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Primeness in Near-rings of Continuous Functions

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Abstract. Various types of primeness have been considered for near-rings. One of these is the concept of equiprime, which was defined in 1990 by Booth, Groenewald and Veldsman. We will investigate when the near-ring $N_0(G)$ of continuous zero-preserving self maps of a topological group G is equiprime. This is the case when G is either T_0 and 0-dimensional or T_0 and arcwise connected. We also give conditions for $N_0(G)$ to be strongly prime and strongly equiprime. Finally, we apply these results to sandwich near-rings of continuous functions.

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1. Preliminaries

In this paper, all near-rings will be right distributive. The notation "$A \lhd N$" means "A is an ideal of N". Let G be an additive topological group. The sets of arbitrary and zero-preserving continuous self-maps of G form near-rings with respect to addition and composition of functions, and are denoted $N(G)$ and $N_0(G)$, respectively. Near-rings of continuous functions have been extensively studied. See for example Magill [8], [9]. We remark that $N_0(G)$ is *zerosymmetric*, i.e. $n0 = 0n = 0$ for all $n \in N_0(G)$.

There are a number of definitions of primeness for near-rings in the literature. The classical definition is given in Pilz [10]: A near-ring N is called *prime* (resp. *semiprime*) if $A, B \lhd N$ (resp. $A \lhd N$), $AB = 0$ implies $A = 0$ or $B = 0$ (resp. $A^2 = 0$ implies $A = 0$). N is called *equiprime* (cf. Booth, Groenewald and Veldsman [1]) if $a, x, y \in N$, $anx = any$ for all $n \in N$ implies $a = 0$ or $x = y$. Note also an equiprime near-ring is zerosymmetric [11, p. 2750]. Both of these definitions of primeness generalise the usual notion of primeness for associative rings. Equiprimeness is of particular interest from the radical-theoretic perspective in that it leads to a Kurosh-Amitsur prime radical for both zerosymmetric and arbitrary near-rings [1].

There are two generalisations to near-rings of the notion of strongly prime (cf. Handelman and Lawrence [4]). A near-ring N is *strongly prime* [3] if $0 \neq a \in N$ implies that there exists a finite subset F of N such that $aFx = 0$ implies $x = 0$, for all $x \in N$. N is *strongly equiprime* [2] if $0 \neq a \in N$ implies that there exists a finite subset F of N such that $x, y \in N, afx = afy$ for all $f \in F$ implies $x = y$. Note that equiprime $\Longrightarrow$ prime and strongly equiprime $\Longrightarrow$ strongly prime. To prove the first implication, let N be an equiprime near-ring and let $0 \neq A, B \lhd N$. Let $a \in A\backslash\{0\}, b \in B\backslash\{0\}$. Then by the equiprimeness of N, and hence also its zerosymmetry, there exists $n \in N$ such that $anb \neq an0 = 0$. Moreover, $an \in A$ and so $AB \neq 0$. Hence N is prime. The second implication is proved by a similar argument, after noting that a strongly equiprime near-ring is equiprime and hence zerosymmetric. We note also (cf. [3]) that a strongly prime near-ring is prime. We refer to Pilz [10] for all undefined concepts concerning near-rings.

For all notions relevant to topological groups, we refer to any of the standard texts, e.g. Higgins [5] and Husain [6]. We will make frequent use of the well-known result that every T_0 topological group is T_3 (and hence Hausdorff). Composition of functions will be denoted by juxtaposition, e.g. ab rather than $a \circ b$. For basic topological notions we refer to any of the standard texts, for example Kelley [7].

Veldsman [11] has noted that the near-ring $M_0(G)$ of all zero-preserving self-maps of an additive group G is always equiprime. This is not in general true for $N_0(G)$, where G is a topological group. In fact $N_0(G)$ need not even be semiprime, as the next result shows.

Proposition 1.1. *Let G be a disconnected topological group, with open components which contain more than one element. Then $N_0(G)$ is not semiprime.*

Proof. Let H be the component of G which contains 0. As is well-known, H is a normal subgroup of G and the remaining components of G are the cosets of H in G. Let $I := \{x \in N_0(G) \mid x(G) \subseteq H\}$. It is straightforward to check that I is a right ideal of G. Now let $m, n \in N_0(G), a \in I, g \in G$. Since $a(g) \in H$, $(a + n)(g)$ and $n(g)$ are contained in the same component (coset of H). By the continuity of m, $m(a + n)(g)$ and $mn(g)$ are contained in the same component. Hence $(m(a + n) - mn)(g) \in H$, and so $m(a + n) - mn \in I$. Thus $I \lhd N_0(G)$. Let $J := \{n \in N \mid nx = 0 \text{ for all } x \in I\}$. Then J is a left ideal of $N_0(G)$. Since $N_0(G)$ is zerosymmetric, I is left invariant, and hence J is also a right ideal of $N_0(G)$. Thus $J \lhd N_0(G)$. Moreover $(I \cap J)^2 = 0$, but $I \cap J \neq 0$. For let $0 \neq h \in H$ and let a be defined by

$$a(g) := \begin{cases} 0 & g \in H \\ h & g \in G\backslash H \end{cases} .$$

Since H is a component of G, it is closed. But by the hypothesis of this proposition, H is also open. It follows that G is the union of the disjoint open sets H and $G\backslash H$. Hence a is continuous. Clearly, $a \in I$. Let $g \in G, x \in I$. Then $ax(g) = 0$, since $x(g) \in H$. Hence $ax = 0$, so $a \in J$, whence $a \in I \cap J$. It follows that $N_0(G)$ is not semiprime. □

We remark that there are abundant examples of topological groups which satisfy the conditions of Proposition 1.1. For example, let $\mathbb{R}$ denote the real numbers with the usual topology and let $\mathbb{Z}_2$ denote the residue classes modulo 2 with the discrete topology. Then $G := \mathbb{R} \times \mathbb{Z}_2$ with the product topology is an example of such a topological group.

2. 0-dimensional topological groups

We recall that a topological space X is called *0-dimensional* if the topology on X has a base consisting of clopen (i.e. both open and closed) sets. In this section we will provide information on the primeness of $N_0(G)$ in the case that the topology on G is 0-dimensional. Let X, Y be nonempty sets and let F be a set of functions from X into Y. We recall that F is said to *separate points* if $x_1, x_2 \in X, x_1 \neq x_2$ implies that there exists $f \in F$ such that $f(x_1) \neq f(x_2)$.

Lemma 2.1. *Let X be an infinite set and let F be a finite set of functions of X into a set Y. If each element of F has finite range, then F cannot separate points.*

Proof. Let $F := \{f_1, \dots, f_n\}$. Since f_1 has finite range, there exists $y_1 \in Y$ such that $f(x) = y_1$ for infinitely many points x of X. Let $X_1 := \{x \in X \mid f_1(x) = y_1\}$. Since f_2 has finite range, there exists $y_2 \in Y$ such that $f_2(x) = y_2$ for infinitely many points x of X_1. Let $X_2 := \{x \in X_1 \mid f_2(x) = y_2\}$. Continuing in this way we obtain a nested sequence of infinite sets $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n$ such that $f_i(x) = y_i$ for all $x \in X_i, 1 \leq i \leq n$. In particular $f_i(x) = y_i$ for all $x \in X_n$. Hence F does not separate points in X_n, and so cannot separate points in X. □

Proposition 2.2. *Let G be a 0-dimensional, T_0 topological group with more than one element. Then*

(a) *$N_0(G)$ is equiprime.*
(b) *$N_0(G)$ is strongly prime if and only if the topology on G is discrete.*
(c) *$N_0(G)$ is strongly equiprime if and only if G is finite.*

Proof. (a) Let $a, x, y \in N_0(G)$, $a \neq 0, x \neq y$. Then there exist $g, h \in G$ such that $a(g) \neq 0, x(h) \neq y(h)$. Without loss of generality we may assume that $x(h) \neq 0$. Since G is T_0, and hence Hausdorff, there exists a clopen set U which contains $x(h)$, but not $y(h)$ or 0. Let n be defined by

$$n(k) := \begin{cases} g & k \in U \\ 0 & k \in G \backslash U \end{cases} .$$

Since U is clopen, n is continuous. Clearly, $n \in N_0(G)$. Moreover $anx(h) = a(g) \neq 0$ and $any(h) = a(0) = 0$. Hence $anx \neq any$ so $N_0(G)$ is equiprime.

(b) Suppose the topology on G is discrete. Let $0 \neq a \in N_0(G)$ and let $g \in G$ be such that $a(g) \neq 0$. Define f by

$$f(k) := \begin{cases} g & k \neq 0 \\ 0 & k = 0 \end{cases} .$$

Then $f \in N_0(G)$. Let $0 \neq x \in N_0(G)$ and let $h \in G$ be such that $x(h) \neq 0$. Then $afx(h) = a(g) \neq 0$. If we let $F := \{f\}$ we see that $N_0(G)$ is strongly prime.

Conversely, suppose that the topology on G is not discrete. Let U be a nonempty clopen subset of G such that $0 \notin U$. Let $0 \neq g \in G$ and let $a \neq 0$ be defined by

$$a(k) := \begin{cases} g & k \in U \\ 0 & k \in G \backslash U \end{cases} .$$

Then $a \in N_0(G)$. Let $F := \{f_1, \dots, f_n\}$ be a finite subset of $N_0(G)$. Since $G\backslash U$ is clopen and f_i is continuous $f_i^{-1}(G\backslash U)$ is clopen. Let $V_i := f_i^{-1}(G\backslash U)\backslash U$. Then V_i is clopen and $0 \in V_i$. Let $V := \bigcap_{i=1}^n V_i$. Then V is clopen and $0 \in V$. Since G is T_0 and not discrete, V is infinite. Let $0 \neq h \in V$. Then $f_i(h) \notin U$ for $1 \leq i \leq n$. Define $x \neq 0$ by

$$x(k) := \begin{cases} h & k \in U \\ 0 & k \in G \backslash U \end{cases} .$$

Then $af_ix = 0$, $1 \leq i \leq n$, whence $aFx = 0$. Hence $N_0(G)$ is not strongly prime.

(c) If G is finite, so is $N_0(G)$. Since $N_0(G)$ is equiprime by (a), it follows easily that it is strongly equiprime.

Conversely, suppose that G is infinite. Let U be a proper clopen subset of G which contains 0, and let $0 \neq g \in G$. Define

$$a(k) := \begin{cases} 0 & k \in U \\ g & k \in G \backslash U \end{cases} .$$

Then $0 \neq a \in N_0(G)$. Let $F := \{f_1, \dots, f_n\}$ be a finite subset of $N_0(G)$. Now the range of af_i has at most two points for $1 \leq i \leq n$. It follows from Lemma 2.1 that $\{af_1, \dots af_n\}$ does not separate points. Let $g_1, g_2 \in G$ be such that $g_1 \neq g_2$ and $af_i(g_1) = af_i(g_2)$ for $1 \leq i \leq n$. Let x and y be defined by

$$x(k) := \begin{cases} 0 & k \in U \\ g_1 & k \in G \backslash U \end{cases} , \quad y(k) := \begin{cases} 0 & k \in U \\ g_2 & k \in G \backslash U \end{cases} .$$

Then $x, y \in N_0(G)$ and $x \neq y$. However $af_ix = af_iy$ for $1 \leq i \leq n$. Hence $N_0(G)$ is not strongly equiprime. □

3. Arcwise connected topological groups

In this section, G will be a T_0, arcwise connected topological group with more than one element. As is well known, this implies that G is completely regular (cf. Husain [6, pp 48-49, Theorems 4 and 5]).

Lemma 3.1. *Let $0 \neq a \in N_0(G)$. Then $aN_0(G)$ separates points.*

Proof. Let $g_1, g_2 \in G, g_1 \neq g_2$. Let $h \in G$ be such that $a(h) = k \neq 0$. Let $p, q \in \mathbb{R}$ with $p < q$. Since G is completely regular and T_0 (and hence T_1, so one-point sets are closed), there exists a continuous function $\theta : G \longrightarrow [p, q]$ such that $\theta(g_1) = p, \theta(g_2) = q$. Moreover, p, q can be chosen such that $p \leq 0 \leq q$ and $\theta(0) = 0$. (If this is not so, replace θ with φ where $\varphi(z) := \theta(z) - \theta(0)$ and replace p and q with $p - \theta(0)$ and $q - \theta(0)$, respectively. Clearly φ will map G into $[p - \theta(0), q - \theta(0)]$, $\varphi(0) = 0$, and since 0 is in the range of φ, $p - \theta(0) \leq 0 \leq q - \theta(0)$.) Now either $p < 0$ or $0 < q$. Assume the latter. Since G is arcwise connected, there exists a continuous function $\lambda : [0, q] \longrightarrow G$ such that $\lambda(0) = 0$, $\lambda(q) = h$. Define $\mu : [p, q] \longrightarrow G$ by

$$\mu(t) := \begin{cases} 0 & p \leq t \leq 0 \\ \lambda(t) & 0 < t \leq q \end{cases} .$$

Then μ is continuous. Let $n := \mu\theta$. Then $n(0) = \mu\theta(0) = \mu(0) = 0$. Hence $n \in N_0(G)$. Moreover, $an(g_1) = a\mu\theta(g_1) = a\mu(p) = a(0) = 0$ and $an(g_2) = a\mu\theta(g_2) = a\mu(q) = a\lambda(q) = a(h) = k \neq 0$. Hence $aN_0(G)$ separates points. □

Proposition 3.2. *$N_0(G)$ is equiprime.*

Proof. Let $a, x, y \in N_0(G)$, $a \neq 0, x \neq y$. Let $g \in G$ be such that $x(g) \neq y(g)$. By Lemma 3.1, $aN_0(G)$ separates points. Hence there exists $n \in N_0(G)$ such that $anx(g) \neq any(g)$ whence $anx \neq any$. Hence $N_0(G)$ is equiprime. □

Proposition 3.3. *Suppose that the topology on G has a base $\mathcal{B}$ consisting of arcwise connected open sets. Then $N_0(G)$ is not strongly prime (and hence not strongly equiprime).*

Proof. Let U be an open set containing 0 whose closure $\mathrm{cl}(U)$ is not G. Let $g \in G \backslash \mathrm{cl}(U)$. Since G is completely regular, there exists a continuous function $\alpha : G \longrightarrow [0, 1]$ such that $\alpha(\mathrm{cl}(U)) = 0$ and $\alpha(g) = 1$. Since G is arcwise connected, there exists a continuous function $\beta : [0, 1] \rightarrow G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $a := \beta\alpha$. Then $0 \neq a \in N_0(G)$ and $a(U) = 0$.

Now let $F := \{f_1, \ldots, f_n\}$ be a finite subset of $N_0(G)$. Let $V_i := f_i^{-1}(U)$ for $1 \leq i \leq n$ and $V := \bigcap_{i=1}^{n} V_i$. Note that $0 \in V$. If $V = G$, $af_i = 0$ for $1 \leq i \leq n$ so $aFx = 0$ for any $0 \neq x \in N_0(G)$ and we are done. Suppose that $V \neq G$. Let W be an element of $\mathcal{B}$ such that $0 \in W \subseteq V$. We have that $W \neq 0$, since then G would be discrete; however, by the hypothesis at the beginning of this section, G has more than one element, and is connected, and thus cannot be discrete. Let $0 \neq h \in W$. Then there exists a continuous function $\lambda : G \longrightarrow [0, 1]$ such that $\lambda(0) = 0$ and $\lambda(h) = 1$. Since W is arcwise connected, there exists a continuous function $\mu : [0, 1] \longrightarrow W$ with $\mu(0) = 0$ and $\mu(1) = h$. Let $x := \mu\lambda$. Then $x \in N_0(G)$, $x(h) = h$ and $x(G) \subseteq W \subseteq V$. It follows that $aFx = 0$ but $x \neq 0$. Hence $N_0(G)$ is not strongly prime. □

4. Sandwich near-rings

Let X and G be a topological space and a topological group respectively, and let $\theta : G \longrightarrow X$ be a continuous map. The *sandwich near-ring* $N_0(G, X, \theta)$ is the set $\{a : X \longrightarrow G \mid a$ is continuous and $a\theta(0) = 0\}$. Addition is pointwise and multiplication is defined by $a \cdot b := a\theta b$. If the topologies on X and G are discrete we denote the near-ring by $M_0(G, X, \theta)$.

Proposition 4.1. *Suppose that X is a 0-dimensional, T_0 topological space and G is a T_0 topological group, both of which have more than one element. Then $N_0(G, X, \theta)$ is equiprime if and only if θ is injective and $cl(\theta(G)) = G$.*

Proof. Suppose that θ is injective and that $\mathrm{cl}(\theta(G)) = X$. Let $a, b, c \in N_0(G, X, \theta), a \neq 0, b \neq c$. Let $x, y \in X$ be such that $a(x) \neq 0, b(y) \neq c(y)$. Note that we may assume, without loss of generality that $x \in \theta(G)$. (For if $x \notin \theta(G)$, it is a limit point of $\theta(G)$, since $\mathrm{cl}(\theta(G)) = X$. By continuity of x, there exists an open set U of X such that $x \in U, a(t) \neq 0$ for all $t \in U$. Then $U \cap \theta(G) \neq \emptyset$. Now replace x with any point $z \in U \cap \theta(G)$.) Let $g \in G$ be such that $\theta(g) = x$. Since $b(y) \neq c(y)$ either $b(y) \neq 0$ or $c(y) \neq 0$. Assume the former. Since θ is injective, $\theta b(y) \neq \theta(0)$. Since X is T_0 and 0-dimensional, it is T_1. Hence there exists a clopen subset V of X such that $\theta b(y) \in V,\ \theta(0) \notin V,\ \theta c(y) \notin V$. Define $n : X \to G$ by

$$n(t) := \begin{cases} g & t \in V \\ 0 & t \in X \backslash V \end{cases}.$$

Then $n\theta(0) = 0$ and $a \cdot n \cdot b(y) = a\theta n\theta b(y) = a\theta(g) = a(x) \neq 0$ and $a \cdot n \cdot c(y) = a\theta n\theta c(y) = a\theta(0) = 0$. Hence $a \cdot n \cdot b \neq a \cdot n \cdot c$, and so $N_0(G, X, \theta)$ is equiprime.

Conversely, suppose that $N_0(G, X, \theta)$ is equiprime. Let $g_1, g_2 \in G$ be such that $\theta(g_1) = \theta(g_2)$. Let U be a clopen, proper subset of X such that $\theta(0) \in U$. Define $a, b : X \longrightarrow G$ by

$$a(x) := \begin{cases} 0 & x \in U \\ g_1 & x \in X \backslash U \end{cases}, \quad b(x) = \begin{cases} 0 & x \in U \\ g_2 & x \in G \backslash U \end{cases}.$$

Then $a, b \in N_0(G, X, \theta)$, $a \neq 0$, and it is easily verified that $a \cdot n \cdot a = a \cdot n \cdot b$ for all $n \in N_0(G, X, \theta)$. Since $N_0(G, X, \theta)$ is equiprime, $a = b$ and so $g_1 = g_2$. Hence θ is injective.

Suppose $\mathrm{cl}(\theta(G)) \neq X$. Then there exists an element x of X which is not a limit point of $\theta(G)$. Since X is 0-dimensional, there exists a clopen subset V of X such that $x \in V, V \cap \theta(G) = \emptyset$. Let $0 \neq h \in G$ and define $c : X \longrightarrow G$ by

$$c(x) := \begin{cases} h & x \in V \\ 0 & x \in X \backslash V \end{cases}.$$

Then $c \in N_0(G, X, \theta)$ and $c \cdot n \cdot c = 0 = c \cdot n \cdot 0$ for all $n \in N_0(G, X, \theta)$. Since $c \neq 0$, this implies that $N_0(G, X, \theta)$ is not equiprime. This contradiction shows that $\mathrm{cl}(\theta(G)) = X$. □

Remark 4.2. 1. Proposition 4.1 generalises Proposition 9.1 of [11] if we take the topologies on G and X to be discrete. In this case the condition $\mathrm{cl}(\theta(G)) = X$ becomes $\theta(G) = X$, i.e. θ is surjective.

2. If the conditions of Proposition 4.1 are satisfied, it need not hold that $\theta(G) = X$. For example, let $\mathbb{Q}$ be the additive group of the rationals and let $X := \mathbb{Q} \cup \{\sqrt{2}\}$, both with the relative topology with respect to the usual topology on the real numbers. Then X is a 0-dimensional T_0 space. Let $\theta : \mathbb{Q} \longrightarrow X$ be the inclusion map. Then θ is injective and $\mathrm{cl}(\theta(G)) = X$. Thus $N_0(\mathbb{Q}, X, \theta)$ is equiprime but $\theta(\mathbb{Q}) = \mathbb{Q} \neq X$.

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