Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 1, 61-74.

# Pure Submodules of Multiplication Modules

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**Abstract.** The purpose of this paper is to investigate pure submodules of multiplication modules. We introduce the concept of idempotent submodule generalizing idempotent ideal. We show that a submodule of a multiplication module with pure annihilator is pure if and only if it is multiplication and idempotent. Various properties and characterizations of pure submodules of multiplication modules are considered. We also give two descriptions for the trace of a pure submodule of a multiplication module.

MSC 2000: 13C13, 13C05, 13A15
Keywords: multiplication module, pure submodule, idempotent submodule, flat module, projective module, radical of a module, trace of a module

## 0. Introduction

Let $R$ be a ring and $M$ a unital $R$-module. Then $M$ is called a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$, [3], [5] and [7]. An ideal $I$ of $R$ which is a multiplication module is called a multiplication ideal.

Let $R$ be a ring and $K$ and $L$ be submodules of an $R$-module $M$. The residual of $K$ by $L$ is $[K : L]$, the set of all $x$ in $R$ such that $xL \subseteq K$. The annihilator of $M$, denoted by ann$M$, is $[0 : M]$. For each $m \in M$, the annihilator of $m$, denoted by ann$(m)$, is $[0 : Rm]$. $M$ is called faithful if ann$M = 0$.

Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. Then $N = IM$ for some ideal $I$ of $R$. Note that $I \subseteq [N : M]$ and hence $N = IM \subseteq [N : M]M \subseteq N$, so that $N = [N : M]M$. Note also that $K$ is a multiplication submodule of $M$ if and only if $N \cap K = [N : K]K$ for every submodule $N$ of $M$, [18, Lemma 3.1]. Finitely generated faithful multiplication modules are cancellation modules, [20, Theorem 9, Corollary].

We introduce the concept of idempotent submodule as follows. A submodule $N$ of $M$ is an idempotent submodule of $M$ if and only if $N = [N : M]N$. If $M$ is a finitely generated faithful multiplication $R$-module, then $N$ is an idempotent submodule of $M$ if and only if $[N : M]$ is an idempotent ideal of $R$.

Our objective is to investigate pure submodules of multiplication modules. In Section 1 we give several characterizations and properties of such submodules. For example in Theorems 1.1 and 1.4 we prove that a submodule $N$ of a multiplication module $M$ with pure annihilator is pure if and only if $N$ is multiplication and idempotent, equivalently $N_P = 0_P$ or $N_P = M_P$ for every maximal ideal $P$ of $R$. If $M$ is a finitely generated faithful multiplication module, then a submodule $N$ of $M$ is pure if and only if $[N : M]$ is a pure ideal of $R$.

Proposition 1.5 adds assumptions of finite generation and shows that if $N$ is a finitely generated pure submodule of a finitely generated faithful multiplication $R$-module, then $N = eM = eN$ for some idempotent $e$ of $R$. Some results about the radical of a pure submodule of a multiplication module are given in Proposition 1.7.

Let $R$ be a ring and $M$ an $R$-module. Recall that $\mathrm{trace}M = T(M) = \sum_{f\in\mathrm{Hom}(M,R)} f(M)$. In Section 2 we study the trace of pure submodules of multiplication modules. We show (Theorem 2.1) that if $N$ is a pure submodule of a finitely generated faithful multiplication module $M$, then $T(N) = [N : M] = \sum_{a\in N} \mathrm{ann}(\mathrm{ann}(a))$. Using this result we are able to prove that a pure submodule $N$ of a finitely generated faithful multiplication module $M$ is finitely generated if and only if $T(N)$ is a finitely generated ideal of $R$, and pure submodules of finitely generated faithful multiplication modules are torsionless (Corollaries 2.3 and 2.4).

All rings are commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [9]–[13] and [19].

## 1. Pure submodules

Let $R$ be a ring and $M$ an $R$-module. Cohn [8] called a submodule $N$ of $M$ a pure submodule if the sequence $0 \to N \otimes E \to M \otimes E$ is exact for every $R$-module $E$. Anderson and Fuller [6] called the submodule $N$ a pure submodule of $M$ if $IN = N \cap IM$ for every ideal $I$ of $R$. Ribenboim [19] defined $N$ to be pure in $M$ if $rM \cap N = rN$ for each $r \in R$. Although the first condition implies the second [14, p.158] and the second obviously implies the third, these definitions are not equivalent in general, see [14, p.158] for an example. In this paper, our definition of purity will be that of Cohn [8].

Pure submodules have been studied extensively. It is well known that if $M/N$ is flat then $N$ is pure in $M$, [9]. Also pure submodules of flat modules are flat, from which it follows that pure ideals are flat ideals [13]. Any direct summand $N$ of an $R$-module $M$ is pure, while the converse is true if $M/N$ is of finite presentation [16].

It is proved [9, Corollaries 11.21 and 11.23] that if $M$ is a flat module then for any submodule $N$ of $M$ the following are equivalent:

(1) $M/N$ is flat.

(2) $IN = IM \cap N$ for each ideal $I$ of $R$.

(3) $rN = rM \cap N$ for each $r \in R$.

It follows that the three definitions of purity given above are equivalent if $M$ is flat. Thus, for example, the definitions coincide for purity of ideals of $R$. Since multiplication modules with pure annihilators are flat [1, Corollary 2.7], we infer that these definitions are equivalent if $M$ is multiplication and ann$M$ is a pure ideal of $R$, in particular if $M$ is a faithful multiplication module.

We will say that a submodule $N$ of $M$ is idempotent in $M$ if $N = [N : M]N$. It is easy to check that this generalizes the concept of idempotent ideal of $R$, the product of an idempotent ideal and an idempotent submodule is an idempotent submodule, the sum of submodules idempotent in $M$ is idempotent in $M$, and the tensor product of two submodules idempotent in $M$ is idempotent in $M \otimes M$.

Let $M$ be a multiplication module and $N$ a submodule of $M$. If $[N : M]$ is an idempotent ideal, then $N = [N : M]M = [N : M]^2M = [N : M]N$, and $N$ is idempotent in $M$. Conversely, if $M$ is a finitely generated faithful multiplication module and $N$ is idempotent in $M$, then $N = [N : M]M = [N : M]N$, and hence $N = [N : M]^2M = [N : M]M$, which shows that $[N : M]^2 = [N : M]$ is an idempotent ideal.

Our first theorem gives several characterizations of pure submodules of faithful multiplication modules with pure annihilator.

**Theorem 1.1.** *Let $R$ be a ring and $M$ a multiplication $R$-module with pure annihilator. Let $N$ be a submodule of $M$. Then statements* (1) *to* (9) *are equivalent, and further if $M$ is finitely generated and faithful then* (1) *to* (10) *are equivalent.*

(1) *$N$ is a pure submodule of $M$.*

(2) *$N$ is multiplication and is idempotent in $M$.*

(3) *$N$ is multiplication and $K = [N : M]K$ for each submodule $K$ of $N$.*

(4) *$N$ is multiplication and $[K : N]N = [K : M]N$ for each submodule $K$ of $M$.*

(5) *$Rx = [N : M]x$ for each $x \in N$.*

(6) *$R = [N : M] + \mathrm{ann}(x)$ for each $x \in N$.*

(7) *$R = \sum_{n\in N} [Rn : M] + \mathrm{ann}(x)$ for each $x \in N$.*

(8) *For each $x \in N$, there exists $a \in [N : M]$ such that $x = ax$.*

(9) *For each maximal ideal $P$ of $R$ either $N_P = 0_P$ or $N_P = M_P$.*

(10) *$I[N : M] = I \cap [N : M]$ for every ideal $I$ of $R$.*

*Proof.* (1) $\Rightarrow$ (2) Let $K$ be a submodule of $M$. Then $K = [K : M]M$. Since $N$ is a pure submodule of $M$, we infer that

$$\begin{aligned}[K : N]N &= N \cap [K : N]M \supseteq N \cap [K : M]M \\ &= N \cap K \supseteq [K : N]N,\end{aligned}$$

so that $[K : N]N = K \cap N$, and $N$ is multiplication. Since $N$ is pure in $M$, we have that $[N : M]N = N \cap [N : M]M = N$, and hence $N$ is idempotent in $M$.

(2) $\Rightarrow$ (3) Let $K$ be a submodule of $N$. Then

$$K = [K : N]N = [K : N][N : M]N = [N : M]K.$$

(2) $\Rightarrow$ (4) Let $K$ be a submodule of $M$. Then

$$[K : N]N = [K : N][N : M]N \subseteq [K : M]N \subseteq [K : N]N,$$

so that $[K : N]N = [K : M]N$.

(4) $\Rightarrow$ (2) Take $K = N$.

(3) $\Rightarrow$ (7) Let $x \in N$. Then $Rx = [N : M]x$. Since $N$ is multiplication, it follows by [1, Lemma 1.1 (iv)] that

$$Rx = [N : M]x = \left(\sum_{n \in N}[Rn : M]\right)x,$$

and then [20, Corollary to Theorem 9] gives that

$$R = \sum_{n \in N}[Rn : M] + \operatorname{ann}(x).$$

(5) $\Leftrightarrow$ (8) Clear.

(7) $\Rightarrow$ (6) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (9) Let $P$ be any maximal ideal of $R$. We discuss two cases.

Case 1: $[N : M] \subseteq P$. Then for each $x \in N$, $(Rx)_P = [N : M]_P(Rx)_P \subseteq P_P(Rx)_P \subseteq (Rx)_P$, so that $(Rx)_P = P_P(Rx)_P$. By Nakayama's Lemma, $(Rx)_P = 0_P$ and hence $N_P = 0_P$.

Case 2: $[N : M] \not\subseteq P$. Then exists $p \in P$ such that $1 - p \in [N : M]$ and hence $(1 - p)M \subseteq N$. It follows that $M_P = N_P$.

(9) $\Rightarrow$ (1) If $N_P = 0_P$ or $N_P = M_P$ for every maximal ideal of $R$, then for every ideal $I$ of $R$, $IN = N \cap IM$ is true locally and hence globally. Thus $N$ is a pure submodule of $M$.

(1) $\Rightarrow$ (10) Let $M$ be a finitely generated faithful multiplication $R$-module. Let $I$ be any ideal of $R$. Then $IN = N \cap IM$. Hence

$$[IN : M] = [(N \cap IM) : M] = [N : M] \cap [IM : M] = [N : M] \cap I.$$

We need to show that $[IN : M] = I[N : M]$. Obviously, $I[N : M] \subseteq [IN : M]$. Conversely, let $x \in [IN : M]$. Then $xM \subseteq IN = I[N : M]M$. But $M$ is cancellation. Thus $x \in I[N : M]$, and hence $[IN : M] \subseteq I[N : M]M$.

(10) $\Rightarrow$ (2) For this part it is not necessary to assume $M$ is finitely generated or faithful. Assume $I[N : M] = I \cap [N : M]$ for all ideals $I$ of $R$. Take $I = [N : M]$. Then $[N : M]^2 = [N : M]$ and hence $[N : M]$ is an idempotent ideal of $R$. It follows by the remark made before the theorem that $N$ is idempotent in $M$. To prove that $N$ is multiplication, let $K$ be any submodule of $M$. Let $I = [K : M]$. Then

$$[(K \cap N) : M] = [K : M] \cap [N : M] = [K : M][N : M] \subseteq [K : N][N : M],$$

and hence

$$K \cap N = [(K \cap N) : M]M \subseteq [K : N][N : M]M \subseteq [K : N]N \subseteq K \cap N,$$

so that $K \cap N = [K : N]N$ and $N$ is multiplication. This completes the proof of the theorem. $\square$

We make some observations on our theorem. First, every direct summand of a multiplication module is multiplication since as observed earlier it is pure. This proves part of [1, Theorem 3.3]. Second, the assumption that $M$ is multiplication in the theorem can not be discarded. For example, the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$ is not multiplication and the non-zero cyclic submodule $N = \mathbb{Z}(a, b)$ is a direct summand (and hence a pure submodule of $M$) if and only if $\gcd(a, b) = 1$. Also $N$ has the property that $[N : M] = 0$, and hence $[N : M]N = 0 \neq N$. This shows that $N$ is not an idempotent submodule of $M$. Third, if $N$ is a pure submodule of a multiplication $R$-module $M$ such that $N$ is a finitely generated faithful or non-torsion submodule of $M$, then $N = M$. Fourth, an ideal $I$ of a ring $R$ is pure if and only if $I$ is multiplication and idempotent. This is equivalent to $I_P = 0_P$ or $I_P = R_P$ for every maximal ideal $P$ of $R$. It is also clear that $I$ is a pure ideal of $R$ if and only if $Ra = Ia$ for every $a \in I$. Every ideal $I$ generated by idempotents is a pure ideal since it is multiplication [5, p.466] and idempotent. We may then conclude that a ring $R$ is von Neumann regular if and only if every ideal of $R$ is pure. For if $R$ is a von Neumann regular ring, the $R_P$ is a field for every maximal ideal $P$ of $R$. It follows that for every ideal $I$ of $R$, $I_P = 0_P$ or $I_P = R_P$ for each maximal ideal $P$ of $R$. Hence every ideal of a von Neumann regular ring is a pure ideal. Conversely, let $a \in R$. Then $Ra = Ra^2$, and hence $a \in Ra^2$. There exists $x \in R$ such that $a = axa$, and hence $R$ is von Neumann regular.

Projective modules are characterized as direct summand of free modules. Hence they are pure submodules of free modules. The trace ideal (see [10]) of a module $M$ is defined as

$$T(M) = \sum_{f \in \mathrm{Hom}(M,R)} f(M).$$

It is known [22],[23] that if $M$ is a projective $R$-module, then

(1) $M = T(M)M$.

(2) $\mathrm{ann}M = \mathrm{ann}(T(M))$.

(3) $T(M)$ is a pure ideal.

We return to investigate the trace of a pure submodule in Part 2, but for now we apply Theorem 1.1 to obtain properties of $[N : M]$ analogous to (1), (2), (3) above when $N$ is a pure submodule of a multiplication module $M$.

**Corollary 1.2.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a pure submodule of $M$. Then the following statements hold.*

(1) $N = [N : M]N$

(2) $\mathrm{ann}N = \mathrm{ann}[N : M]$

(3) $[N : M]$ *is a pure ideal of $R$.*

*Proof.* (1) follows by Theorem 1.1.

(2) As $N$ is pure in $M$, we have $IN = N \cap IM$ for every ideal $I$ of $R$. Taking $I = \mathrm{ann}N$, we get that $0 = N \cap (\mathrm{ann}N)M$, and hence

$$\begin{aligned} 0 = [0 : M] &= [(N \cap (\mathrm{ann}N)M) : M] = [N : M] \cap [(\mathrm{ann}N)M : M] \\ &= [N : M] \cap \mathrm{ann}N = [N : M]\mathrm{ann}N. \end{aligned}$$

Hence $\mathrm{ann}N \subseteq \mathrm{ann}[N : M]$. Conversely, if $x \in \mathrm{ann}[N : M]$ then $x[N : M] = 0$, and hence $xN = x[N : M]N = 0$, so that $x \in \mathrm{ann}N$ and $\mathrm{ann}[N : M] \subseteq \mathrm{ann}N$ and (2) follows.

(3) Let $a \in [N : M]$. Then $aM \subseteq N$, and by Theorem 1.1 (3), $aM = [N : M]aM$, and hence $Ra = [N : M]a$, This shows that $[N : M]$ is a pure ideal of $R$. □

The $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$ is free, finitely generated and faithful but not multiplication. $N = \mathbb{Z} \oplus 0$ a direct summand of $M$, and hence a pure (in fact projective) submodule of $M$. $[N : M] = 0$. Hence $N \neq [N : M]N$ and $0 = \mathrm{ann}N \neq \mathrm{ann}[N : M] = \mathbb{Z}$.

Another consequence of Theorem 1.1 is the following:

**Corollary 1.3.** *Let $R$ be a ring and $M$ a multiplication $R$-module with pure annihilator.*

(1) *If $I$ is a pure ideal of $R$ and $N$ is a pure submodule of $M$, then $IN$ is a pure submodule of $M$.*

(2) *If $N_1$ and $N_2$ are pure submodules of $M$, then so too are $N_1 + N_2$ and $N_1 \cap N_2$.*

(3) *If $N_1$ and $N_2$ are pure submodules of $M$ then $N_1 \otimes N_2$ is a pure submodule of $M \otimes M$.*

*Proof.* (1) As each of $I$ and $N$ is a pure ideal (submodule) of $R$ ($M$), then by Theorem 1.1 we infer that each of $I$ and $N$ is a multiplication and an idempotent ideal (submodule) of $R$ ($M$). Hence $IN$ is multiplication [5, Corollary to Theorem 2], and moreover $IN$ is an idempotent submodule of $M$. By Theorem 1.1, $IN$ is pure.

(2) Let $x \in N_1 \cap N_2$. By Theorem 1.1, there exist $y_1 \in [N_1 : M]$ and $y_2 \in [N_2 : M]$ such that $x = xy_1 = xy_2$. Hence $x = xy_2 = x(y_1y_2)$. As $y_1y_2 \in [(N_1 \cap N_2) : M]$, $N_1 \cap N_2$ is a pure submodule of $M$. Suppose now that $x \in N_1 + N_2$. There exist $y \in N_1$ and $z \in N_2$ such that $x = y + z$. By Theorem 1.1, $Ry = [N_1 : M]y$, $Rz = [N_2 : M]z$, and hence

$$\begin{aligned} Rx &= R(y + z) \subseteq Ry + Rz = [N_1 : M]y + [N_2 : M]z \\ &\subseteq [(N_1 + N_2) : M](y + z) = [(N_1 + N_2) : M]x, \end{aligned}$$

and by Theorem 1.1, $N_1 + N_2$ is a pure submodule of $M$.

(3) Let $n_1 \in N_1$ and $n_2 \in N_2$. Then $Rn_1 = [N_1 : M]n_1$ and $Rn_2 = [N_2 : M]n_2$ and hence

$$\begin{aligned} R(n_1 \otimes n_2) &= Rn_1 \otimes Rn_2 = [N_1 : M]n_1 \otimes [N_2 : M]n_2 \\ &= [N_1 : M][N_2 : M](n_1 \otimes n_2). \end{aligned}$$

It is easy to check that

$$[N_1 : M][N_2 : M] \subseteq [N_1 \otimes N_2 : M \otimes M],$$

and this finally implies that

$$R(n_1 \otimes n_2) \subseteq [N_1 \otimes N_2 : M \otimes M](n_1 \otimes n_2) \subseteq R(n_1 \otimes n_2),$$

so that $R(n_1 \otimes n_2) = [N_1 \otimes N_2 : M \otimes M](n_1 \otimes n_2)$, and by Theorem 1.1, the result follows. $\square$

It is not true that every submodule of a pure submodule of a module $M$ is pure in $M$, but in case $R$ is von Neumann regular, it is true. For example if $R$ is a Prüfer domain and $I$ a non-finitely generated ideal of $R$, then $I$ is not multiplication and hence not pure. $R$ is a pure submodule of $R$, but $I$ is not a pure submodule of $R$. On the other hand suppose $R$ is von Neumann regular and $M$ is a multiplication $R$-module. Let $N$ be a pure submodule of $M$ and $K$ a submodule of $N$. By Theorem 1.1, $N$ is idempotent in $M$, and multiplication. Hence $K = [K : N]N$. But since $R$ is von Neumann regular, $[K : N]$ is generated by idempotents and is hence idempotent. Moreover, it is multiplication [5]. It follows that $[K : N]$ is a pure ideal of $R$, and by Corollary 1.3(1), $K$ is a pure submodule of $M$. We note also that $K$ is pure in $N$. For this, it is enough to show that $K$ is idempotent in $N$, which is true since

$$K = [K : M]K \subseteq [K : N]K \subseteq K,$$

so that $K = [K : N]K$.

The next result shows some properties of pure submodules of finitely generated faithful multiplication modules. It may be compared with [15, Lemma 1.4] and [20, Theorem 10].

**Theorem 1.4.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Then the following statements are true:*

(1) *A submodule $N$ of $M$ is pure if and only if $[N : M]$ is a pure ideal of $R$.*

(2) *An ideal $I$ of $R$ is pure if and only if $IM$ is a pure submodule of $M$.*

(3) *If $K$ is a pure submodule of $N$ and $N$ is a pure submodule of $M$, then $K$ is a pure submodule of $M$.*

(4) *If $R$ is a p.p.ring (a ring in which every projective ideal is principal) and $N$ is a pure submodule of $M$, then* ann$N$ *is a pure ideal of $R$.*

*Proof.* (1) Let $N$ be a submodule of $M$ such that $[N : M]$ is a pure ideal of $R$. Then $[N : M]$ is multiplication and idempotent. By [20, Theorem 10], $N = [N : M]M$ is multiplication and by the remark made before Theorem 1.1, $N$ is idempotent in $M$. By Theorem 1.1, $N$ is pure in $M$. Alternatively, $M$ is flat by [1, Lemma 2.7] and $M/N$ is multiplication. For let $K/N$ be a submodule of $M/N$. Then $K$ is a submodule of $M$. Hence $K = [K : M]M$ and it is easy to check that $K/N = [K/N : M/N]M/N$. If $[N : M] = \text{ann}(M/N)$ is a pure ideal of $R$, then by [1, Corollary 2.7] $M/N$ is flat and finally by [16, p.54] and [9, Corollary 11.21] $N$ is pure in $M$. The converse is Corollary 1.2(3).

(2) Let $I$ be an ideal of $R$ such that $IM$ is a pure submodule of $M$. Since $I = [IM : M]$, $I$ is a pure ideal of $R$, by (1). Conversely, let $I$ be a pure ideal of $R$. Then $I$ is multiplication and hence $IM$ is a multiplication submodule of $M$, [20, Theorem 10]. But $I$ is an idempotent ideal and $I = [IM : M]$. Thus $IM = I^2M = [IM : M]IM$, and hence $IM$ is idempotent in $M$ and is a pure submodule by Theorem 1.1.

(3) Let $K$ be a pure submodule of $N$ and $N$ a pure submodule of $M$. By [20, Theorem 10], $[K:M]=[K:N][N:M]$. By Theorem 1.1, for each $x \in K$ (and hence $x \in N$),

$$[K:M]x=[K:N][N:M]x=[K:N]x=Rx.$$

Hence $K$ is a pure submodule of $M$.

(4) Let $N$ be a pure submodule of $M$. Then $[N:M]$ is a pure ideal of $R$ by (1). As $R$ is a p.p. ring, we infer from [22, Example 3.3], that $[N:M]$ is generated by idempotents. Suppose that $[N:M]=\sum\limits_{\alpha\in\Lambda} Re_\alpha$, where $e_\alpha=e_\alpha^2$. Then $\operatorname{ann}[N:M]=\bigcap\limits_{\alpha\in\Lambda}\operatorname{ann}(e_\alpha)=\bigcap\limits_{\alpha\in\Lambda}R(1-e_\alpha)$. Let $x\in\operatorname{ann}[N:M]$. Then $xe_\alpha=0$ and hence $x(1-e_\alpha)=x$ for all $\alpha\in\Lambda$. Since $Rx$ is a projective ideal, we infer from [4, p.332] that

$$Rx\subseteq\bigcap_{\alpha\in\Lambda}Rx(1-e_\alpha)=\left(\bigcap_{\alpha\in\Lambda}R(1-e_\alpha)\right)x=\operatorname{ann}[N:M]x\subseteq Rx,$$

so that $Rx=\operatorname{ann}[N:M]x$, and hence $\operatorname{ann}N=\operatorname{ann}[N:M]$ is a pure ideal of $R$. $\square$

The next result generalizes some properties of pure ideals to pure submodules. It may be compared with [15, Corollary 1 to Lemma 1.5], [21, Corollary 11].

**Proposition 1.5.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module.*

(1) *A pure submodule $N$ of $M$ is finitely generated if and only if $\operatorname{ann}N=\operatorname{ann}K$ for some finitely generated submodule $K$ of $N$.*

(2) *If $N$ is a finitely generated pure submodule of $M$, then*

  (i) *$N=eN=eM$ for some idempotent element $e$ of $R$,*

  (ii) *$M=N\oplus(\operatorname{ann}N)M$,*

  (iii) *$R=\operatorname{ann}N\oplus\operatorname{ann}(\operatorname{ann}N)$.*

*Proof.* (1) We have from Theorem 1.1 that $R=[N:M]+\operatorname{ann}(x)$ for each $x\in N$. Let $K$ be a finitely generated submodule of $N$ with $\operatorname{ann}K=\operatorname{ann}N$. Then $R=[N:M]+\operatorname{ann}K=[N:M]+\operatorname{ann}N$. By Corollary 1.2, $\operatorname{ann}N=\operatorname{ann}[N:M]$. Thus $R=[N:M]+\operatorname{ann}[N:M]$. There exist $x\in[N:M]$ and $y\in\operatorname{ann}[N:M]$ such that $1=x+y$. This implies that

$$[N:M]=[N:M]x\subseteq Rx\subseteq[N:M],$$

so that $[N:M]=Rx$, and hence $N=[N:M]M=xM$ is a finitely generated submodule of $M$. The converse is obvious.

(2) (i) Since $N$ is finitely generated, it follows by [15, Corollary 1 to Lemma 1.5] that $[N:M]$ is a finitely generated ideal. Hence $[N:M]$ is a principal pure ideal of $R$, and hence it is generated by an idempotent element, say $e$. It follows that $N=[N:M]N=eN=[N:M]M=eM$.

(ii) As $R=[N:M]+\operatorname{ann}N$, we infer that $M=N+(\operatorname{ann}N)M$. But $N\cap(\operatorname{ann}N)M=0$. Thus $M=N\oplus(\operatorname{ann}N)M$.

(iii) By Corollary 1.2, $[N:M]\mathrm{ann}N = 0$, and hence $[N:M] \subseteq \mathrm{ann}(\mathrm{ann}N)$. Hence

$$R = [N:M] + \mathrm{ann}N \subseteq \mathrm{ann}(\mathrm{ann}N) + \mathrm{ann}N \subseteq R,$$

so that $R = \mathrm{ann}N + \mathrm{ann}(\mathrm{ann}N)$. But $\mathrm{ann}N$ is generated by an idempotent. Thus $\mathrm{ann}N \cap \mathrm{ann}(\mathrm{ann}N) = 0$, and the result follows. □

A submodule $P$ of a free $R$-module $F$ is projective if and only if it is a pure submodule of $F$. But not all pure submodules of multiplication modules are projective. However, it follows by the above Proposition and [1, Corollary 2.4], [20, Corollary 1 to Theorem 10] that every finitely generated pure submodule of a finitely generated faithful multiplication module is projective.

Every pure ideal $I$ of $R$ has the property that $I_P = 0_P$ or $I_P = R_P$ for each maximal ideal $P$ of $R$. It follows that for every non-empty collection $J_\lambda (\lambda \in \Lambda)$ of ideals of $R$, the equality $\bigcap_{\lambda\in\Lambda} J_\lambda I = (\bigcap_{\lambda\in\Lambda} J_\lambda) I$ is true locally and hence globally. The next result generalizes the above fact to pure submodules of multiplication modules and should be compared with [4, Theorem 1].

**Theorem 1.6.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a pure submodule of $M$. Then the following statements are true.*

(1) *For every non-empty collection $I_\lambda (\lambda \in \Lambda)$ of ideals of $R$, $\bigcap_{\lambda\in\Lambda} I_\lambda N = (\bigcap_{\lambda\in\Lambda} I_\lambda) N$.*

(2) *$[N:M]$ is the intersection of all ideals $I$ of $R$ such that $N = IN$.*

(3) *$N$ is a pure $[N:M]$-submodule of $M$.*

*Proof.* (1) By Corollary 1.2, $[N:M]$ is a pure ideal of $R$. Let $I_\lambda (\lambda \in \Lambda)$ be any collection of ideals of $R$. Then

$$[(\bigcap_{\lambda\in\Lambda} I_\lambda N) : M] = \bigcap_{\lambda\in\Lambda} [I_\lambda N : M] = \bigcap_{\lambda\in\Lambda} (I_\lambda [N:M]) = (\bigcap_{\lambda\in\Lambda} I_\lambda)[N:M],$$

and hence

$$\bigcap_{\lambda\in\Lambda} I_\lambda N = [\bigcap_{\lambda\in\Lambda} I_\lambda N : M]M = (\bigcap_{\lambda\in\Lambda} I_\lambda)[N:M]M = (\bigcap_{\lambda\in\Lambda} I_\lambda)N,$$

and (1) is proved.

(2) Let $S$ be the collection of all ideals $I$ of $R$ with the property that $N = IN$. Then $N = \bigcap_{I\in S} IN = (\bigcap_{I\in S} I)N$, by (1). It follows that

$$[N:M] = [(\bigcap_{I\in S} I)N : M] = (\bigcap_{I\in S} I)[N:M],$$

and hence $[N:M] \subseteq \bigcap_{I\in S} I$. But $N$ is pure, and hence an idempotent. Thus $N = [N:M]N$, and this means that $[N:M] \in S$. So $[N:M] \subseteq \bigcap_{I\in S} I$, and hence is the smallest element of $S$.

(3) Let $T = [N : M]$. It is clear that $[N :_R M] = [N :_T M]$. Hence $N = [N :_R M]N = [N :_T M]N$, and $N$ is an idempotent $T$-submodule of $M$. On the other hand $N = [N :_R M]M = [N :_T M]M$, and $N$ is a multiplication $T$-submodule of $M$. By Theorem 1.1, $N$ is a pure $T$-submodule of $M$. □

Let $R$ be a ring and $M$ an $R$-module. A submodule $P$ of $M$ is called a prime submodule of $M$ if $P \neq M$ and whenever $rm \in P$, for some $m \in M$ and $r \in R$, then $m \in P$ or $r \in [P : M]$. The $M$-radical, rad $N$, of a submodule $N$ of $M$ is defined as the intersection of all prime submodules of $M$ containing $N$, [17]. If $I$ is an ideal of $R$, then $\sqrt{I}$ is defined as the intersection of all prime ideals of $R$ containing $I$. If $I$ is a pure (and hence idempotent) ideal of $R$, then $I = I\sqrt{I}$, and if $I$ is a finitely generated pure ideal (principal ideal generated by an idempotent element) of $R$, then $I = \sqrt{I}$. The next result generalizes the above facts to pure submodules of multiplication modules.

**Proposition 1.7.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a pure submodule of $M$. Then*

(1) $N = \sqrt{[N : M]}N$,

(2) $[N : M]\mathrm{rad}N = N = [\mathrm{rad}N : M]N$,

(3) *If $N$ is finitely generated, then rad $N$ is a pure submodule of $M$ and moreover, $N =$ rad $N$.*

*Proof.* (1) Let $S$ be the set of all prime ideals of $R$ that contain $[N : M]$. Then $\sqrt{[N : M]} = \bigcap_{P \in S} P$, and by Theorem 1.6,

$$\sqrt{[N : M]}N = (\bigcap_{P \in S} P)N = \bigcap_{P \in S} PN.$$

For each $P \in S$, $N = [N : M]N \subseteq PN \subseteq N$, so that $N = PN$, and hence

$$N = \bigcap_{P \in S} PN = \sqrt{[N : M]}N.$$

(2) It follows from (1) and [17, Theorem 4] that

$$N = \sqrt{[N : M]}N = \sqrt{[N : M]}[N : M]M = [N : M]\mathrm{rad}\, N.$$

But rad $N \subseteq M$ and $M$ is a multiplication module. Thus rad $N = [\mathrm{rad}\, N : M]M$, and hence

$$[N : M]\mathrm{rad}\, N = [N : M][\mathrm{rad}\, N : M]M = [\mathrm{rad}\, N : M]N.$$

(3) Suppose $N$ is a finitely generated pure submodule of $M$. Then by Proposition 1.5, $[N : M]$ is a finitely generated pure ideal, and hence $[N : M] = \sqrt{[N : M]}$. It follows that

$$N = [N : M]M = \sqrt{[N : M]}M = \mathrm{rad}\, N,$$

and hence the result. This finishes the proof of the proposition. □

## 2. Trace of pure submodules

Let $R$ be a ring and $M$ an $R$-module. The trace of $M$, denoted by $T(M)$, is the ideal generated by the set $\{f(m) : f \in \text{Hom}(M,R), m \in M\}$, [10], [23]. It is well-known that for any fractional ideal $I$ of $R$, $T(I) = II^{-1}$, [10, Lemma 4.2.2] from which it follows that $T(I) = R$ if and only if $I$ is an invertible fractional ideal of $R$. Let $M$ be a projective $R$-module. Then $M \oplus N = F$ for some $R$-module $N$ and free $R$-module $F$. In this case $T(M)$ is the ideal of $R$ generated by the coordinates of the elements in $M$, for any basis chosen in $F$. It is proved [22] that if $M$ is a projective $R$-module, then $M = T(M)M$, $\text{ann}M = \text{ann}(T(M))$, and $T(M)$ is a pure ideal of $R$. If $M$ is a finitely generated faithful multiplication $R$-module, then $M$ is projective [1, Corollary 2.4] and [20, Theorem 10], and hence $M = T(M)M$. This gives that $T(M) = R$. In this section, we investigate the trace of pure submodules of multiplication modules. Let $N$ be a submodule of an $R$-module $M$. Let $H(N) = \sum\limits_{a\in N}\text{ann}(\text{ann}(a))$. We start with the following result which gives two descriptions of the trace of pure submodules of multiplication modules.

**Theorem 2.1.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a pure submodule of $M$. Then*

$$[N : M] = T(N) = H(N).$$

*Proof.* As $N$ is pure, $N = [N : M]N$, and hence for all $f \in \text{Hom}(N,R)$,

$$f(N) = [N : M]f(N).$$

It follows that

$$T(N) = \sum_f f(N) = [N : M]\sum_f f(N) = [N : M]T(N) = [T(N)N : M].$$

To prove that $T(N) = [N : M]$, it is enough to show that $N = T(N)N$. Let $P$ be a maximal ideal of $R$. If $N_P = 0_P$, then both sides collapse to $0_P$. Otherwise $N_P = M_P$ and hence $N_P$ is a finitely generated faithful multiplication $R_P$-module. By [1, Corollary 2.4] and [20, Theorem 10 Corollary 1], $N_P$ is a projective $R_P$-module, and

$$N_P = T(N_P)N_P = T(N)_P N_P.$$

This implies that $N = T(N)N$. Suppose now that $a \in N$ and $f \in \text{Hom}(N,R)$. Clearly, $\text{ann}(a) \subseteq \text{ann}(f(a))$, hence $\text{ann}(a)f(a) = 0$, and $f(a) \in \text{ann}(\text{ann}(a))$. This gives that $Rf(a) \subseteq \text{ann}(\text{ann}(a))$ and

$$f(N) = \sum_{a\in N} Rf(a) \subseteq \sum_{a\in N}\text{ann}(\text{ann}(a)) = H(N).$$

Hence, $T(N) \subseteq H(N)$, and $[N : M] \subseteq H(N)$. On the other hand, let $a \in N$. From Theorem 1.1, we have that $R = [N : M] + \text{ann}(a)$, and hence

$$\text{ann}(\text{ann}(a)) = [N : M]\text{ann}(\text{ann}(a)),$$

so that $H(N) = [N : M]H(N)$. This finally gives that $H(N) \subseteq [N : M]$, and hence $H(N) = [M : M]$. This finishes the proof of the theorem. □

Let $R$ be a ring and $I$ a pure ideal of $R$. By the above theorem, $T(I) = I$, and from [10, p.72] it follows that $I = II^{-1}$. If $N$ is a pure submodule of a finitely generated faithful multiplication $R$-module $M$, then $T(N) = [N : M]$. Hence by Corollary 1.2, $N = T(N)N$, and $\operatorname{ann}N = \operatorname{ann}(T(N))$, and $T(N)$ is a pure ideal of $R$. As $[N : M]$ is a pure (idempotent) ideal of $R$, $T([N : M]) = [N : M] = T(N)$. Also, $[N : M] = [N : M]^{-1}[N : M]$, and hence $[N : M]N = [N : M]^{-1}[N : M]N$. This implies that $N = [N : M]^{-1}N$. Also $N$ is a multiplication submodule of $M$. Thus for all $x \in N$, we have that $Rx = [N : M]^{-1}x$, equivalently, $R = [N : M]^{-1} + \operatorname{ann}(x)$. These are two futher characterizations of pure submodules of multiplication modules which could be added to Theorem 1.1. It is now clear that the trace of a pure submodule of a finitely generated faithful multiplication module is the smallest element of the collection of all ideals $I$ with the property that $N = IN$ (Theorem 1.6).

We mention four corollaries to Theorem 2.1. The first gives the description of the trace of finitely generated pure submodules of multiplication modules, the second may be compared with [23, Lemma 1.2], and the third should be compared with [15, Theorem 1.3(ii)] and shows that pure submodules of multiplication modules are torsionless.

**Corollary 2.2.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a finitely generated submodule of $M$. Then $T(N) = [N : M] = \operatorname{ann}(\operatorname{ann}N)$.*

*Proof.* Let $A(N) = \operatorname{ann}(\operatorname{ann}N)$. We only need to show that $[N : M] = A(N)$. As $N$ is finitely generated, we infer from Theorem 1.1 that $R = [N : M] + \operatorname{ann}N$, and hence $A(N) = [N : M]A(N)$. It follows that $A(N) \subseteq [N : M]$. Let $x \in [N : M]$. Then $xM \subseteq N$, and hence $x\operatorname{ann}(N)M = 0$. If follows that $x\operatorname{ann}N \subseteq \operatorname{ann}M = 0$, and $x \in A(N)$. Hence $[N : M] \subseteq A(N)$ so that $[N : M] = A(N)$, and the result is proved. □

Without the assumption that $N$ is finitely generated in Theorem 2.1, the conclusion fails. For example, let $I$ be a proper ideal generated by a countably infinite set of idempotents in a ring $R$ with $\operatorname{ann}I = 0$. Then $I$ is multiplication and idempotent. Hence it is a pure ideal. We have $T(I) = H(I) = I$, but $\operatorname{ann}(\operatorname{ann}I) = R$.

**Corollary 2.3.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a pure submodule of $M$. Then $N$ is finitely generated if and only if $T(N)$ is finitely generated.*

*Proof.* See [15, Lemma 1.4 (ii)]. □

Low and Smith [15] proved that for a faithful multiplication module $M$, $\bigcap_{f \in \operatorname{Hom}(M,R)} \ker f = 0$, from which it follows that $M$ is torsionless in the sense that it can be embedded in a direct product of copies of $R$. We can conclude that if in addition $M$ is finitely generated, then the same is true for pure submodules of $M$.

**Corollary 2.4.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. If $N$ is a pure submodule of $M$, then $N$ is torsionless.*

*Proof.* Let $K = \bigcap_{f\in \mathrm{Hom}(N,R)} \ker f$. By Theorem 1.1, $N$ is multiplication and hence $K = [K : N]N$. It follows that $0 = f(K) = [K : N]f(N)$ for all $f \in \mathrm{Hom}(N, R)$. This implies that $0 = [K : N]T(N)$, and hence using Theorem 2.1 and Corollary 1.2(2), $[K : N] \subseteq \mathrm{ann}(T(N)) = \mathrm{ann}([N : M]) = \mathrm{ann}N$. This implies that $K = [K : N]N = 0$, as required. □

**Corollary 2.5.** *Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a finitely generated pure submodule of $M$. Let $N = K \oplus L$ for submodules $K$ and $L$ of $M$. Then*

(1) $T(N) = T(K) \oplus T(L)$,
(2) $N = T(K)N \oplus T(L)N$.

*Proof.* (1) $K$ and $L$ are pure submodules of $N$, and by Theorem 1.4 (3), they are pure submodules of $M$. From Theorem 2.1, we get that $T(N) = [N : M]$, $T(K) = [K : M]$, and $T(L) = [L : M]$. It follows from [2, p.560] and [20, Proposition 4] that

$$T(N) = [K + L : M] = [K : M] + [L : M] = T(K) + T(L).$$

But

$$0 = [0 : M] = [(K \cap L) : M] = [K : M] \cap [L : M] = T(K) \cap T(L).$$

Thus $T(N) = T(K) \oplus T(L)$.
(2) $N = T(N)N = T(K)N + T(L)N$. From Theorem 1.6 we have that

$$T(K)N \cap T(L)N = (T(K) \cap T(L))\, N = 0,$$

and hence $N = T(K)N \oplus T(L)N$, as required. □

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Received October 30, 2002