Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 2, 493-498.

On k^+-neighbour Packings and One-sided Hadwiger Configurations

Károly Bezdek[1] **Peter Brass**[2]

Department of Geometry, Eötvös University
H-1117 Budapest, Pázmány Péter sétány 1/c
e-mail: kbezdek@ludens.elte.hu

Department of Computer Science, The City College, CUNY
138th Street at Convent Avenue, New York NY-10031, USA
e-mail: peter@cs.ccny.cuny.edu

Abstract. We show that in d-dimensional Euclidean space the maximum number of non-overlapping translates of a d-dimensional convex body K that can touch K and can lie in a closed supporting half-space of K is always at most $2\cdot 3^{d-1}-1$, with this bound to be reached only if K is an affine image of a d-cube. Such one-sided Hadwiger configurations occur at the boundary of finite packings or near the holes in packings of density 0, so this implies that in d-dimensional Euclidean space any k^+-neighbour packing by translates of a d-dimensional convex body has positive density for all $k \geq 2\cdot 3^{d-1}$; and there is a $(2\cdot 3^{d-1}-1)^+$-neighbour packing by translates of a d-cube that has density 0. (A packing is called a k^+-neighbour packing if each packing element has at least k neighbours.) This answers an old question of L. Fejes Tóth (1973).

1. Introduction

All packings considered in this note will be packings by translates of a convex body (i.e. of a compact convex set with nonempty interior) in d-dimensional Euclidean space with $d \geq 2$. Packings with restrictions on the number of neighbours of each body were studied in a number of papers, see e.g. [6, section 7] for a survey. By a k^+-neighbour packing we mean a packing by translates in which each translate touches at least k other translates of the packing. k^+-neighbour packings of a given convex body C are of course only possible if k is at most the Hadwiger number of C, which is the maximum number of neighbours

[1] supported by the Hung. Nat. Sci. Found. (OTKA), grant no. T029786 and T037752
[2] research done is supported by DFG grant BR 1465/5-2

0138-4821/93 $ 2.50

of C in any packing by translates of C. For several special sets C, and numbers k, it has been studied whether there are finite k^+-neighbour packings, or infinite packings of this kind of density 0, or whether any k^+-neighbour packing has positive density. It is known that in the plane, for any C that is not a parallelogram, there are finite 3^+-neighbour packings, that any 4^+-neighbour packing is infinite but possibly of density 0, and that any 5^+-neighbour packing has positive density (this follows, e.g., quite simply from the angular measure constructed in [2], in which each interior angle of the graph of touching pairs is at least $\frac{\pi}{3}$). For parallelograms there are finite 4^+-neighbour packings, 5^+-neighbour packings of density 0. Moreover, any 6^+-neighbour packing has positive density. Perhaps it is not surprising that in the plane also the minimum density of 5^+-neighbour packings is known [10]. In 3-space, these questions were studied for the ball. There is a 6-neighbour packing of 240 congruent balls constructed by G. Wegner (see [6]), but the existence of finite 7^+ and 8^+-neighbour packings is open. It is known that any 9^+-neighbour packing of congruent balls has to be infinite [9], but density 0 is possible, and any 10^+-neighbour packing has to be of positive density [4], [11], [1]. Also the minimum number of balls in such a finite k^+-neighbour packing was studied [5].

For several of the above mentioned theorems one has to show that a k^+-neighbour packing of translates of a convex body C has positive density. This is done by proving that it is impossible to have all k neighbours of C 'on one side' of C. For if the packing has density 0, then there must be arbitrary large spherical 'holes' in the packing. More exactly, for any $r > 0$ there must be a ball B of radius larger than or equal to r with the property that B does not intersect any of the translates of C in the packing but, it touches at least one translate of C in the packing say, C itself. Then, by compactness, it must be possible to choose k translates $(C+t_i)_{i=1}^k$ of C and one hyperplane h such that each of the $C+t_i$ touches C, they form a packing, and they are all contained in one closed half-space h^+ bounded by h, and C touches h. Equivalently, there are non-overlapping translates $C+t_1, \dots, C+t_k$ of C that all touch C and a nonzero vector v such that $\langle v, t_i \rangle \geq 0$ for all i, where $\langle\ \rangle$ denotes the standard inner product of the given Euclidean space. We call such a packing a one-sided Hadwiger configuration, and the maximum k for which such a packing exists the one-sided Hadwiger number of C. So the one-sided Hadwiger number of C is the maximum number of translates of C that can touch C in a packing, if all the translates are contained in a closed supporting half-space of C.

Theorem. *If C is a convex body in d-dimensional Euclidean space, then the one-sided Hadwiger number of C (i.e. the maximum number of non-overlapping translates of C that can touch C and can lie in a closed supporting half-space of C) is at most $2 \cdot 3^{d-1} - 1$. Moreover, this bound is reached only if C is an affine d-cube.*

Corollary. *If C is a d-dimensional convex body, then any k^+-neighbour packing by translates of C with $k \geq 2 \cdot 3^{d-1}$ must have a positive density in d-dimensional Euclidean space. Moreover, there is a $(2 \cdot 3^{d-1} - 1)^+$-neighbour packing by translates of an affine d-cube with density 0 in d-dimensional Euclidean space.*

Remark 1. The Corollary answers an old question of L. Fejes Tóth raised in [3].

2. Proof of the Theorem

Let $C = C + t_0, C + t_1, \ldots, C + t_k$ be a one-sided Hadwiger configuration, with the normal vector v. Note first that we can assume C to be centrally symmetric, replacing it by $\frac{1}{2}(C + (-C))$, which preserves the packing property, touching pairs, and one-sidedness. As in the classical proof for the Hadwiger number [8], we use that $\bigcup_{i=0}^{k}(C + t_i) \subseteq 3C$. Let the whole configuration be scaled such that the normal vector v of the hyperplane $\{p \mid \langle p, v\rangle = -1\}$ touching C and containing the whole arrangement in one halfspace is a unit vector. Then, if $h(x)$ denotes the hyperplane $\{p \mid \langle p, v\rangle = x\}$, C is between $h(-1)$ and $h(1)$ and touches both, and all $C + t_i$ are between $h(-1)$ and $h(3)$. Let $\mathrm{Vol}_{d-1}(\cdot)$ (resp., $\mathrm{Vol}_d(\cdot)$) denote the $(d-1)$-dimensional (resp., d-dimensional) volume measure. Then $\int_{-1}^{1} \mathrm{Vol}_{d-1}(C \cap h(x))dx = \mathrm{Vol}_d(C)$, and

$$\int_{-1}^{3} \mathrm{Vol}_{d-1}\Big(\Big(\bigcup_{i=0}^{k}(C + t_i)\Big) \cap h(x)\Big)dx = (k+1)\mathrm{Vol}_d(C). \tag{1}$$

Next we write the above integral as a sum of two integrals from -1 to 0 and from 0 to 3, and estimate them separately. First, notice that

$$\int_{0}^{3} \mathrm{Vol}_{d-1}\Big(\Big(\bigcup_{i=0}^{k}(C + t_i)\Big) \cap h(x)\Big)dx \le \int_{0}^{3} \mathrm{Vol}_{d-1}\Big(3C \cap h(x)\Big)dx = \frac{1}{2}3^d\mathrm{Vol}_d(C). \tag{2}$$

Second, for the estimation in what follows we need a bound for the intersection volume function $f(x) := \mathrm{Vol}_{d-1}(C \cap h(x-1)), 0 \le x \le 1$. By its definition $f(0) = 0$, f is positive and monotone increasing on $]0, 1]$, and by the Brunn-Minkowski inequality the function $(f(\cdot))^{\frac{1}{d-1}}$ is concave.

Lemma. *If f is a function on $[0, 1]$ with the properties $f(0) = 0$, f positive and monotone increasing on $]0, 1]$, and $f(x) = (g(x))^k$ for some concave function g, then $h(y) := \frac{1}{f(y)}\int_0^y f(x)dx$ is strictly monotone increasing on $]0, 1]$.*

Let now $a_i := \langle v, t_i\rangle$ for $i = 0, \ldots, k$, then the Lemma implies the following:

$$\begin{aligned}
&\int_{-1}^{0} \mathrm{Vol}_{d-1}\Big(\Big(\bigcup_{i=0}^{k}(C + t_i)\Big) \cap h(x)\Big)dx = \sum_{i=0}^{k}\int_{-1}^{0} \mathrm{Vol}_{d-1}\Big((C + t_i) \cap h(x)\Big)dx \\
&= \sum_{i=0}^{k}\int_{0}^{1} \mathrm{Vol}_{d-1}\Big(C \cap (h(x-1) - t_i)\Big)dx = \sum_{i=0}^{k}\int_{0}^{1-a_i} f(x)dx \\
&\le \sum_{i=0}^{k}\int_{0}^{1} f(x)dx\frac{f(1-a_i)}{f(1)} = \int_{0}^{1} f(x)dx\frac{1}{f(1)}\sum_{i=0}^{k} f(1-a_i) \\
&= \int_{0}^{1} f(x)dx\frac{1}{f(1)}\sum_{i=0}^{k} \mathrm{Vol}_{d-1}\Big(C \cap h(-a_i)\Big) = \int_{0}^{1} f(x)dx\frac{1}{f(1)}\sum_{i=0}^{k} \mathrm{Vol}_{d-1}\Big((C + t_i) \cap h(0)\Big)
\end{aligned}$$

$$= \int_0^1 f(x)dx \frac{1}{f(1)} \mathrm{Vol}_{d-1}\Big((\bigcup_{i=0}^{k} (C+t_i)) \cap h(0) \Big) \le \int_0^1 f(x)dx \frac{1}{f(1)} \mathrm{Vol}_{d-1}\Big(3C \cap h(0) \Big)$$
$$= \frac{1}{2}\mathrm{Vol}_d(C) \frac{1}{\mathrm{Vol}_{d-1}(C \cap h(0))} \mathrm{Vol}_{d-1}\Big(3C \cap h(0)\Big) = \frac{1}{2} 3^{d-1} \mathrm{Vol}_d(C).$$

Combining this with equations (1) and (2) we get

$$(k+1)\mathrm{Vol}_d(C) \le \frac{1}{2} 3^d \mathrm{Vol}_d(C) + \frac{1}{2} 3^{d-1} \mathrm{Vol}_d(C),$$

so $k \le 2 \cdot 3^{d-1} - 1$ as claimed in the Theorem.

To prove that equality can only be reached for affine d-cubes, notice first that, by the strict monotonicity in the Lemma, all those a_i that correspond to translates $C + t_i$ contributing nontrivially to the above integral have to be 0. So there are many translates who have their centers in $h(0)$, and only these contribute to $\bigcup_i (C+t_i) \cap h(0)$. Hence these $(d-1)$-dimensional intersections are all translates of $C \cap h(0)$, and in the equality case they have to tile $3C \cap h(0)$, which implies that these intersections are translates of a $(d-1)$-dimensional affine cube [8]. Also there is only the obvious way to tile the $(d-1)$-dimensional affine cube $3C \cap h(0)$ by 3^{d-1} translates of $C \cap h(0)$, so the set of translation vectors $\{t_i \mid t_i \in h(0)\}$ is centrally symmetric. But then the translates

$$\begin{aligned} &\{C + t_i \mid t_i \in h(0)\} \cup \{C + t_i \mid t_i \notin h(0)\} \cup \{C + (-t_i) \mid t_i \notin h(0)\} \\ &= \{C + t_i\} \cup \{C + (-t_i)\} \end{aligned}$$

form a packing that exactly tiles $3C$, so C is a convex body with Hadwiger number $3^d - 1$, thus it is an affine d-cube.

Proof of the Lemma. Without loss of generality we can assume that f is differentiable. To prove that $h(y) := \frac{1}{f(y)} \int_0^y f(x)dx$ is monotone increasing, we have to show that $h' \ge 0$ or equivalently $\int_0^y f(z)dz \le \frac{(f(y))^2}{f'(y)}$. Keep now $y > 0$ fixed.

If $f = g^k$ for some concave g, then there is a linear function $l(z) = (b_1 + b_2(z-y))$ with $b_1 = (f(y))^{\frac{1}{k}}$ and $b_2 = \frac{f'(y)}{k(f(y))^{\frac{k-1}{k}}}$ such that $g(z) \le l(z)$, and so $f(z) \le (l(z))^k$. So for all z we have

$$f(z) \le \left((f(y))^{\frac{1}{k}} + \frac{f'(y)}{k(f(y))^{\frac{k-1}{k}}}(z-y) \right)^k = f(y)\left(1 + \frac{f'(y)}{kf(y)}(z-y)\right)^k.$$

Integrating this we obtain

$$\int_0^y f(z)dz \le \int_0^y f(y)\Big(1 + \frac{f'(y)}{kf(y)}(z-y)\Big)^k dz = \frac{k}{k+1}\,\frac{(f(y))^2}{f'(y)}\left(1 - \Big(1 - \frac{yf'(y)}{kf(y)}\Big)^{k+1}\right).$$

Now since the first factor of the expression on the right is smaller than one, it is sufficient to show that the last factor is also smaller than one. Suppose this is not true, then

$\left(1 - \frac{yf'(y)}{kf(y)}\right)^{k+1} < 0$. Consider now $\left(1 + \frac{f'(y)}{kf(y)}(z-y)\right)^{k+1}$ as function of z: This is positive for $z = y$ and by the assumption negative for $z = 0$, so there must be a $z_0 \in]0, y[$ with $\left(1 + \frac{f'(y)}{kf(y)}(z_0-y)\right)^{k+1} = 0$, but then also $f(z_0) \leq f(y)\left(1 + \frac{f'(y)}{kf(y)}(z_0-y)\right)^{k} = 0$. However, by the assumption of the Lemma we have $f(z_0) > 0$ for all $z_0 \neq 0$, a contradiction. This proves the Lemma.

Remark 2. The key inequality of the proof, $\frac{1}{f(1-a_i)}\int_0^{1-a_i} f(x)dx \leq \frac{1}{f(1)}\int_0^1 f(x)dx$, which is here obtained from the Brunn-Minkowski inequality by the Lemma, can also be deduced in several different ways, using the Schwarz-symmetrization (which reduces to check the inequality for centrally symmetric convex bodies C that are solids of revolution with an additional plane symmetry whose symmetry hyperplane is orthogonal to the axis of revolution), or using the Prekopa-Leindler inequality. We thank K. Swanepoel for pointing out that last possibility.

Remark 3. The Corollary (resp., the Theorem) can be extended in a straightforward way to obtain a generalization of a theorem of L. Fejes Tóth and Sauer [7] on higher-order neighbours in packings. They define in a packing each body as 0-neighbour of itself, and a convex body B as k-neighbour of A if B is neighbour of a $k-1$-neighbour of A, but not of a neighbour of smaller order. Equivalently, B is k-neighbour of A, if the distance of A and B in the graph of touching pairs of the packing is k. Now our extension of the Corollary is as follows. If in a packing of translates of a convex body in d-dimensional Euclidean space each translate has more than $(k+1)\cdot(2k+1)^{d-1}$ j-neighbours with $0 \leq j \leq k$, then the packing must have positive density. The case $k = 1$ is the Corollary proved above. [7] proves this claim only for translates of affine cubes, whereas our result holds for arbitrary translative packings.

Remark 4. We mention the following open problem. Find the smallest positive integer $n(d)$ with the property that if C is an arbitrary convex body in Euclidean d-space, then the maximum number of non-overlapping translates of C that can touch C and can lie in an *open* supporting half-space of C is at most $n(d)$. Obviously, $n(d) < 2\cdot 3^{d-1} - 1$. Moreover, it is clear from the definition of $n(d)$ that any k^+-neighbour packing of translates of a convex body in Euclidean d-space with $k \geq n(d) + 1$ must be an infinite packing.

References

[1] Bezdek, A.; Bezdek, K.: *A note on the ten-neighbour packings of equal balls.* Beiträge Algebra Geom. **27** (1988), 49–53. Zbl 0752.52013

[2] Braß, P.: *Erdős distance problems in normed spaces.* Comput. Geom.: Theory Appl. **6** (1996), 195–214. Zbl 0860.52008

[3] Fejes Tóth, L.: *Five-neighbour packing of convex discs.* Periodica Math. Hung. **4** (1973), 221–229. Zbl 0274.52015

[4] Fejes Tóth, G.: *Ten-neighbour packing of equal balls.* Periodica Math. Hung. **12** (1981), 125–127. Zbl 0438.52014

[5] Fejes Tóth, G.; Harborth, H.: *Kugelpackungen mit vorgegebenen Nachbarnzahlen*. Studia Sci. Math. Hung. **22** (1981), 79–82. Zbl 0564.52007

[6] Fejes Tóth, G.; Kuperberg, W: *A survey of recent results in the theory of packing and covering*. In: New trends in discrete and computational geometry. J. Pach, ed.: Algorithms and Combinatorics Series **10**, Springer-Verlag 1993, 251–279. Zbl 0798.52020

[7] Fejes Tóth, L.; Sauer, N.: *Thinnest packing of cubes with a given number of neighbours*. Canad. Math. Bull. **20** (1977), 501-507. Zbl 0377.52010

[8] Groemer, H.: *Abschätzungen für die Anzahl der konvexen Körper, die einen konvexen Körper berühren*. Monatsh. Math. **65** (1961), 74–81. Zbl 0096.16701

[9] Kertész, G.: *Nine points on the hemisphere*. 'Intuitive geometry', Colloq. Math. Soc. János Bolyai **63** (1994), 189–196. Zbl 0822.52005

[10] Makai, E.: *Five-neighbour packing of convex plates*. In: 'Intuitive geometry', Colloq. Math. Soc. János Bolyai **48** (1987), 373–381. Zbl 0628.52014

[11] Sachs, H.: *No more than nine unit balls can touch a closed unit hemisphere*. Studia Sci. Math. Hung. **21** (1986), 203–206. Zbl 0618.52010

Received April 11, 2001