha$, and assume $a \ne -1$, so K spans the same line as $\begin{pmatrix} 0 & 1 \\ \frac{1-\bar{a}}{1+a} & 0 \end{pmatrix}$. The fraction $\frac{1-\bar{a}}{1+a}$ can assume the value 1, or any other

complex number not on the unit circle. Congruence by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ can rotate the value of the fraction by $\frac{x}{\bar{x}}$, to some β on the nonnegative real axis. As in case 1, congruence by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ transforms β to $1/\beta$, so that the congruence classes can be represented by $\beta \in [0,1]$. However, the $\beta = 1$ case is Hermitian, and congruent to the $\alpha = -1$ matrix from case 1. The $a = -1$ case turns out to be congruent to the $\beta = 0$ case. It is straightforward to check the representatives for different values of β are pairwise inequivalent and also not equivalent to the matrices from cases 1 and 2.

5. $K = \begin{pmatrix} 0 & 1+i\lambda \\ 1+i\lambda & i \end{pmatrix}$ spans the same line as $\begin{pmatrix} 0 & 1 \\ 1 & \frac{i}{1+i\lambda} \end{pmatrix}$. Then using $A = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda}{2} & 1+\lambda^2 \end{pmatrix}$, $AK\bar{A}^T$ is proportional to $\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$. It is easy to check this matrix does not belong to the previous two families of equivalence classes.
6. This case is the same as the previous.

The second main case, not addressed in [13], is where the Hermitian part of K is singular. For some K, its Hermitian part could be singular, while the Hermitian part of a complex scalar multiple $c \cdot K$ is nonsingular. Such cases fall into the above classes, so it is enough to consider those $K \neq 0$ such that $\det(\frac{1}{2}(c \cdot K + \bar{c} \cdot \bar{K}^T)) = 0$ for all $c \in \mathbb{C}$. The theorem will follow from the claim that any such K spans a line congruent to the span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The proof of the claim involves some elementary matrix calculations, as in the previous paragraphs, but here the details will be given. Let $K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an arbitrary nonzero matrix with complex entries, and let $c = x + iy$ be a nonzero complex number. For the Hermitian part to be singular for all values of c, the following equation must hold for all x and y.

$$\begin{aligned}
0 &= \det(c \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \bar{c}\begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}) \\
&= c^2(\alpha\delta - \beta\gamma) + \bar{c}^2(\bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma}) + c\bar{c}(\bar{\alpha}\delta + \alpha\bar{\delta} - \beta\bar{\beta} - \gamma\bar{\gamma}) \\
&= x^2(\alpha\delta - \beta\gamma + \bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma} + \bar{\alpha}\delta + \alpha\bar{\delta} - \beta\bar{\beta} - \gamma\bar{\gamma}) \\
&\quad + y^2(-\alpha\delta + \beta\gamma - \bar{\alpha}\bar{\delta} + \bar{\beta}\bar{\gamma} + \bar{\alpha}\delta + \alpha\bar{\delta} - \beta\bar{\beta} - \gamma\bar{\gamma}) \\
&\quad + 2ixy(\alpha\delta - \beta\gamma - \bar{\alpha}\bar{\delta} + \bar{\beta}\bar{\gamma}).
\end{aligned}$$

Subtracting the coefficients on x^2 and y^2 shows $\mathrm{Re}(\det(K)) = 0$, and the coefficient on the xy term must also be zero, so $\mathrm{Im}(\det(K)) = \det(K) = 0$. The matrix K is rank 1 and also satisfies

$$\bar{\alpha}\delta + \alpha\bar{\delta} = \beta\bar{\beta} + \gamma\bar{\gamma}. \tag{2}$$

If $\alpha \neq 0$, then K is proportional to $\begin{pmatrix} 1 & \beta \\ \gamma & \beta\gamma \end{pmatrix}$, and by equation (2), $\beta\gamma + \bar{\beta}\bar{\gamma} = |\beta|^2 + |\gamma|^2$, so $|\bar{\beta} - \gamma|^2 = 0$ and $\gamma = \bar{\beta}$. If $\beta = 0$, K is as in the claim, and otherwise, the claim follows since

$$\begin{pmatrix} 1 & 0 \\ -1 & 1/\bar\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ \bar\beta & \beta\bar\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1/\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\alpha = 0$, then by equation (2), $\beta = \gamma = 0$, and K is proportional to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which is also congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, using $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To see that this normal form for K is not congruent to the other rank 1 matrix from the above case 4, where $\beta = 0$, suppose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix} = \begin{pmatrix} a\bar a & a\bar c \\ \bar a c & c\bar c \end{pmatrix} = \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix},$$

for some $\zeta \in \mathbb{C}$. This would imply $a = c = 0$, contradicting the requirement that A is invertible. $\square$

The classification from Theorem 4.3 can be interpreted geometrically. In terms of Theorem 4.1, the rank 2 case of c-equivalence splits into infinitely many r-equivalence classes, and the rank 1 case breaks up into two r-equivalence classes. The off-diagonal rank 1 case, where $\beta = 0$ in the matrix $\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$, is in the same r-equivalence class as the stereographic projection from Example 2.4, where $\ker(P)$ intersects the image of s, but not the image of $s \circ \Delta$.

Example 4.4. The other rank 1 case from the theorem is $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and a matrix P, such that K spans $k(\ker(P))$, is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The parametric map is of the form

$$P \circ s \circ \Delta : [z_0 : z_1] \mapsto [z_0\bar z_1 : z_1\bar z_0 : z_1\bar z_1],$$

differing from the stereographic case in that the image $s \circ \Delta(\mathbb{C}P^1)$ meets the kernel of P, so $[1 : 0]$ is not in the domain of $P \circ s \circ \Delta$. This map is one-to-one with domain $\mathbb{C}$, $[z : 1] \mapsto [z : \bar z : 1]$, and its image is a totally real plane in an affine neighborhood, $\{[Z_0 : Z_1 : Z_2] : Z_2 = 1, Z_1 = \bar Z_0\}$.

Example 4.5. The isolated rank 2 case, $K = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$, corresponds to a matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & i \end{pmatrix},$$

and a parametric map of the form

$$P \circ s \circ \Delta : [z_0 : z_1] \mapsto [z_0\bar z_0 : z_0\bar z_1 - z_1\bar z_0 : z_1\bar z_0 + iz_1\bar z_1].$$

The map is defined for all $[z_0 : z_1]$, and restricting to the affine neighborhood $[1 : z]$ in the domain gives $z \mapsto [1 : \bar{z} - z : z + iz\bar{z}]$, or a real map

$$(x, y) \mapsto (X, Y, Z) = (-2iy, x, y + x^2 + y^2) \in (i \cdot \mathbb{R}) \times \mathbb{R}^2 \subseteq \mathbb{C}^2.$$

The image of this restriction is a paraboloid in three real dimensions, with no tangent planes parallel to the complex line $X = 0$. The set $\Delta(\mathbb{C}P^1)$ meets the singular locus of $P \circ s$ at only one point, $\Delta([0 : 1])$. Considering the restriction of $P \circ s \circ \Delta$ to the neighborhood $[z : 1]$, checking its (real) Jacobian shows that it drops rank at $[0 : 1]$. $P \circ s \circ \Delta$ is not an immersion at that point, and the map's image does not have a well-defined tangent plane at the singular value $[0 : 0 : 1]$. It is possible to choose a different representative coefficient matrix, Q, equal to $C \cdot P$ for some automorphism C,

$$Q = \begin{pmatrix} 1 & 1 & 1 & 2i \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}$$

so that the image of $Q \circ s \circ \Delta$ is contained in the $Z_0 \neq 0$ affine neighborhood. Note that

$$z_0\bar{z}_0 + z_0\bar{z}_1 + z_1\bar{z}_0 + 2iz_1\bar{z}_1 = |z_0 + z_1|^2 + (-1 + 2i)|z_1|^2$$

is complex-valued but never 0.

Example 4.6. The matrices $\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$, $0 \leq \beta < 1$, correspond to inequivalent embeddings of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. Representative matrices are

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & -2\beta & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

which define representative parametric maps:

$$[z_0 : z_1] \mapsto [z_0\bar{z}_0 + z_1\bar{z}_1 : 2z_0\bar{z}_1 - 2\beta z_1\bar{z}_0 : z_1\bar{z}_1 - z_0\bar{z}_0].$$

The $\beta = 0$ case is a parametrization of a sphere in a real hyperplane inside the $Z_0 = 1$ affine neighborhood, and is r-equivalent to the stereographic projection from Example 2.4. For any β, $(P \circ s \circ \Delta)([0 : 1]) = [1 : 0 : 1]$. Using coordinates $z = x + iy$ on the $z_0 = 1$ neighborhood in the domain, and considering the $Z_0 = 1$ neighborhood in the target, $P \circ s \circ \Delta$ restricts to

$$\begin{aligned} [1 : z] &\mapsto [1 + z\bar{z} : 2\bar{z} - 2\beta z : z\bar{z} - 1] \\ (x, y) &\mapsto (X, Y, Z) = \left(\frac{2x(1-\beta)}{1 + x^2 + y^2}, \frac{-2y(1+\beta)}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}\right). \end{aligned}$$

The image $(P \circ s \circ \Delta)(\mathbb{C}P^1)$ is an ellipsoid,

$$\left(\frac{X}{1-\beta}\right)^2 + \left(\frac{Y}{1+\beta}\right)^2 + Z^2 = 1.$$

Solving for Z defines two hemispheres in the ellipsoid, each as a graph over the XY-plane,

$$\begin{aligned} Z &= \pm\sqrt{1-\frac{X^2}{(1-\beta)^2}-\frac{Y^2}{(1+\beta)^2}} \\ &= \pm(1-\frac{X^2}{2(1-\beta)^2}-\frac{Y^2}{2(1+\beta)^2}+O(3)) \\ &= \pm(1-\frac{1+\beta^2}{2(1-\beta^2)^2}(X^2+Y^2+\frac{2\beta}{1+\beta^2}(X^2-Y^2))+O(3)) \\ z_2 &= \pm 1 \mp \frac{1+\beta^2}{2(1-\beta^2)^2}(z_1\bar{z}_1+\frac{\beta}{1+\beta^2}(z_1^2+\bar{z}_1^2))+O(3). \end{aligned}$$

The coefficient $\frac{\beta}{1+\beta^2}$ has values in $[0,\frac{1}{2})$ for $\beta\in[0,1)$. It is Bishop's quadratic invariant for elliptic complex tangents ([3]).

Example 4.7. A special case of the diagonal normal form is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is congruent to the $\beta = 1$ case of the previous example. Geometrically it is the $\beta \to 1^-$ limit of the ellipsoids, which deflate into a closed, elliptical disc contained in a real 2-plane. A representative coefficient matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and the parametric map

$$[z_0 : z_1] \mapsto [z_0\bar{z}_0 + z_1\bar{z}_1 : z_0\bar{z}_1 : z_1\bar{z}_0]$$

is two-to-one except for a circular singular locus. The image is contained in the totally real plane $Z_2 = \bar{Z}_1$ inside the affine neighborhood $Z_0 = 1$.

Example 4.8. The Hermitian matrix $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is also a special case, where $\alpha = 1$ in Theorem 4.3. Considering a coefficient matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the parametric map

$$[z_0 : z_1] \mapsto [z_0\bar{z}_0 - z_1\bar{z}_1 : z_0\bar{z}_1 : z_1\bar{z}_0]$$

is a two-to-one submersion, where antipodal points are identified:

$$(P\circ s\circ\Delta)([\bar{z}_1 : -\bar{z}_0]) = [z_1\bar{z}_1 - z_0\bar{z}_0 : -z_0\bar{z}_1 : -z_1\bar{z}_0] = (P\circ s\circ\Delta)([z_0 : z_1]).$$

The image is a (totally real) real projective plane in $\mathbb{C}P^2$.

Example 4.9. The only remaining cases from Theorem 4.3 are the diagonal matrices with $\alpha = \cos(\theta) + i\sin(\theta)$, $0 < \theta < \pi$. For each α, a representative coefficient matrix is

$$P = \begin{pmatrix} -\alpha & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{i}{2} & \frac{i}{2} & 0 \end{pmatrix},$$

which defines a parametric map

$$P \circ s \circ \Delta : [z_0 : z_1] \mapsto [z_1\bar{z}_1 - \alpha z_0\bar{z}_0 : \frac{1}{2}(z_0\bar{z}_1 + z_1\bar{z}_0) : \frac{i}{2}(-z_0\bar{z}_1 + z_1\bar{z}_0)].$$

The points $[0 : 1]$ and $[1 : 0]$ both have image $[1 : 0 : 0]$, but otherwise the map is one-to-one. The singular locus of $P \circ s$ is the set $\{([z_0 : z_1], [w_0 : w_1]) : z_1w_1 + \alpha z_0w_0 = 0\}$, which does not meet the image of Δ. The composition $P \circ s \circ \Delta$ is an immersion with one double point, and the image is totally real in $\mathbb{C}P^2$, and contained in the $Z_0 \neq 0$ affine neighborhood. Restricting to the $[1 : z]$ neighborhood in the domain, with $z = x + iy$, $\alpha = a + bi$, defines a parametric map $\mathbb{R}^2 \to \mathbb{R}^4$, with target coordinates $Z_1 = x_1 + iy_1$, $Z_2 = x_2 + iy_2$:

$$\begin{aligned} z &\mapsto (Z_1, Z_2) = (\frac{z + \bar{z}}{2(z\bar{z} - \alpha)}, \frac{\bar{z} - z}{2i(z\bar{z} - \alpha)}) \\ (x, y) &\mapsto (x_1, y_1, x_2, y_2) \\ &= \frac{1}{(x^2 + y^2 - a)^2 + b^2}(x^3 + xy^2 - xa, xb, -yx^2 - y^3 + ya, -yb). \end{aligned}$$

The image in $\mathbb{C}^2 = \mathbb{R}^4$ is exactly the common zero locus V_α of the following real polynomials:

$$y_1x_2 - x_1y_2 \tag{3}$$
$$b^2(x_1^2x_2^2 + 2x_1^2y_2^2 + y_1^2y_2^2 + (x_2^2 + y_2^2)^2) - bx_2y_2 - ay_2^2 \tag{4}$$
$$b^2((x_1^2 + y_1^2)^2 - (x_2^2 + y_2^2)^2) - bx_1y_1 + bx_2y_2 + ay_2^2 - ay_1^2. \tag{5}$$

All three equations are necessary to define V_α, for example, the zero locus of just (3), (4) is $V_\alpha \cup \{x_2 = y_2 = 0\}$. As $b \to 0^+$, the affine variety V_α approaches the totally real plane $\{y_1 = y_2 = 0\}$, and the two limiting cases $\alpha = \pm 1$ were described in the previous two examples. At the double point, the tangent cone is the union of two totally real planes, $\{y_1 = y_2 = 0\} \cup \{bx_1 + ay_1 = bx_2 + ay_2 = 0\}$. Totally real spheres with a single point of self-intersection in $\mathbb{C}^2$ have also been considered in [21] and [2]. Pairs of totally real subspaces (M, N) which meet only at the origin have been considered by D. Burns and [22]. The pair appearing in this example is $N = \mathbb{R}^2$, with coordinates x_1, x_2, and $M = (A + i)\mathbb{R}^2$, where

$$A = \begin{pmatrix} -\frac{a}{b} & 0 \\ 0 & -\frac{a}{b} \end{pmatrix}.$$

A $\mathbb{C}$-linear transformation of $\mathbb{C}^2$ which fixes $N = \mathbb{R}^2$ has a matrix representation S with real entries, and transforms A into SAS^{-1}; the quantity $-a/b = \frac{1}{2}Tr(A)$ is clearly a similarity invariant.

As in Example 4.5, there is complex affine neighborhood in which part of the image is a real quadric in a real hyperplane. Setting $Z_1 = 1$ gives the parametrization

$$\begin{aligned}[1 : x+iy] &\mapsto [x^2+y^2-a-ib : x : y] \\ (X,Y,Z) &= (\frac{x^2+y^2-a}{x}, \frac{-b}{x}, \frac{y}{x}).\end{aligned}$$

The implicit equation in (X,Y,Z) is

$$bXY - aY^2 + b^2Z^2 + b^2 = 0,$$

which is a two-sheeted hyperboloid for $b > 0$.

To summarize, the r-equivalence class of a coefficient matrix P can be recognized by inspecting the image of the map $P \circ s \circ \Delta : \mathbb{C}P^1 \to \mathbb{C}P^2$. The r-equivalence classes are represented by the following cases, starting with the two rank 1 cases.

- $P \circ s \circ \Delta$ is the stereographic projection map, where the kernel of P is spanned by a rank one matrix, and $P \circ s \circ \Delta$ is defined for all points in $\mathbb{C}P^1$.
- The image is a totally real affine plane, where the kernel of P is spanned by a rank one matrix, and $P \circ s \circ \Delta$ is not defined at one of the points of $\mathbb{C}P^1$.
- $P \circ s \circ \Delta$ is singular at one point, and is totally real away from this point.
- $\mathbb{C}P^1$ is embedded in $\mathbb{C}P^2$. There are two elliptic complex tangents, with the same Bishop invariant $\gamma = \frac{\beta}{1+\beta^2}$. γ can attain any value in the interval $(0, \frac{1}{2})$. (The $\gamma = 0$ case is the stereographic sphere.)
- The image is a disc contained in a totally real plane, and $P \circ s \circ \Delta$ is two-to-one, except along a singular curve.
- $P \circ s \circ \Delta$ is two-to-one, and its image is a real projective plane.
- The image is a totally real immersed sphere with one point of self-intersection. A parameter $-a/b$, determined by the tangent planes at that point, can attain any real value and classifies such maps up to r-equivalence.

This section concludes with two remarks on Theorem 4.3.

It is interesting that in the two cases with continuous parameters, inequivalent immersions can be easily distinguished by finding holomorphic invariants in the coefficients of the defining functions of the images near the exceptional points. In higher codimensions, $2m < n$, the situation will be different, since it was observed in [6] that the nondegenerate complex tangents are "stable," with no continuous invariants under formal biholomorphic transformations.

The $0 \le \beta < 1$ matrices of the theorem are congruent to symmetric matrices:

$$\begin{aligned}\begin{pmatrix} 1 & i \\ i\frac{1+\beta}{1-\beta} & \frac{1+\beta}{1-\beta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} 1 & -i\frac{1+\beta}{1-\beta} \\ -i & \frac{1+\beta}{1-\beta} \end{pmatrix} &\propto \begin{pmatrix} -i\left(\frac{1-\beta}{1+\beta}\right)^2 & 1 \\ 1 & i \end{pmatrix} \\ &= \begin{pmatrix} -it^2 & 1 \\ 1 & i \end{pmatrix},\end{aligned}$$

with $0 < t \le 1$. This shows that every r-equivalence class of 3×4 matrices has a representative P so that $k(\ker(P))$ is spanned by a complex symmetric matrix. It also shows that the classification of 2×2 pencils in Theorem 4.3 gives exactly the same results as a classification of [20] of complex quadratic forms up to real congruence.

5. $\mathbb{C}P^2$ in $\mathbb{C}P^5$

By the codimension calculation from Example 3.2, the next pair (m, n) where complex tangents are expected to be isolated is $m = 2$, $n = 5$. In contrast to Theorem 4.1, there are infinitely many c-equivalence classes; some naïve counting will suggest that the dimension of the parameter space exceeds the dimension of the group acting on it. By Theorem 2.3, the c-equivalence problem is equivalent to classifying three-dimensional subspaces K of $M(3, \mathbb{C})$, under the action $K \mapsto B_{3\times 3} K A^T_{3\times 3}$. The r-equivalence problem, or the classification of K up to the congruence of Theorem 2.8, seems to be even more difficult.

Rather than attempt higher-dimensional analogues of Theorems 4.1 or 4.3, this final section will consider just a few examples, and scrutinize only the following simple one in detail.

Example 5.1. Consider the following coefficient matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1+i & 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}_{6\times 9}.$$

It is intentionally rather sparse, to simplify some calculations, and the non-zero entries play specific roles, as follows. The top row is chosen so that $P \circ s \circ \Delta$ will have an image contained in the $Z_0 \neq 0$ neighborhood. Deleting the top row and the first, middle, and last columns leaves a 5×6 submatrix, in row-echelon form so that P has rank 6 and $\ker(P)$ is a 3-dimensional subspace of $\mathbb{C}^9$. Its last column (the eighth of nine in P) is chosen so that $k(\ker(P))$, which is the following subspace of $M(3, \mathbb{C})$:

$$\{\begin{pmatrix} (1+i)c_1 + ic_3 & 0 & -c_2 \\ c_2 & -c_1 & c_2 \\ 0 & 0 & -c_3 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C}\},$$

contains no matrices of rank 1, and so $P \circ s$ is defined for all $(z, w) \in \mathbb{C}P^2 \times \mathbb{C}P^2$.

The goal of this example is to show that this choice of P defines an immersion of $\mathbb{C}P^2$ in an affine neighborhood of $\mathbb{C}P^5$, which has exactly three complex tangents. The computations were initially carried out using MAPLE software ([19]), but the following paragraphs will outline the main steps in human-readable format. This immersion will be rather peculiar in that it is not a one-to-one mapping, which is unexpected, considering the high codimension.

The composition $P \circ s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^5$ is defined for all of $\mathbb{C}P^2$. By inspection of the parametric map taking $[z_0 : z_1 : z_2]$ to:

$$[z_0\bar{z}_0 + (1+i)z_1\bar{z}_1 + iz_2\bar{z}_2 : (z_0 - z_2)\bar{z}_1 : z_0\bar{z}_2 : z_1\bar{z}_0 : z_1\bar{z}_2 : z_2(\bar{z}_0 + \bar{z}_1)],$$

the image of $P \circ s \circ \Delta$ does not meet the $Z_0 = 0$ hyperplane. (This is as in Examples 4.5 and 4.9, where the first component is not real-valued, but it doesn't vanish for any $(z_0, z_1, z_2) \neq \vec{0}$.)

The singular locus of $P \circ s$ is a complex algebraic subvariety of the domain $\mathbb{C}P^2 \times \mathbb{C}P^2$. In order to find its intersection with the image of Δ, it will be enough to check the Jacobian matrix of $P \circ s$, considered as a map $\mathbb{C}^4 \to \mathbb{C}^5$ when it is restricted to three of the nine affine charts in the domain, and the $Z_0 \neq 0$ chart in the target. For example, the restriction of $P \circ s$ to the $z_0 = 1$, $w_0 = 1$ neighborhood defines a map

$$(z_1, z_2, w_1, w_2) \mapsto \left(\frac{P_1(z,w)}{P_0(z,w)}, \dots, \frac{P_5(z,w)}{P_0(z,w)}\right). \tag{6}$$

The locus where the rank drops is the common zero locus of five 4×4 determinants, which will be inhomogeneous rational functions in z_1, z_2, w_1, w_2. Since the image of Δ does not meet the zero locus of the denominators (which are powers of P_0), it is enough to consider the numerators of these rational functions, and re-introduce z_0 and w_0 to get five bihomogeneous polynomials which define a subset of $\{(z,w) \in \mathbb{C}P^2 \times \mathbb{C}P^2 : z_0 \neq 0, w_0 \neq 0, P_0(z,w) \neq 0\}$. Repeating this procedure for the other charts in the domain will give other subsets of $\mathbb{C}P^2 \times \mathbb{C}P^2$, but with significant overlaps, and which satisfy the same bihomogeneous polynomial equations. According to MAPLE, these polynomials are:

$$z_1 w_2 (z_2 w_2 + (i-1) z_1 w_0 + (i-1) z_1 w_1) \tag{7}$$
$$z_1 w_2 (z_2 w_2 - z_0 w_2 + (i-1) z_1 w_1) \tag{8}$$
$$z_1 (z_2 w_1 w_2 + i z_2 w_0^2 + (i-1) z_1 w_1 (w_0 + w_1) - i z_0 w_0 w_1 - i w_0^2 z_0) \tag{9}$$
$$w_2 (z_2^2 w_2 - z_0 z_2 w_2 + i z_0 z_2 w_0 + (i-1) z_1 z_2 w_1 - i z_0^2 w_0 - i z_0^2 w_1) \tag{10}$$
$$z_0 ((i-1) z_1 w_1 - i z_0 w_0)(w_0 + w_1) + z_2 w_0 (z_2 w_2 - z_0 w_2 + i z_0 w_0). \tag{11}$$

The real diagonal image of Δ, $[w_0 : w_1 : w_2] = [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$, meets this locus in a real algebraic variety, which (again, according to MAPLE) consists of exactly three points: $x_1 = \Delta([1:0:1])$, $x_2 = \Delta([1:-1:0])$, and $x_3 = \Delta([1:i:1-i])$.

Since getting an exact count of the number of complex jump points is the important part of this example, and since computations such as finding all the solutions of a system of polynomial equations should be checked by hand whenever possible, the following calculations will verify MAPLE's claim. First, it is easy to check that these three points are in the common zero locus of equations (7)–(11), and are indeed elements of the singular locus of $P \circ s$.

Second, suppose there is some $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$ with $z_0 = 0$ and $z_1 \neq 0$, $z_2 \neq 0$, whose image under Δ satisfies equation (7), so that

$$z_1 \bar{z}_2 (z_2 \bar{z}_2 + (i-1) z_1 \bar{z}_1) = 0.$$

However, none of the three factors vanishes, so there are no such points in the singular locus.

The next case is where $z_1 = 0$. Any point $\Delta([z_0 : 0 : z_2])$ satisfies (7)–(9), and (10) then implies

$$\begin{aligned} &\bar{z}_2 (z_2^2 \bar{z}_2 - z_0 z_2 \bar{z}_2 + i z_0 z_2 \bar{z}_0 - i z_0^2 \bar{z}_0) \\ &= \bar{z}_2 (z_2 - z_0)(z_2 \bar{z}_2 + i z_0 \bar{z}_0) = 0, \end{aligned}$$

where the last factor is nonzero, and the only solutions are $z_0 = z_2$, which gives the point x_1, or $z_1 = z_2 = 0$, in which case (11) would imply z_0 is also zero.

The next case is $z_2 = 0$, $z_1 \neq 0$, so that (11) becomes

$$z_0(\bar{z}_0 + \bar{z}_1)((i-1)z_1\bar{z}_1 - iz_0\bar{z}_0) = 0.$$

As in the previous case, the last factor is nonzero, one of the solutions is $z_1 = -z_0$, which gives x_2, and $z_0 = z_2 = 0$ in (9) would imply $z_1 = 0$.

Finally, the remaining case is that all three projective coordinates are nonzero, so that z_0 can be assumed to be 1, and subtracting (8) from (7) implies $\bar{z}_2 + (i-1)z_1 = 0$. Plugging $z_0 = 1$ and $z_2 = (1+i)\bar{z}_1$ into (8) gives

$$z_1(1-i)z_1(-(1-i)z_1 + (1+i)\bar{z}_1(1-i)z_1 + (i-1)z_1\bar{z}_1) = 0,$$

where the only nonzero solution is $z_1 = i$, which gives the point x_3.

The images of the three points, x_1, x_2, x_3, under $P \circ s$ are

$$X_1 = [1 : 0 : \frac{1-i}{2} : 0 : 0 : \frac{1-i}{2}],$$

$$X_2 = [1 : \frac{i-2}{5} : 0 : \frac{i-2}{5} : 0 : 0],$$

$$X_3 = [1 : \frac{2-3i}{13} : \frac{5-i}{13} : \frac{3+2i}{13} : \frac{1+5i}{13} : \frac{-6-4i}{13}],$$

which are the candidates for complex jump points in the image of $\mathbb{C}P^2$. (They are also candidates for differential-topological singularities, as in Example 4.5.)

The real tangent planes at these points are found by considering the restriction of $P \circ s \circ \Delta$ to the $z_0 = 1$ affine neighborhood, so that $P \circ s \circ \Delta : \mathbb{R}^4 \to \mathbb{R}^{10}$ is given by

$$(z_1, z_2) \mapsto (\mathrm{Re}(\frac{P_1}{P_0}), \mathrm{Im}(\frac{P_1}{P_0}), \ldots, \mathrm{Im}(\frac{P_5}{P_0})).$$

At each point z in the domain, there is a real 10×4 Jacobian matrix D_z of derivatives, whose image is a four-dimensional subspace T_z of $\mathbb{R}^{10}$.

It turns out (according to calculations left to MAPLE) that at each point x_1, x_2, x_3, the real Jacobian matrix has full rank. This is enough to prove that $P \circ s \circ \Delta$ is an immersion (although it could also be checked directly that the real Jacobian has full rank at every point).

In $\mathbb{C}^5$, the scalar multiplication map $\vec{v} \mapsto i \cdot \vec{v}$ is real-linear, and induces a complex structure operator J on $\mathbb{R}^{10}$, which is a 10×10 block matrix with five $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ blocks on the diagonal. The concatenation of D_z with $J \cdot \mathrm{D}_z$ results in a 10×8 matrix, which maps $\mathbb{R}^8$ to $\mathbb{R}^{10}$ so that the image subspace is the sum $T_z + JT_z$. At the totally real points z, where T_z and JT_z meet only at the origin, $T_z + JT_z$ is 8-dimensional. At the three points x_1, x_2, x_3, it can be calculated that the 10×8 matrix has rank 6, which proves that x_1, x_2, x_3 are not "exceptionally exceptional," that is, none of the tangent spaces is a complex 2-plane, but instead each contains exactly one complex line. In the notation of Section 3, $N_1 = \{x_1, x_2, x_3\}$, and $N_2 = \emptyset$.

To illustrate the idea, the procedure for finding T_z will be recorded here only for $z = x_1$.

$$\mathrm{D}_{x_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}_{10\times 4}$$

has rank 4, but $[\mathrm{D}_{x_1}, J \cdot \mathrm{D}_{x_1}]_{10\times 8}$ has rank 6. A basis for its kernel is $\{(0, -1, 1, -1, 1, 0, 0, 0)^T$, $(1, 0, 0, 0, 0, 1, -1, 1)^T\}$, and the following equation shows that the image of D_{x_1} contains a J-invariant subspace:

$$\mathrm{D}_{x_1} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = J \cdot \mathrm{D}_{x_1} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = J \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

The span of $(0, 0, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$ and its image under J is the complex line $\{Z_1 = Z_2 = Z_3 - Z_4 = Z_3 - Z_5 = 0\}$. The same calculations for x_2 and x_3 yield different complex lines tangent to $\mathbb{C}P^2$ in $\mathbb{C}^5$.

It remains to check that $P \circ s \circ \Delta$ is one-to-one except for a triple point. The following calculation is similar to that of [23]. First, consider those points in the image such that all three domain coordinates, z_0, z_1, z_2, are nonzero, so that $P \circ s \circ \Delta$ restricts to a map:

$$[z_0 : z_1 : 1] \mapsto [\frac{P_0}{z_1} : \frac{(z_0 - 1)\bar{z}_1}{z_1} : \frac{z_0}{z_1} : \bar{z}_0 : 1 : \frac{\bar{z}_0 + \bar{z}_1}{z_1}].$$

This map is clearly one-to-one from $\{(z_0, z_1) \in \mathbb{C}^2 : z_0 \neq 0, z_1 \neq 0\}$ to $\mathbb{C}^5$. Points on the line $\{[0 : z_1 : 1]\}$ are mapped to

$$[(1 + i)z_1 z_1 + i : -\bar{z}_1 : 0 : 0 : z_1 : \bar{z}_1],$$

and points on the line $\{[z_0 : 0 : 1]\}$ are mapped to

$$[z_0\bar{z}_0 + i : 0 : z_0 : 0 : 0 : \bar{z}_0];$$

both of these restrictions are also one-to-one, with images disjoint from the previous image (and each other, except at their point of intersection in the domain, $[0:0:1]$). Another line in the domain is $\{[1:z_1:0]\}$, whose points are mapped to

$$[1+(1+i)z_1\bar{z}_1 : \bar{z}_1 : 0 : z_1 : 0 : 0].$$

This restriction is also one-to-one, but its image meets the previous images when $z_1 = 0$. The only remaining point in the domain is $[0:1:0]$, whose image is $[1:0:0:0:0:0]$, which is the same as the image of $[1:0:0]$ and $[0:0:1]$. In fact, $P \circ s \circ \Delta$ maps the complex projective lines $\{[z_0:0:z_2]\}$ and $\{[z_0:z_1:0]\}$ in the domain into two-dimensional complex subspaces in the range, falling into the Example 4.9 case of the classification from Theorem 4.3.

Example 5.2. Consider the following family of coefficient matrices, as a perturbation of the previous example by varying two of the entries:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1+i & 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 & t & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & t \end{pmatrix}.$$

The $t = 0$ case is the previous example, where there are three complex tangents, and a single triple point. The two changed entries contribute $z_1 w_1$ and $z_2 w_2$ terms to the numerators of the map (6). It is expected that for t close to zero, the perturbed immersion will still have exactly three complex tangents. For example, at $t = -1/2$, the diagonal $\Delta(\mathbb{C}P^2)$ intersects the singular set of $P \circ s$ at $x_1 = \Delta([1:0:1])$, $x_2 = \Delta([1:-1:0])$, and $x_3 = \Delta([\frac{i}{2} : -i : 1])$. Also, the triple point breaks up into two double points: both domain points $[\frac{1}{4} + (-2 \pm \frac{3}{4}\sqrt{7})i : 0 : 1]$ are mapped to the same image under $P \circ s \circ \Delta$, and similarly, $[1 : \frac{7+\sqrt{47}}{8} + \frac{-9-\sqrt{47}}{8}i : 0]$ and $[1 : \frac{7-\sqrt{47}}{8} + \frac{-9+\sqrt{47}}{8}i : 0]$ have the same image.

At $t = -1$, there are five complex tangents (and again two double points). Topologically, the two new jump points forming in this homotopy are expected to have opposite orientation indices, so that the index sum of [9] and [4] is still 3. It would be interesting to understand the local geometry of this "pair creation." Two concepts which might be useful analogies are the homotopical construction of [11] for surfaces in $\mathbb{C}^2$ with complex tangents, and the cancellation of Whitney cross-cap singularities, described as a "confluence of umbrellas" in [1].

Example 5.3. Consider the following coefficient matrix, as a perturbation of the $t = -1/2$ case of the previous example.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1+i & 0 & 0 & 0 & i \\ -1 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -1 & \frac{1}{9} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} \end{pmatrix}.$$

The new entries contribute more $z_i\bar{z}_i$ terms to the quadratic polynomials in the parametrization. The composite map $(P \circ s \circ \Delta)([z_0 : z_1 : z_2])$ is:

$$\begin{aligned}[z_0\bar{z}_0 + (1+i)z_1\bar{z}_1 + iz_2\bar{z}_2 \ &: \ (z_0 - \frac{1}{2}z_1 - z_2)\bar{z}_1 - z_0\bar{z}_0 + \frac{1}{9}z_2\bar{z}_2 \\ &: \ z_0\bar{z}_2 \\ &: \ z_1\bar{z}_0 \\ &: \ z_1\bar{z}_2 \\ &: \ z_2(\bar{z}_0 + \bar{z}_1 - \frac{1}{2}\bar{z}_2) - \frac{3}{2}z_0\bar{z}_0].\end{aligned}$$

This is an embedding with exactly three complex tangents. The diagonal $\Delta(\mathbb{C}P^2)$ intersects the singular set of $P \circ s$ at $x_1 = \Delta([1 : 0 : 3])$, $x_2 = \Delta([1 : 2 : 0])$, and $x_3 = \Delta([9 + 28i : -18 - 63i : 54 - 30i])$. The new coefficients were chosen so these points would have Gaussian integer coordinates, since the rank calculations, as in Example 5.1, require exact arithmetic. The check that it is one-to-one is also as in Example 5.1.

While embeddings with an odd number of complex tangents are the generic case, images of quadratic maps $\mathbb{C}P^2 \to \mathbb{C}P^5$ can exhibit some unusual behavior. The first two examples showed there can be isolated double or triple points. The next two will demonstrate some different unstable geometric properties, as well as algebraic degeneracy in the rank of $\ker P$. The diversity of CR singular and topologically singular phenomena perhaps speaks to the complexity of the problem of classifying subspaces of $M(d, \mathbb{C})$ up to congruence.

Example 5.4. The first five rows of the matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

are the coefficients of Whitney's embedding of $\mathbb{C}P^2$ in $\mathbb{C}^4$, from Example 2.5. Adding the last row makes a rank 6 matrix, and defines a map $P \circ s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^5$ taking $[z_0 : z_1 : z_2]$ to:

$$[z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 : z_1\bar{z}_2 : z_2\bar{z}_0 : z_0\bar{z}_1 : z_0\bar{z}_0 - z_1\bar{z}_1 : z_2\bar{z}_1].$$

The kernel's image under k is the set

$$\left\{ \begin{pmatrix} c_1 & c_3 & 0 \\ 0 & c_1 & 0 \\ c_2 & 0 & -2c_1 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

The matrices in this subspace with rank ≤ 1 form exactly two lines, where $c_1 = c_2 = 0$, or $c_1 = c_3 = 0$. So, $P \circ s$ is not defined at the points $x_0 = ([0 : 1 : 0], [1 : 0 : 0])$ or $x_1 = ([1 : 0 : 0], [0 : 0 : 1])$. The image of $P \circ s \circ \Delta$ is contained in the the 7-dimensional real subspace $\{Z_4 = \bar{Z}_4, Z_5 = \bar{Z}_1\}$ of the $Z_0 = 1$ affine neighborhood. (However, the image is not contained in any complex hyperplane.) $P \circ s \circ \Delta$ is a one-to-one immersion, since it is a smooth graph over Whitney's example. The real diagonal image of Δ meets the singular locus of $P \circ s$ at exactly three points, $x_2 = \Delta([1 : 0 : 0])$, $x_3 = \Delta([0 : 1 : 0])$, and $x_4 = \Delta([0 : 0 : 1])$. Their images under $P \circ s$ are

$$\begin{aligned} X_2 &= [1 : 0 : 0 : 0 : 1 : 0], \\ X_3 &= [1 : 0 : 0 : 0 : -1 : 0], \\ X_4 &= [1 : 0 : 0 : 0 : 0 : 0]. \end{aligned}$$

At X_3, the real tangent 4-plane is $\{Z_2 = Z_4 = 0, Z_5 = \bar{Z}_1\}$, which contains the Z_3-axis. At X_4, the tangent 4-plane is $\{Z_3 = Z_4 = 0, Z_5 = \bar{Z}_1\}$, which contains the Z_2-axis. The unusual point is X_2, where the tangent space is the complex 2-plane $\{Z_1 = Z_4 = Z_5 = 0\}$, which, by the codimension formula for complex tangents, is a topologically unstable phenomenon; this submanifold is not in general position.

It is worth pointing out that for the map from Example 5.3, the image in $\mathbb{C}^5$ is not contained in any real hyperplane (it is easy to pick one point in the image to translate to the origin, and then pick 10 more points with $\mathbb{R}$-independent coordinates), so the embedding from Example 5.4 is unusually flat.

Example 5.5. The following coefficient matrix has the same top row as the previous example, and contains the identity matrix as a 6×6 submatrix.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The kernel's image under k is the set

$$\{\begin{pmatrix} -c_3 - c_2 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & c_1 & c_3 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C}\}.$$

The matrices in this subspace with rank ≤ 1 form exactly one line, where $c_2 = c_3 = 0$. The composition $P \circ s \circ \Delta$ defines the parametric map

$$[z_0 : z_1 : z_2] \mapsto [\bar{z}_0 z_0 + \bar{z}_1 z_1 + \bar{z}_2 z_2 : \bar{z}_1 z_0 : \bar{z}_2 z_0 : \bar{z}_0 z_1 : \bar{z}_0 z_2 : \bar{z}_1 z_2].$$

The image is contained in the 6-dimensional real subspace $\{Z_1 = \bar{Z}_3, Z_2 = \bar{Z}_4\}$ of the $Z_0 = 1$ affine neighborhood. It is known (see [8]) that $\mathbb{C}P^2$ cannot be immersed in $\mathbb{R}^6$, so this map will have a non-empty singular locus, which generically, for smooth maps from a 4-manifold to $\mathbb{R}^6$, is expected to be one-dimensional. In this case, the domain $\mathbb{C}P^2$ contains two spheres which are mapped onto flat discs, as in Example 4.7:

$$\begin{aligned} [z_0 : 0 : z_2] &\mapsto [z_0 \bar{z}_0 + z_2 \bar{z}_2 : 0 : z_0 \bar{z}_2 : 0 : z_2 \bar{z}_0 : 0], \\ [z_0 : z_1 : 0] &\mapsto [z_0 \bar{z}_0 + z_1 \bar{z}_1 : z_0 \bar{z}_1 : 0 : z_1 \bar{z}_0 : 0 : 0]. \end{aligned}$$

Finally, we recall a construction by G. Mannoury (circa 1898), of an embedding of $\mathbb{C}P^2$ ([17], [18]).

Example 5.6. The map

$$\begin{aligned}[z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 \quad &: \quad \sqrt{2}z_0\bar{z}_0 : \sqrt{2}z_1\bar{z}_1 : \sqrt{2}z_2\bar{z}_2 \\ &: \quad z_1\bar{z}_2 + z_2\bar{z}_1 : i(z_1\bar{z}_2 - z_2\bar{z}_1) \\ &: \quad z_2\bar{z}_0 + z_0\bar{z}_2 : i(z_2\bar{z}_0 - z_0\bar{z}_2) \\ &: \quad z_0\bar{z}_1 + z_1\bar{z}_0 : i(z_0\bar{z}_1 - z_1\bar{z}_0)]\end{aligned}$$

has ten components, so it could be defined as $P \circ s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^9$, using a 10×9 coefficient matrix P, composed with the map $s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^8$. In this case, P is not a projection, but a linear inclusion, and it would be easy to construct a 9×10 matrix Q so that $Q \circ P$ is the identity. So, all the previous examples in this section are projections of this embedding.

The image is contained in the intersection of the $\{Z_0 = 1\}$ affine neighborhood in $\mathbb{C}P^9$ and the real projective space $\mathbb{R}P^9$, since all the components are real. Restricting the target to the affine neighborhood $\{X_0 = 1\}$ in $\mathbb{R}P^9$ gives Mannoury's embedding of $\mathbb{C}P^2$ in $\mathbb{R}^9$, with image contained in the intersection of the real hyperplane $X_1 + X_2 + X_3 = \sqrt{2}$ and the hypersphere with center $\vec{0}$ and radius $\sqrt{2}$.

Some of the calculations omitted from this section, and some of the unpublished papers in the references, are available from the author's web site, www.ipfw.edu/math/Coffman/.

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