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Squaring the Circle by Dissection

To Professor Johannes Böhm on the occasion of his 75th birthday

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Abstract. The set-theoretical circle-squaring problem goes back to Tarski: Can a circle be partitioned into sets that can be reassembled to form a square? We give a short survey on results to this question and add new claims concerning “scissor congruence” of circle and square with respect to particular affine transformations.

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1. Equidecomposability and congruence by dissection

The classical form of the circle-squaring problem of the ancient Greek geometers, to construct a square with the same area as a given circle only with a straightedge and a compass, had been solved negatively in the 19th century. But a new view on the old problem was opened by Tarski [7]: Can a circle be partitioned into sets that can be reassembled to form a square (having the same area)? The answer to this set-theoretical question depends on the particular types of partitions and on the groups of transformations which are allowed in the piecewise congruence of circle and square.

Given two sets $A, B \subseteq \mathbb{R}^d$ and a group $\mathcal{G}$ of bijections of $\mathbb{R}^d$, A and B are called *equidecomposable with respect to* $\mathcal{G}$ if A and B can each be partitioned into the same finite number of respectively $\mathcal{G}$-congruent pieces. Formally,

$$A = \bigcup_{i=1}^{n} A_i \qquad \text{and} \qquad B = \bigcup_{i=1}^{n} B_i,$$

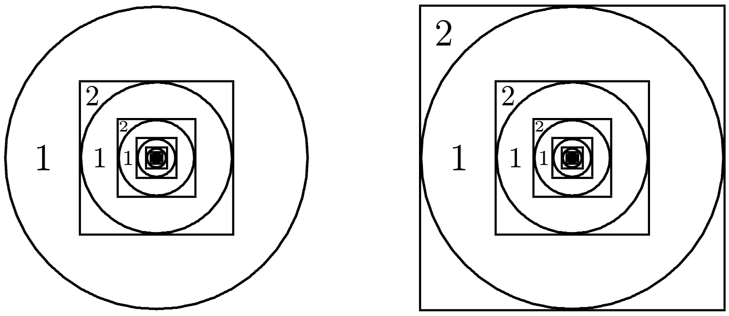

Figure 1. Equidecomposability via dyadic homotheties

$A_i \cap A_j = \emptyset = B_i \cap B_j$ if $1 \leq i < j \leq n$, and there are $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{G}$ such that, for each $1 \leq i \leq n$, $\gamma_i(A_i) = B_i$. Similarly, A and B are called *countably equidecomposable with respect to* $\mathcal{G}$ if the same applies to countable partitions of A and B.

Theorem 1. [2] *Any two sets $A, B \subseteq \mathbb{R}^d$ with nonempty interior are countably equidecomposable with respect to the group of translations.*

This gives a first positive solution of the set-theoretical circle-squaring problem. It was one of the most surprising discoveries in the last decade that a similar decomposition result can be gained using finitely many pieces only.

Theorem 2. [3] *Let $C, S \subseteq \mathbb{R}^2$ be a circle and a square, respectively, of the same area. Then C and S are equidecomposable with respect to the group of translations.*

In his paper [4] Laczkovich extended the circle-squaring result to pairs of measurable sets $A, B \subseteq \mathbb{R}^d$ which have the same positive Lebesgue measure and fulfil a weak boundary condition. In addition to that, the well-known paradox of Banach, Tarski, and Hausdorff even can be used for manipulating the volume of subsets of $\mathbb{R}^d$, provided that $d \geq 3$.

Theorem 3. [8] *Any two bounded sets $A, B \subseteq \mathbb{R}^d$, $d \geq 3$, with nonempty interior are equidecomposable with respect to the group of proper Euclidean motions.*

The previous claims rest on deep investigations using the axiom of choice. The situation becomes essentially easier if the group of transformations contains contractive maps. A very small group of that type consists of the *dyadic homotheties*. It is generated by the translations and one central dilatation with similarity coefficient 2. (In the following the coefficient 2 does not play an important role. One can replace it by any other real number larger than 1.)

Proposition 4. (cf. Corollary 3.7 from [8]) *Any two sets $A, B \subseteq \mathbb{R}^d$, each of which is bounded and has nonempty interior, are equidecomposable with respect to the group of dyadic homotheties. More precicely, there exist decompositions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $\alpha_1(A_1) = B_1$ and $\alpha_2(A_2) = B_2$ with suitable dyadic homotheties α_1, α_2.*

Proof. We choose dyadic homotheties α and β such that $\alpha(A) \subseteq B$ and $\beta(B) \subseteq A$. Then we put $A_1 = \bigcup_{i=0}^{\infty} (\beta\alpha)^i (A \setminus \beta(B))$ and $A_2 = A \setminus A_1$. One easily computes that this setting gives rise to the decompositions $A = A_1 \cup A_2$ and $B = \alpha(A_1) \cup \beta^{-1}(A_2)$. (Figure 1 illustrates the situation for a circle and a square in the Euclidean plane.) $\square$

Theorems 1–3 sound paradoxical to everybody who thinks of partitions of bodies in a physical sense. Now we replace the concept of partitions into disjoint subsets by dissections which allow to ignore boundary points. We say that *the set $A \subseteq \mathbb{R}^d$ is dissected into the subsets* $A_1, A_2, \ldots, A_n$, in symbols $A = A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n$, if $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $\mathrm{int}(A_i) \cap \mathrm{int}(A_j) = \emptyset$ for $1 \le i < j \le n$. Given two sets $A, B \subseteq \mathbb{R}^d$ and a group $\mathcal{G}$ of bijections of $\mathbb{R}^d$, A and B are called *congruent by dissection with respect to the group $\mathcal{G}$* if there exist dissections

$$A = \bigsqcup_{i=1}^{n} A_i, \qquad B = \bigsqcup_{i=1}^{n} B_i,$$

and transformations $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{G}$ such that $\gamma_i(A_i) = B_i$ for $1 \le i \le n$. In this case the n sets $A_1, A_2, \ldots, A_n$ (or $B_1, B_2, \ldots, B_n$, respectively) are called *the pieces of dissection*.

If we do not impose geometrical restrictions on the pieces of dissection, the concept of congruence by dissection generalizes equidecomposability. In this full generality it is far from its physical background and leads to degenerate results.

Proposition 5. *Any two bounded sets $A, B \subseteq \mathbb{R}^d$ with nonempty interior are congruent by dissection with respect to the group of translations if the pieces of dissection are allowed to be arbitrary subsets of $\mathbb{R}^d$.*

Proof. Clearly, there exist coverings $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $B = B_1 \cup B_2 \cup \ldots \cup B_n$ and translations $\tau_1, \tau_2, \ldots, \tau_n$ such that $\tau_i(A_i) = B_i$ if $1 \le i \le n$. We put $A_{i,1} = A_i \cap \mathbb{Q}^d$ and $A_{i,2} = A_i \setminus A_{i,1}$, $1 \le i \le n$. Then all sets $A_{i,j}$ and $\tau_i(A_{i,j})$ have empty interior. Hence we obtain the desired dissections

$$A = \bigsqcup_{\substack{i=1,2,\ldots,n \\ j=1,2}} A_{i,j} \qquad \text{and} \qquad B = \bigsqcup_{\substack{i=1,2,\ldots,n \\ j=1,2}} \tau_i\left(A_{i,j}\right). \qquad \square$$

Congruence by dissection makes sense only if the pieces of dissection have a small boundary. A tool for quantifying the size of the boundary $\mathrm{bd}(A)$ of a set $A \subseteq \mathbb{R}^d$ is the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$. We say that the boundary of A is *rectifiable* if $\mathcal{H}^{d-1}(\mathrm{bd}(A)) < \infty$. The proof of Proposition 4 gives rise to the following conclusion, which in particular can be used for squaring the circle by dissection.

Corollary 6. *Let $A, B \subseteq \mathbb{R}^d$ be bounded sets with nonempty interior and rectifiable boundary. Then A and B are congruent by dissection with respect to the group of dyadic hometheties using two pieces with rectifiable boundary.*

Now we come back to the two-dimensional case. In the following we demand the pieces of dissection to be closed topological discs. These are images of the closed unit circle in $\mathbb{R}^2$ under homeomorphisms of $\mathbb{R}^2$. Of course, circle and square are congruent with respect to homeomorphisms of $\mathbb{R}^2$. We want to discuss congruence by dissection using different types of affine transformations.

Theorem 7. [1] *Let C and S be a circle and a square in $\mathbb{R}^2$, respectively. Then C and S are not congruent by dissection with respect to the group of Euclidean motions using closed topological discs as pieces of dissection.*

In contrast with that, larger groups of transformations give rise to positive results, even if the pieces of dissection underly additional smoothness conditions.

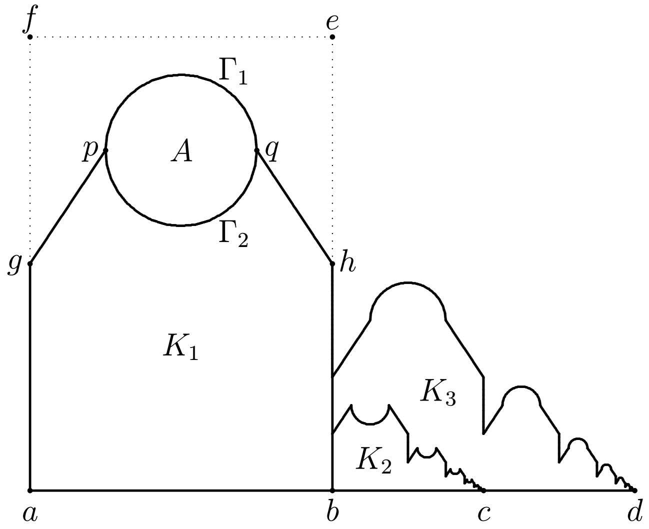

Figure 2. $K = A \sqcup K_1 \sqcup K_2 \sqcup K_3$

2. Dissections into pieces with rectifiable boundary

Theorem 8. *Any two closed topological discs $A, B \subseteq \mathbb{R}^2$ with rectifiable boundary are congruent by dissection with respect to the group of dyadic homotheties using ten closed topological discs with rectifiable boundary as pieces of dissection.*

Proof. First we show that there exist closed topological discs K, K_1, K_2, K_3 with rectifiable boundary and dyadic homotheties $\alpha_1, \alpha_2, \alpha_3$ such that

$$K = A \sqcup K_1 \sqcup K_2 \sqcup K_3 = \alpha_1(K_1) \sqcup \alpha_2(K_2) \sqcup \alpha_3(K_3) \tag{1}$$

(see Figure 2).

We choose a rectangle $\operatorname{conv}(\{a, b, e, f\})$ such that the edge ab is parallel to the first coordinate axis and the set A is contained in the interior of the "upper half". That is, $A \subseteq \operatorname{int}(\operatorname{conv}(\{e, f, g, h\}))$ where $g = \frac{a+f}{2}$ and $h = \frac{b+e}{2}$ are the centres of the edges af and be, respectively. Let p and q be points from $\operatorname{bd}(A)$ with minimal and maximal first coordinate, respectively. Then $\operatorname{bd}(A)$ splits into an "upper arc" Γ_1 and a "lower arc" Γ_2, both connecting p and q. K_1 is to denote the closed topological disc whose boundary consists of Γ_2 and the polygonal chain $pgabhq$. Let $c = \frac{3}{2}b - \frac{1}{2}a$ and $d = 2b - a$, let τ be the translation mapping a onto b, and let α, γ, δ be dilatations with centres a, c, d, respectively, and with similarity coefficient 2. We put $K_2 = \{c\} \cup \bigcup_{i=0}^{\infty} \gamma^{-i}\tau\alpha^{-2}(K_1)$ and $K_3 = \operatorname{cl}\left(\bigcup_{i=1}^{\infty} \delta^{-i}(K_1 \cup A) \setminus K_2\right)$. Both sets are closed topological discs with rectifiable boundary. Finally, $K = A \cup K_1 \cup K_2 \cup K_3$ is a disc of that type as well. One easily verifies that K is dissected as claimed under (1), where $\alpha_1 = \alpha^{-1}$, $\alpha_2 = \delta\gamma^{-1}$, and $\alpha_3 = \delta$.

Now we fix a dyadic homothety ρ such that $\rho(K) \subseteq \operatorname{int}(B)$. Then $\operatorname{cl}(B \setminus \rho(K))$ can be dissected into two closed topological discs K_4 and K_5, both having a rectifiable boundary. Hence

$$B = \rho(K) \sqcup K_4 \sqcup K_5.$$

Exchanging the roles of A and B, we find closed topological discs $L, L_1, L_2, \dots, L_5$ with rectifiable boundary and dyadic homotheties $\beta_1, \beta_2, \beta_3, \sigma$ such that

$$L = B \sqcup L_1 \sqcup L_2 \sqcup L_3 = \beta_1(L_1) \sqcup \beta_2(L_2) \sqcup \beta_3(L_3) \qquad \text{and} \qquad A = \sigma(L) \sqcup L_4 \sqcup L_5.$$

Combining the equations, we obtain

$$\begin{aligned} A &= \sigma(L) \sqcup L_4 \sqcup L_5 \\ &= \sigma(B \sqcup L_1 \sqcup L_2 \sqcup L_3) \sqcup L_4 \sqcup L_5 \\ &= \sigma(\rho(K) \sqcup K_4 \sqcup K_5 \sqcup L_1 \sqcup L_2 \sqcup L_3) \sqcup L_4 \sqcup L_5 \\ &= \sigma(\rho(\alpha_1(K_1) \sqcup \alpha_2(K_2) \sqcup \alpha_3(K_3)) \sqcup K_4 \sqcup K_5 \sqcup L_1 \sqcup L_2 \sqcup L_3) \sqcup L_4 \sqcup L_5 \end{aligned}$$

and, similarly,

$$B = \rho(\sigma(\beta_1(L_1) \sqcup \beta_2(L_2) \sqcup \beta_3(L_3)) \sqcup L_4 \sqcup L_5 \sqcup K_1 \sqcup K_2 \sqcup K_3) \sqcup K_4 \sqcup K_5.$$

This proves Theorem 8. □

The above construction also can be done with sets A and B not having rectifiable boundaries. Of course, then the pieces of dissection do not fulfil the boundary condition, too.

Corollary 9. *Any two closed topological discs $A, B \subseteq \mathbb{R}^2$ are congruent by dissection with respect to the group of dyadic homotheties using ten closed topological discs as pieces of dissection.*

Clearly, Theorem 8 applies to the circle-squaring problem. However, in the particular case of squaring the circle considerations of the above type can be refined so that only four pieces of dissection appear. We do not describe the details. Figure 3 shows corresponding dissections.

Proposition 10. *Any circle C and any square S in $\mathbb{R}^2$ are congruent by dissection with respect to the group of dyadic homotheties using four closed topological discs with rectifiable boundary as pieces of dissection.*

3. Dissections into pieces with smooth boundary

Given a rectifiable Jordan arc $\Gamma = \{x(t) : 0 \le t \le 1\} \subseteq \mathbb{R}^2$, we say that Γ is *twice continuously differentiable* if, for all points $x \in \Gamma$, the tangent vector and its derivative (with respect to the arclength) exist and depend continuously on x. The curve Γ is called *convex* if it is part of the boundary of a convex body in $\mathbb{R}^2$.

The boundary $\mathrm{bd}(A)$ of a topological disc $A \subseteq \mathbb{R}^2$ is to be called *of type* $C^2 \vee c$ if it splits into finitely many Jordan arcs Γ_i, $1 \le i \le n$, such that every arc Γ_i is twice continuously differentiable *or* convex. Similarly, $\mathrm{bd}(A)$ is to be called *of type* $C^2 \wedge c$ if it consists of finitely many arcs Γ_i, all being twice continuously differentiable *and* convex.

Theorem 11. *Let C and S be a circle and a square in $\mathbb{R}^2$, respectively. Then C and S are congruent by dissection neither with respect to the group of similarities nor with respect to the group of equiaffine transformations if the pieces of dissection are restricted to be closed topological discs with boundary of type $C^2 \vee c$.*

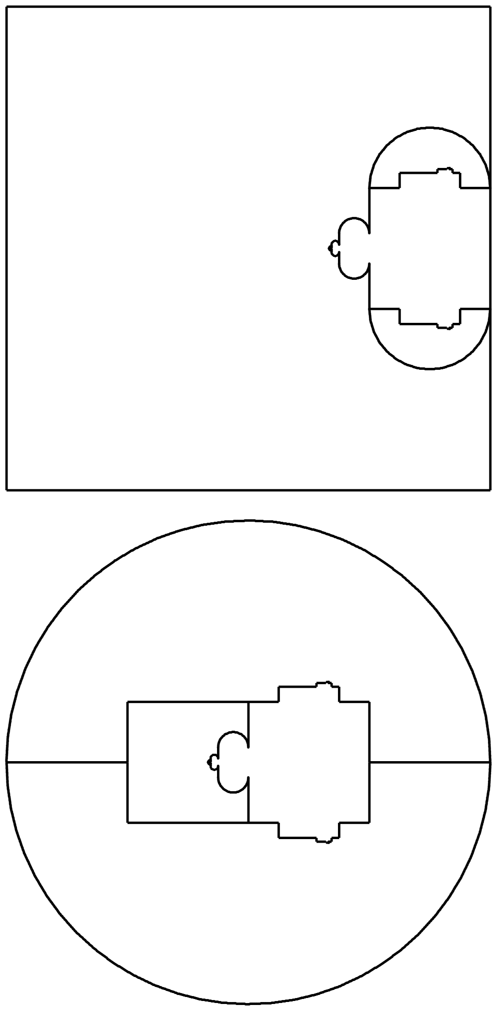

Figure 3. Dissections of a circle and a square

Proof. The proof is motivated by an idea from [1]. We want to show the negative results by the aid of two invariant valuations μ_1 and μ_2 on the family of all closed topological discs fulfilling the above boundary condition.

Let $A \subseteq \mathbb{R}^2$ be a topological disc of that type. For a point $x \in \mathrm{bd}(A)$, $\kappa(x)$ is to denote the curvature of $\mathrm{bd}(A)$ at x, provided that $\mathrm{bd}(A)$ is twice differentiable at x. We define

$$\kappa_1(x) = \begin{cases} \kappa(x) & \text{if, for some neighbourhood } U \text{ of } x,\ \mathrm{bd}(A) \cap U \text{ is a circular arc,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\kappa_2(x) = \begin{cases} \kappa(x) & \text{if, for some neighbourhood } U \text{ of } x,\ \mathrm{bd}(A) \cap U \text{ is an ellipsoidal arc,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if, for some neighbourhood } U \text{ of } x,\ \mathrm{conv}(\mathrm{bd}(A) \cap U) \subseteq A, \\ -1 & \text{otherwise.} \end{cases}$$

First we consider the quantity

$$\mu_1(A) = \int_{\mathrm{bd}(A)} \mathrm{sgn}(x)\kappa_1(x)\, d\mathcal{H}^1(x).$$

The integral is well defined. Indeed, it suffices to show that $\int_\Gamma \kappa_1(x) d\mathcal{H}^1(x) < \infty$ for every arc $\Gamma \subseteq \mathrm{bd}(A)$ which is twice continuously differentiable or convex. If the differentiability condition is fulfilled, then $\kappa : \Gamma \to \mathbb{R}$ is bounded from above by some value $\kappa_{\max}$, since κ is continuous on the compact set Γ. Hence $\int_\Gamma \kappa_1(x) d\mathcal{H}^1(x) \le \kappa_{\max}\mathcal{H}^1(\Gamma) < \infty$. If Γ is convex then $\int_\Gamma \kappa_1(x) d\mathcal{H}^1(x) \le 2\pi$, because the integral $\int_{\Gamma_0} \kappa_1(x) d\mathcal{H}^1(x)$ over a circular subarc $\Gamma_0 \subseteq \Gamma$ agrees with the angle between the normal vectors in the endpoints of Γ_0.

Obviously, μ_1 is invariant under similarities. One easily verifies the additivity of μ_1. That is, $\mu_1(A \sqcup B) = \mu_1(A) + \mu_1(B)$ if A, B, and $A \sqcup B$ are closed topological discs with boundary of type $C^2 \vee c$. By induction, this yields $\mu_1(A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n) = \mu_1(A_1) + \mu_1(A_2) + \ldots + \mu_1(A_n)$, provided that the sets $A_1, A_2, \ldots, A_n$ as well as $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n$ are closed topological discs with boundary of type $C^2 \vee c$. Hence $\mu_1(C) = \mu_1(S)$ would be a necessary condition for congruence by dissection of C and S with respect to similarities using pieces of the desired type. But we have $\mu_1(C) = 2\pi$ and $\mu_1(S) = 0$. This proves the first part of Theorem 11.

Now we define

$$\mu_2(A) = \int_{\mathrm{bd}(A)} \mathrm{sgn}(x)\kappa_2(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x).$$

If an arc $\Gamma \subseteq \mathrm{bd}(A)$ is twice continuously differentiable then $\int_\Gamma \kappa_2(x)^{\frac{1}{3}} d\mathcal{H}^1(x) \le \kappa_{\max}^{\frac{1}{3}}\mathcal{H}^1(\Gamma)$. In the case of a convex arc Γ we estimate $\int_\Gamma \kappa_2(x)^{\frac{1}{3}} d\mathcal{H}^1(x) \le \int_\Gamma \kappa(x)^{\frac{1}{3}} d\mathcal{H}^1(x)$. The last integral is called the *affine arclength* of Γ. Although the curvature $\kappa(x)$ needs to exist almost everywhere only, the affine arclength of Γ is well defined and finite, since it can be expressed as an infimum of a nonempty set of reals (see [6]). This justifies the definition of $\mu_2(A)$.

The functional μ_2 is invariant under equiaffine transformations, because the differential $\kappa(x)^{\frac{1}{3}} d\mathcal{H}^1(x)$ is an equiaffine invariant (see [5], [6]). As above, μ_2 is additive in the sense that $\mu_2(A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n) = \mu_2(A_1) + \mu_2(A_2) + \ldots + \mu_2(A_n)$ if $A_1, A_2, \ldots, A_n$, and $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n$ are closed topological discs with boundary of type $C^2 \vee c$. Accordingly, we would have $\mu_2(C) = \mu_2(S)$ if C and S would be congruent by dissection with respect to equiaffine transformations using pieces fulfilling the boundary condition $C^2 \vee c$. This condition fails, for $\mu_2(S) = 0$, whereas $\mu_2(C) = 2\pi r^{\frac{2}{3}}$, r denoting the radius of C. □

Theorem 12. *Any circle C and any square S in $\mathbb{R}^2$ are congruent by dissection with respect to the group of affine transformations using closed topological discs with boundary of type $C^2 \wedge c$ as pieces of dissection.*

Proof. The circular arc $\Gamma_0 = \left\{ \left(\frac{\sqrt{2}}{2} + \cos(\varphi), \frac{9\sqrt{2}+4\sqrt{6}}{22} + \sin(\varphi) \right) : \frac{19\pi}{12} \le \varphi \le \frac{21\pi}{12} \right\}$ of radius 1 and length $\frac{\pi}{6}$ connects the points $x_0 = (\xi_{0,1}, \xi_{0,2}) = \left(\sqrt{2}, \frac{-2\sqrt{2}+4\sqrt{6}}{22} \right)$ and $x_1 = (\xi_{1,1}, \xi_{1,2}) = \left(\frac{\sqrt{2}+\sqrt{6}}{4}, \frac{7\sqrt{2}-3\sqrt{6}}{44} \right)$ (see Figure 4). The affine contraction $\alpha(\xi_1, \xi_2) = \left(\frac{1+\sqrt{3}}{4}\xi_1, \frac{-1+\sqrt{3}}{4}\xi_2 \right)$ maps x_0 onto x_1. Hence $\Gamma = \{a\} \cup \bigcup_{i=0}^{\infty} \alpha^i(\Gamma_0)$ is a rectifiable Jordan arc between x_0 and $a = (0,0)$. Straightforward (but lengthy) calculations show that the first two derivatives of Γ_0 and the

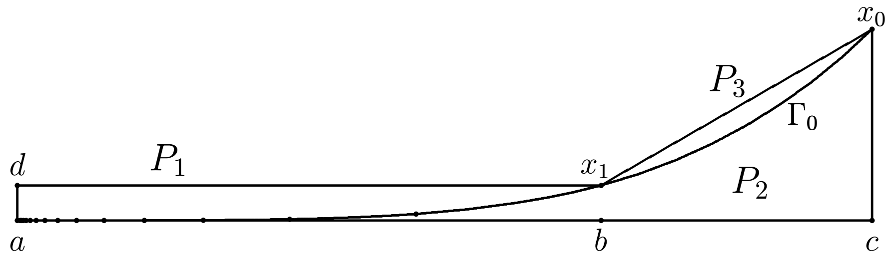

Figure 4. $P = P_1 \sqcup P_2 \sqcup P_3$

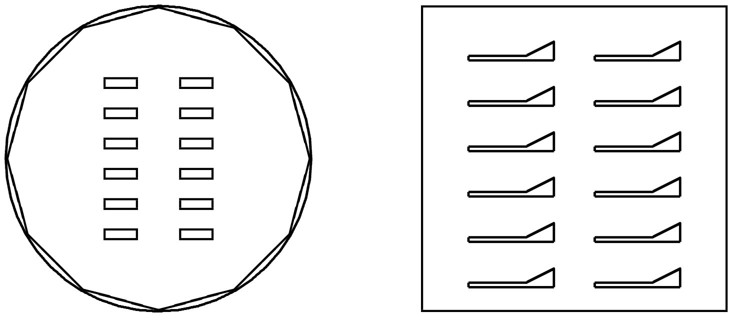

Figure 5.

ellipsoidal arc $\Gamma_1 = \alpha(\Gamma_0)$ coincide at the common endpoint x_1. Repeated application of this argument implies that the arc Γ is convex and twice differentiable at all points except for a. Finally one verifies that, if a point $x \in \Gamma$ approaches a, then the corresponding tangent approaches the first coordinate axis and the curvature tends to 0. Hence the whole arc Γ is convex and twice differentiable.

Let $b = (\xi_{1,1}, 0)$, $c = (\xi_{0,1}, 0)$, and $d = (0, \xi_{1,2})$. The polygon P with vertices a, c, x_0, x_1, d can be dissected into three closed topological discs:

$$P = P_1 \sqcup P_2 \sqcup P_3. \tag{2}$$

P_1 is bounded by the line segments ad, dx_1 and the arc $\mathrm{cl}(\Gamma \setminus \Gamma_0)$. The boundary of P_2 consists of ac, cx_0, and Γ. The remaining piece is $P_3 = \mathrm{conv}(\Gamma_0)$. Of course, the boundaries of P_1, P_2, and P_3 are of type $C^2 \wedge c$.

The affine transformation α maps ac onto ab, cx_0 onto bx_1, and Γ onto $\mathrm{cl}(\Gamma \setminus \Gamma_0)$. Hence the rectangle $\mathrm{conv}(\{a, b, x_1, d\})$ can be written as

$$\mathrm{conv}(\{a, b, x_1, d\}) = P_1 \sqcup \alpha(P_2). \tag{3}$$

Equations (2) and (3) yield the following: given any affine image Q of $P_3 = \mathrm{conv}(\Gamma_0)$ and any rectangle R with $\mathrm{int}(Q) \cap \mathrm{int}(R) = \emptyset$, the polygon P and $Q \sqcup R$ are congruent by dissection with respect to affine transformations using pieces as desired in Theorem 12.

We dissect the circle C into $C = C_1 \sqcup C_2$, C_1 consisting of 12 affine copies of $\mathrm{conv}(\Gamma_0)$ and 12 rectangles, and the square S into $S = S_1 \sqcup S_2$, where S_1 is formed by 12 affine images of P (see Figure 5). The above arguments show that C_1 and S_1 are congruent by dissection. The remaining polygonal regions C_2 and S_2 are congruent by dissection, too. Indeed, C_2 and S_2

can each be partitioned into the same finite number of triangles, which clearly are congruent under affine transformations. Hence C and S are congruent by dissection as claimed in Theorem 12. □

4. Concluding remarks

The above considerations belong to a wide field of problems. Given a group of transformations of $\mathbb{R}^2$, are a circle and a square, or two members form a larger class of sets, congruent by dissection with respect to that group (using pieces with small boundary)? If so, what geometric restrictions can be imposed on the pieces such that congruence by dissection is still possible? Moreover, one may ask for dissections with minimal number of pieces. Are the estimates given in Theorem 8 and Proposition 10 sharp? What is the minimal number of pieces in the claim of Theorem 12?

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