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# Tauvel's Height Formula in Iterated Differential Operator Rings

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**Abstract.** Let $k$ be a field of positive characteristic, $R$ an associative algebra over $k$ and let $\Delta_{1,n} = \{\delta_1, \dots, \delta_n\}$ be a finite set of $k$-linear derivations from $R$ to $R$. Let $A = R_n = R[\theta_1, \delta_1] \cdots [\theta_n, \delta_n]$ be an iterated differential operator $k$-algebra over $R$ such that $\delta_j(\theta_i) \in R_{i-1}\theta_i + R_{i-1}; 1 \le i < j \le n$.
As central result we show that if $R$ is noetherian affine $\Delta_{1,n}$-hypernormal and if Tauvel's height formula holds for the $\Delta_{1,n}$-prime ideals of $R$, then Tauvel's height formula holds in $A$.
In particular, let $g$ be a completely solvable finite-dimensional $k$-Lie algebra acting by derivations on $R$ and let $U(g)$ be the enveloping algebra of $g$. If $R$ is noetherian affine $g$-hypernormal and if Tauvel's height formula holds for the $g$-prime ideals of $R$, then Tauvel's height formula holds in the crossed product of $R$ by $U(g)$.

## 0. Introduction

Throughout the paper, $k$ is a field and all rings (except the Lie algebras over $k$) are associative with identity. Let $A$ be a $k$-algebra. Suppose that $\Delta$ is a set of derivations of $A$. An ideal $I$ of $A$ is a $\Delta$-ideal (or a $\Delta$-invariant ideal) provided $\delta(I) \subseteq I$ for all $\delta \in \Delta$. A $\Delta$-prime ideal of $A$ is any proper $\Delta$-ideal $P$ such that whenever $I$ and $J$ are $\Delta$-ideals of $A$ satisfying $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.

**Remark 0.1.** If $A$ is noetherian and if $k$ has characteristic 0, then by [1, Corollary 2.10], the $\Delta$-prime ideals of $A$ are prime. This result breaks down completely in positive characteristic [2, Lemma 1.2 and the remark below it].

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Let $A$ be noetherian, and let $P$ be a prime ideal of $A$. The height of $P$, denoted $ht(P)$, is the supremum of the lengths of chains of prime ideals with $P$ at the top. If $P$ is not prime, its height is the infimum of the heights of its minimal prime ideals. If $P$ is a $\Delta$-prime ideal of $A$, the $\Delta$-height of $P$ denoted $\Delta$-$ht(P)$ is the supremum of the lengths of chains of $\Delta$-prime ideals with $P$ at the top.

We will denote by $d()$ the Gelfand-Kirillov dimension of a $k$-algebra; for its properties the reader can consult [3] and [4]. Let us recall the following result of P. Tauvel which relates the height of a prime ideal to the corresponding factor algebra:

**Theorem 0.2.** (Tauvel [5]) *Let $g$ be a finite dimensional solvable Lie algebra over an algebraically closed field $k$ of characteristic zero and $A = U(g)$ the enveloping algebra of $g$. If $P$ is a prime ideal in $A$, then*

$$d(A) = ht(P) + d(A/P). \qquad (*)$$

We will call $(*)$ Tauvel's height formula.

By a domain we mean a ring without divisors of zero. A $k$-algebra is called affine if it is finitely generated as a $k$-algebra.

By Schelter's theorem [6], Tauvel's height formula holds in any noetherian affine prime P.I. algebra over a field. We give below some examples of such rings.

**Examples 0.3.** (1) Let $k$ be a field of positive characteristic. Let $R$ be a commutative affine $k$-algebra which is an integral domain and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a finite set of commuting $k$-linear derivations from $R$ to $R$. Let $A = R[\theta_1, \ldots, \theta_n;\ \delta_1, \ldots, \delta_n]$ be the corresponding ring of differential operators. Then $A$ is a noetherian prime affine P.I. algebra [7, Theorem 4.1].

(2) Let $g$ be a finite dimensional Lie algebra over a field $k$ of characteristic $p > 0$ and $U(g)$ its enveloping algebra. By [8, 1.10], $U(g)$ is a finite module over its affine center and so is a noetherian affine P.I. algebra. As a consequence of this result, if $R$ is a noetherian affine prime P.I. algebra then so is $R \otimes U(g)$ [4].

Let $R$ be a $k$-algebra and $A = R[\theta_1, \delta_1] \cdots [\theta_n, \delta_n]$ an iterated differential operator $k$-algebra over $R$. Set $R_i = R[\theta_1, \delta_1] \cdots [\theta_i, \delta_i]$; $0 \leq i \leq n$; so $R_0 = R$ and $R_n = A$. Consider the following three conditions:

(i) each $\delta_i$ is a derivation of $R; 1 \leq i \leq n$.

(ii) $\delta_j(\theta_i) \in R_{i-1}\theta_i + R_{i-1}; 1 \leq i < j \leq n$.

(iii) $\delta_j(R_i) \subseteq R_i; 0 \leq i < j \leq n$.

Note that (i) and (ii)$\Rightarrow$(iii).

**Examples 0.4.** (1) Let $R$ be a $k$-algebra and let $\{\delta_1, \ldots, \delta_n\}$ be a finite set of commuting $k$-linear derivations from $R$ to $R$. Then the ring of differential operators $A = R[\theta_1, \ldots, \theta_n; \delta_1, \ldots, \delta_n]$ is an iterated differential operator $k$-algebra over $R$ and the conditions (i) and (ii) are satisfied (we extend each $\delta_j$ to $A$ by setting $\delta_j(\theta_i) = 0;\ 1 \leq i \leq j \leq n$).

(2) Let $R$ be a $k$-algebra and $g$ a completely solvable $k$-Lie algebra of finite dimension $n$ acting on $R$ by derivations and $A$ the crossed product of $R$ by $U(g)$. Then $A$ is an iterated differential operator $k$-algebra over $R$ and the conditions (i) and (ii) are satisfied.

In [9] we have shown the following generalization of Theorem 0.2:

**Theorem 0.5.** *Let $k$ be a field of characteristic zero, $R$ a noetherian affine $k$-algebra and $A = R_n = R[\theta_1, \delta_1] \cdots [\theta_n, \delta_n]$ an iterated differential operator $k$-algebra over $R$ with conditions* (i) *and* (iii). *Let $P$ be a $\Delta_{m+1,n}$-invariant prime ideal of $B = R_m = R[\theta_1, \delta_1] \cdots [\theta_m, \delta_m]$; $0 \le m \le n$. Assume that for all $\Delta_{1,n}$-invariant prime ideals of $R$*

(1) *Tauvel's height formula holds;*

(2) *the $\Delta_{1,n}$ height and the height coincide.*

*Then*

$$d(B) = d(B/P) + ht(P).$$

The purpose of this paper is to establish Theorem 0.5 when the characteristic of $k$ is $p > 0$. In order to do this we need of some restrictions: we will assume that the conditions (i) and (ii) are satisfied in $R$ and that $R$ is a $\Delta_{1,n}$-hypernormal ring.

**Remarks 0.6.** (1) If our iterated differential operator $k$-algebra is one of the rings of examples 0.3, then our result is not new.

(2) In the proof of Theorem 0.5, we have used the theorem of Goldie which states that every essential right ideal of a semiprime right Goldie ring contains a regular element. In the present context, this theorem of Goldie is not always true (see Remark 0.1). It is for this reason that we have made the $\Delta_{1,n}$-hypernormality assumption on $R$. This assumption enables us to establish an analog to the theorem of Goldie mentioned above (Corollary 1.3) and ensures that the $\Delta_{1,n}$-height and the height coincide for all $\Delta_{1,n}$-prime ideals of $R$ (Proposition 1.8).

## 1. Preliminary results

Let $A$ be a $k$-algebra. Suppose that $\Delta$ is a finite set of derivations of $A$. The ring $A$ is $\Delta$-prime if 0 is a $\Delta$-prime ideal of $A$. We say that $A$ is $\Delta$-simple provided $A^2 \neq 0$ and $A$ has no proper $\Delta$-ideals. An easy consequence of the definitions is that $\Delta$-simple implies $\Delta$-prime.

An element $x$ of $A$ is $\Delta$-normal if $xA = Ax$ and the ideal $Ax$ is a $\Delta$-ideal.

We will say that $A$ is

- $\Delta$-normally separated if for any pair of distinct comparable $\Delta$-prime ideals $P \subset Q$ of $A$, there exists $x \in Q - P$ such that $x + P$ is a $\Delta$-normal element of $A/P$.
- $\Delta$-hypernormal if, whenever $I \subset J$ are two $\Delta$-ideals of $A$, there exists $x \in J - I$ such that $x + I$ is a $\Delta$-normal element of $A/I$.
- $\Delta$-locally finite if every element of $A$ is contained in a finite dimensional $\Delta$-stable subspace of $R$.

Clearly, $\Delta$-simple implies $\Delta$-hypernormal and $\Delta$-hypernormal implies $\Delta$-normally separated.

**Lemma 1.1.** *Let $k$ be algebraically closed. Assume that $\Delta$ is a finite set of commuting derivations of $A$. Let $A$ be $\Delta$-locally finite and let $I$ be a nonzero $\Delta$-ideal of $A$. Then there* *is in $I$ a nonzero element $u$ such that $\delta(u) = \lambda u$ for all $\delta \in \Delta$; where $\lambda \in k$. If furthermore, $A$ is commutative, then $A$ is $\Delta$-hypernormal.*

Let $S$ be a nonempty subset of $A$. The left annihilator of $S$ is defined by

$$lann_A(S) = \{a \in A : as = 0 \quad \text{for all} \quad s \in S\}.$$

We define in a similar way the right annihilator $rann_A(S)$ of $S$. If $S$ is $\Delta$-ideal of $A$, then so is $lann_A(S)$.

The following result is the $\Delta$-invariant version of the well known result which asserts that any nonzero normal element in a prime ring is a regular element.

**Lemma 1.2.** *Let $A$ be a $\Delta$-prime ring and let $x$ be a nonzero $\Delta$-normal element of $A$. Then $x$ is a regular element in $A$.*

*Proof.* Let $v$ be an element of $A$ such that $vx = 0$. If $u$ is an element of $A$, we have $v(ux) = v(xw) = (vx)w = 0$ where $w \in A$ is such that $ux = xw$. Hence $v \in lann_A(Ax)$. Since $Ax$ is a nonzero $\Delta$-ideal in $A$, we have $lann_A(Ax) = 0$, by [10, page 71]. So $x$ is left regular. Let $v$ be an element of $A$ such that $xv = 0$. If $u$ is an element of $A$, we have $(xu)v = (wx)v = w(xv) = 0$ where $w \in A$ is such that $xu = wx$. Hence $v \in rann_A(xA)$. Since $xA$ is a nonzero $\Delta$-ideal in $A$, we have $rann_A(xA) = 0$, by [10, page 71]. So $x$ is right regular. □

**Corollary 1.3.** *Let $A$ be a $\Delta$-prime ring.*

(1) *Then every nonzero $\Delta$-ideal of $A$ is an essential right ideal.*
(2) *If the characteristic of $k$ is $p > 0$ and if $A$ is $\Delta$-hypernormal then any nonzero $\Delta$-ideal of $A$ contains a regular element.*
(3) *If the characteristic of $k$ is $p > 0$ and if $A$ is $\Delta$-normally separated, then any nonzero $\Delta$-prime ideal of $A$ contains a regular element.*

*Proof.* (1) Let $I$ be a right ideal of $A$ and let $J$ be a nonzero $\Delta$-ideal of $A$. Suppose that $I \cap J = 0$. Then $IJ = 0$. If $I \neq 0$, then $lann_A(J) \neq 0$ and is a $\Delta$-ideal. This is a contradiction since $A$ is a $\Delta$-prime ring. It follows that $J$ is an essential right ideal of $A$.

(2) and (3) follow from Lemma 1.2. □

**Remark 1.4.** Even if $A$ is right noetherian, we are unable to prove (1.3) without the normality hypothesis. Of course, in the characteristic 0 case, this follows from the Goldie's theorem which asserts that every essential right ideal of a semiprime right Goldie ring contains a regular element.

By [7, Lemma 2.2], if $A$ is right noetherian and if $P$ is a $\Delta$-prime ideal of $A$, there is exactly one prime ideal in $A$ minimal over $P$. So if $I$ is a $\Delta$-prime ideal of $A$, then $ht(I) = ht(P)$ where $P$ is the unique prime ideal in $A$ minimal over $I$.

Given an ideal $I$ of $A$ we denote by $I^+$ the largest $\Delta$-ideal of $A$ contained in $I$. If $J$ is an ideal of $A$ and if $I$ is a $\Delta$-ideal of $A$ with $I \subseteq J$, then $(J/I)^+ = J^+/I$.

By [1, Lemma 2.4], if $I$ is a prime ideal of $A$, then $I^+$ is a $\Delta$-prime ideal of $A$ and by [1, Lemma 2.11], if $A$ is noetherian, then a prime ideal $P$ of $A$ is a minimal prime ideal in $A$ if and only if $P^+$ is a minimal $\Delta$-prime ideal in $A$.

**Lemma 1.5.** *Let $A$ be noetherian and let $Q$ be a $\Delta$-prime ideal of $A$. If $P$ is the unique prime ideal in $A$ minimal over $Q$, then $P^+ = Q$.*

*Proof.* The ring $A/Q$ is noetherian $\Delta$-prime and $P/Q$ is the unique minimal prime ideal in $A/Q$. It follows from [1, Lemma 2.11] that $P^+/Q$ is a minimal $\Delta$-prime ideal in $A/Q$. Since $A/Q$ is a $\Delta$-prime ring, we must have $P^+ = Q$. □

**Lemma 1.6.** *Let $A$ be noetherian and let $Q$ be a $\Delta$-prime ideal of $A$. Then $\Delta$-$ht(Q) \leq ht(Q)$.*

*Proof.* Set $\Delta$-$ht(Q) = d$. Then there is a saturated chain of $\Delta$-prime ideals

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{d-1} \subset Q_d = Q$$

in $A$. Let $P$ be the unique prime ideal in $A$ minimal over $Q$. Then $ht(Q) = ht(P)$. Suppose that $P$ is also the unique minimal prime ideal over $Q_{d-1}$. Then by Lemma 1.5, $P^+ = Q = Q_{d-1}$ which is a contradiction. So $P/Q_{d-1}$ contains strictly a prime ideal $P_{d-1}/Q_{d-1}$ (the only one) in $A/Q_{d-1}$ minimal over $Q/Q_{d-1}$. Hence, $P_{d-1}$ is the unique prime ideal in $A$ minimal over $Q_{d-1}$ and $P_{d-1} \subset P$. Continuing this processus, we get in $A$ a chain of prime ideals

$$P_0 \subset P_1 \subset \cdots \subset P_{d-1} \subset P_d = P$$

where each $P_i$ is the unique minimal prime ideal over $Q_i$. It follows that $d \leq ht(P)$. □

**Lemma 1.7.** *Let $A$ be noetherian $\Delta$-normally separated and let $P$ be a $\Delta$-prime ideal of $A$. Then $ht(P) < \infty$.*

*Proof.* Let $I$ be the unique minimal prime ideal over $P$ and let $J$ be a prime ideal of $A$ strictly contained in $I$. After factoring out $A$ by $J^+$, we may assume that $J^+ = 0$. So $A$ is a $\Delta$-prime ring and $J$ cannot contain a nonzero $\Delta$-normal element of $A$. Since $A$ is $\Delta$-normally separated, $P$ contains a nonzero $\Delta$-normal element $x$. So, for any element $a \in A$, we have $ax - xa_1 = 0 \in J$ for some $a_1 \in A$. It follows that $x + J \in I/J$ and $x + J$ is a nonzero normal element in $A/J$. By [11, Theorem 3.5], we have $ht(I) < \infty$. But $ht(P) = ht(I)$, so the result follows. □

**Proposition 1.8.** *Let $A$ be noetherian $\Delta$-normally separated and let $P$ be a $\Delta$-prime ideal of $A$. Then $\Delta$-$ht(P) = ht(P) < \infty$.*

*Proof.* By Lemma 1.7, $ht(P) < \infty$. Set $ht(P) = d$. If $d = 0$, the result is true by Lemma 1.6. Suppose that $d \neq 0$ and let $I$ be the unique minimal prime ideal over $P$ in $A$. Then we have $d = ht(P) = ht(I)$. Let

$$Q_0 \subset Q_1 \subset \cdots \subset Q_d = I$$

be a saturated chain of prime ideals in $A$ ending at $I$. We have $(Q_0)^+ \subseteq I^+ = P$; $ht(Q_0)^+ = ht(Q_0) = 0$; $ht(I) = ht(I/(Q_0)^+)$ and $ht(P) = ht(P/(Q_0)^+)$. So, by passing to the ring

$A/(Q_0)^+$, we can suppose that $(Q_0)^+ = 0$; i.e. $A$ is a $\Delta$-prime ring. Since $P$ is nonzero, there exists in $P \subseteq I$ a nonzero element $x$ such that $x$ is a $\Delta$-normal element of $A$. Since $x$ is a $\Delta$-normal element in the $\Delta$-prime ring $A$, Lemma 1.2 implies that $x$ is regular in $A$. Set $\bar{A} = A/Ax$ and $\bar{I} = I/Ax$. By the Principal Ideal Theorem, $ht\bar{I} = d-1$. But $\bar{I}$ is the unique minimal prime ideal over $\bar{P}$; so $ht(\bar{I}) = ht(\bar{P})$. From this, we deduce that $ht(\bar{P}) = d-1$. Suppose by an induction hypothesis that the result is true for $\bar{P}$ in $\bar{A}$. So there exists a saturated chain of $\Delta$-prime ideals in $\bar{A}$ ending at $\bar{P}$ of length $d-1$

$$\bar{P}_1 \subset \bar{P}_2 \subset \cdots \subset \bar{P}_{d-1} \subset \bar{P}_d = \bar{P}.$$

By taking the inverse images, we get a chain of $\Delta$-ideals in $A$

$$0 \subset Ax \subset P_1 \subset P_2 \subset \cdots \subset P_d = P$$

where the $P_i$ are $\Delta$-prime ideals. Hence, $d \leq \Delta\text{-}ht(P)$; and the result follows from Lemma 1.6. □

**Lemma 1.9** *Let $A$ be noetherian $\Delta$-normally separated and let $P$ be a $\Delta$-prime ideal of $A$. Then $d(A) \geq d(A/P)+\Delta\text{-}ht(P)$.*

*Proof.* Let

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_l = P$$

be a chain of $\Delta$-prime ideals in $A$. For each $i$, $P_{i+1}/P_i$ is a nonzero $\Delta$-prime ideal of the $\Delta$-prime $\Delta$-normally separated $k$-algebra $A/P_i$. By Corollary 1.3, it contains a regular element of $A/P_i$. It follows from [3, Proposition 3.15] that $d(A/P_{i+1}) + 1 \leq d(A/P_i)$, and induction yields

$$d(A/P) + l \leq d(A/P_0) \leq d(A)$$

by [3, Lemma 3.1]. Taking the supremum of $l$ gives the result. □

**Lemma 1.10.** *Let $A$ be a noetherian $\Delta$-prime affine P.I. algebra. Let $P$ be a $\Delta$-prime ideal of $A$. Then we have*

$$d(A) = d(A/P) + ht(P).$$

*Proof.* Clearly, the result is true if $P = 0$. Now suppose that $P \neq 0$. Let $I$ be the unique minimal prime ideal over $P$ and $J$ the unique minimal prime ideal in $A$. So $J \subset I$ and $ht(P) = ht(I) = ht(I/J)$. Since $A/J$ is a noetherian affine prime P.I. algebra, by Schelter's theorem we have $d(A/J) = d(A/I) + ht(I/J)$. On the other hand, $A$ is a noetherian P.I. algebra and $J$ is its only minimal prime ideal. Hence, by [3, Theorem 10.15], $d(A) = d(A/J)$. Also, $A/P$ is a noetherian P.I. algebra and $I/P$ is its only minimal prime ideal. Hence, by [3, Theorem 10.15] again, $d(A/P) = d(A/I)$. □

## 2. The main result

Let $R$ be a $k$-algebra. Let $\delta$ be a $k$-derivation of $R$. The left $R$-module structure of $R[\theta]$, with $\theta$ being an indeterminate, can be extended to a $k$-algebra structure by setting $\theta r = r\theta + \delta(r)$; $r \in R$. The ring thus obtained is denoted $R[\theta, \delta]$ and is called a differential operator $k$-algebra over $R$.

Let $n$ be a positive integer. A $k$-algebra $A = R_n$ is said to be an iterated differential operator $k$-algebra over $R$ if there exists a chain of subalgebras

$$R = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_{n-1} \subseteq R_n = A$$

such that each $R_i$ is (isomorphic to) a differential operator $k$-algebra over $R_{i-1}$. So for each $1 \leq i \leq n$ there is a derivation $\delta_i$ of $R_{i-1}$ such that $R_i = R_{i-1}[\theta_i, \delta_i]$. Note that each $R_l$ is a free left and right $R$-module with basis $\theta_1^{i_1}\theta_2^{i_2}\cdots\theta_{l-1}^{i_{l-1}}\theta_l^{i_l}$ and that for all positive integer $l$ and for all $u \in R_{i-1}$

$$\theta_i^l u = u\theta_i^l + \sum_{0<j<l} \binom{l}{j} \delta_i^j(u)\theta_i^{l-j}.$$

We will set $\Delta_{i,j} = \{\delta_i, \delta_{i+1}, \ldots, \delta_j\}$; $1 \leq i < j \leq n$.

If $R$ is a prime ring, then it is $\Delta_{i,j}$-prime for all $1 \leq i < j \leq n$.

From now on we fix a $k$-algebra $R$, a positive integer $n$ and an iterated differential operator $k$-algebra $A = R_n$ with the following conditions:

(i) each $\delta_i$ is a derivation of $R; 1 \leq i \leq n$

(ii) $\delta_j(\theta_i) \in R_{i-1}\theta_i + R_{i-1}; 1 \leq i < j \leq n$.

From (i) and (ii) we deduce that

$$\delta_j(\theta_i^l) \in R_{i-1}\theta_i^l + \sum_{q<l} R_{i-1}\theta_i^q;\ 1 \leq i < j \leq n$$

and

$$\delta_j(R_i) \subseteq R_i;\ 0 \leq i < j \leq n.$$

In the sequel we shall use the conventions that $\Delta_{n+1,n} = \emptyset$ and $\Delta_{n,n} = \delta_n$, so the $\Delta_{n+1,n}$-ideals (resp. the $\Delta_{n+1,n}$-prime ideals) of $A = R_n$ are precisely the ideals (resp. the prime ideals) of $R_n$.

**Lemma 2.1.** *For a fixed $i$; $1 \leq i \leq n$, if $R_{i-1}$ is $\Delta_{i,n}$-hypernormal, then $R_i$ is $\Delta_{i+1,n}$-hypernormal.*

*Proof.* Since $R_i = R_{i-1}[\theta_i, \delta_i]$, every element of $R_i$ has a unique expression $\sum_j r_j\theta_i^j$ (the sum is finite). We shall set $\theta_i = \theta$ and $\delta_i = \delta$. Let $I_1 \subset I_2$ be two $\Delta_{i+1,n}$-ideals of $R_i$. Choose an element $f \in I_2 - I_1$ of minimal degree $m$ and, set for $j = 1; 2$

$$K_j = \{c \in R_{i-1} :\ c\theta^m + \sum_{l=0}^{m-1} c^l\theta^l \in I_j\ ;\ \text{where the } c_l \text{ are elements of } R_{i-1}\}.$$

Clearly, $K_1$ and $K_2$ are ideals of $R_{i-1}$ and $K_1 \subset K_2$.

Let $c \in K_1$. Hence, there are elements $c_q \in R_{i-1}$ such that $p = c\theta^m + \sum_{q=0}^{m-1} c_q\theta^q$ is an element of $I_1$. Clearly, $\theta p - p\theta$ is an element of $I_1$ and the coefficient of $\theta^m$ in $\theta p - p\theta$ is $\delta(c)$. So $\delta(c) \in K_1$. Let $\delta_l \in \Delta_{i+1,n}$. Then $\delta_l(p)$ is an element of $I_1$. If we denote by $s_l$ the coefficient of $\delta_l(\theta^m)$, then the coefficient of $\theta^m$ in $\delta_l(p) - ps_l$ is $\delta_l(c)$ and $\delta_l(p) - ps_l \in I_1$; so $\delta_l(c) \in K_1$. It follows that $K_1$ is a $\Delta_{i,n}$-ideal of $R_{i-1}$.

In a similar way, we show that $K_2$ is a $\Delta_{i,n}$-ideal of $R_{i-1}$. Suppose that $R_{i-1}$ is $\Delta_{i,n}$-hypernormal. Hence there exists $b \in K_2 - K_1$ such that $ub - bv \in K_1$ and $\delta_j(b) - r_j b \in K_1$ for all $u \in R_{i-1}$ and for each $\delta_j \in \Delta_{i,n}$, where $v$ and $r_j$ are some elements of $R_{i-1}$. Let $t = b\theta^m + q \in I_2 - I_1$ with $deg(q) < m$. The coefficient of $\theta^m$ in $ut - tv$ is $ub - bv \in K_1$. The minimality of $m$ enables us to conclude that $ut - tv \in I_1$ for all $u \in R_{i-1}$ (where $v$ is some element of $R_{i-1}$). Let $\delta_l \in \Delta_{i+1,n} \subseteq \Delta_{i,n}$. If we denote by $s_l$ the coefficient of $\delta_l(\theta^m)$, then the coefficient of $\theta^m$ in $\delta_l(t) - r_l t - ts_l$ is $\delta_l(b) - r_l b \in K_1$. By the minimality of $m$, we have $\delta_l(t) - r_l t - ts_l \in I_1$. So $\delta_l(t) - r_l t - s'_l t \in I_1$ where $s'_l \in R_{i-1}$ is such that $ts_l - s'_l t \in I_1$. On the other hand, the coefficient of $\theta^m$ in $\theta t - t\theta - r_i t$ is $\delta_i(b) - r_i b \in K_1$. So the minimality of $m$ implies that $\theta t - t\theta - r_i t \in I_1$; i.e. $\theta t - t(\theta + r'_i) \in I_1$, where $r'_i \in R_{i-1}$ is such that $r_i t - tr'_i \in I_1$. We deduce easily from all this that $t + I_1$ is a $\Delta_{i+1,n}$-normal element in $A/I_1$.□

**Proposition 2.2.** *If $R$ is $\Delta_{1,n}$-hypernormal, then each $R_i$ is $\Delta_{i+1,n}$-hypernormal; in particular, $A = R_n$ is hypernormal.*

**Lemma 2.3.** *Fix two integers $i$ and $j$ such that $0 \le i < j \le n$.*

(1) *If $I$ is a $\Delta_{j+1,n}$-ideal of $R_j$, then $I \cap R_i$ is a $\Delta_{i+1,n}$-ideal of $R_i$ and $(I \cap R_i)R_j \subseteq I$.*

(2) *If $I$ is a $\Delta_{i+1,n}$-ideal of $R_i$, then $IR_j$ is a $\Delta_{j+1,n}$-ideal of $R_j$. Moreover, $(IR_j) \cap R_i = I$ and $R_j/(IR_j) \simeq (R_i/I)[\theta_{i+1}, \delta_{i+1}] \cdots [\theta_j, \delta_j]$.*

(3) *An ideal $Q$ of $R_i$ is $\Delta_{i+1,n}$-prime if and only if $Q = P \cap R_i$ for a $\Delta_{j+1,n}$-prime ideal $P$ of $R_j$.*

(4) *An ideal $Q$ of $R_i$ is $\Delta_{i+1,n}$-prime if and only if $QR_j$ is a $\Delta_{j+1,n}$-prime ideal of $R_j$.*

*Proof.* (1) and (2) straightforward.

(3) Adapt the proof of [1, Lemma 4.3].

(4) We know that every element of $R_{i+1}$ has a unique expression $\sum_l r_l\theta_i^l$ with $r_l \in R_i$ (the sum is finite). If $I$ is an ideal of $R_{i+1}$ we denote by $\tau(I)$ the set of leading coefficients of elements of $I$. If $I$ is $\Delta_{i+2,n}$-invariant then $\tau(I)$ is a $\Delta_{i+1,n}$-ideal of $R_i$ (see the proof of Lemma 2.1). Now assume that $R_i$ is $\Delta_{i+1,n}$-prime and let $I$ and $J$ be $\Delta_{i+2,n}$-ideals of $R_{i+1}$ such that $IJ = 0$. One shows easily that $\tau(I)\tau(J) = o$. Since $R_i$ is $\Delta_{i+1,n}$-prime we have $\tau(I) = 0$ or $\tau(J) = 0$, and this clearly implies that $I = 0$ or $J = 0$; i.e. $R_{i+1}$ is $\Delta_{i+2,n}$-prime. So if $R_i$ is $\Delta_{i+1,n}$-prime then $R_j$ is $\Delta_{j+1,n}$-prime. □

From now on we assume that $k$ is a field of positive characteristic. Suppose that $R$ is noetherian $\Delta_{1,n}$-hypernormal. Let $P$ be a $\Delta_{m+1,n}$-prime ideal of $B = R_m$; $0 \le m \le n$. Set $Q = P \cap R$. By Lemma 1.7 and Proposition 1.8, $\Delta_{1,n}$-$ht(Q) = ht(Q) < \infty$. Thus there exists a saturated chain of $\Delta_{1,n}$-prime ideals of $R$ with $Q$ at the top

$$Q_0 \subset Q_1 \subset \cdots \subset Q_l = Q$$

where $l$=$\Delta_{1,n}$-$ht(Q)$. Set $P_i = Q_iB$; $0 \le i \le l$ and $P_{l+i} = (P \cap R_i)B$; $0 \le i \le m$. So $P_l = Q_lB = QB$ and $P_{l+m} = P$. By Lemma 2.3, all the $P_i$ are $\Delta_{m+1,n}$-prime ideals of $B$. Consider the chain of $\Delta_{m+1,n}$-prime ideals of $B$ ending at $P$

$$P_0 \subset P_1 \subset \cdots \subset P_l \subseteq P_{l+1} \subseteq \cdots \subseteq P_{l+m} = P. \qquad (\alpha)$$

**Proposition 2.4.** *Let $R$ be noetherian, affine and $\Delta_{1,n}$-hypernormal with finite Gelfand-Kirillov dimension and let $P$ be a $\Delta_{m+1,n}$-prime ideal of $B = R_m; 0 \le m \le n$. Assume that Tauvel's height formula is valid for all $\Delta_{1,n}$-prime ideals of $R$. Then the length of the chain $(\alpha)$ is $d(B) - d(B/P)$.*

*Proof.* The proof is similar to that of [9, Proposition 3.1]. We proceed by induction on $m$. If $m = 0$, the result is true by the hypotheses. Assume the result true in $R_i$, $0 \le i < m$. Set $B' = R_{m-1}$ and $P' = P \cap B'$; so $P \cap R_i = P' \cap R_i$ for $0 \le i \le m-1$. Set $P'_i = Q_iB'$; $0 \le i \le l$ and $P'_{l+i} = (P' \cap R_i)B'$; $0 \le i \le m-1$; thus $P' = P'_{l+m-1}$. By the induction hypothesis, the chain

$$P'_0 \subset P'_1 \subset \cdots \subset P'_l = QB' \subseteq P'_{l+1} \subseteq \cdots \subseteq P'_{l+m-1} = P' \qquad (\beta)$$

has length $d(B') - d(B'/P')$. By [12], its length is $d(B) - d(B/P'B)$. Clearly $P_i = P'_iB$ for $0 \le i \le l$; $P_{l+i} = P'_{l+i}B$ and $P_{l+i} \cap B' = P'_{l+i}$ for $0 \le i \le m-1$. From this we deduce that $P_i = P_{i+1}$ if and only if $P'_i = P'_{i+1}$, $0 \le i \le l$ and $P'_{l+i+1} = P'_{l+i}$ if and only if $P_{l+i+1} = P_{l+i}$. It follows that the chain of $\Delta_{m+1,n}$-prime ideals of $B$

$$P_0 \subset P_1 \subset \cdots \subset P_l \subseteq P_{l+1} \subseteq \cdots \subseteq P_{l+m-1} = P'B$$

has the same length as $(\beta)$. So its length is $d(B) - d(B/P'B)$. If $P = P'B$, the result is true. If $P'B \subset P$, the chain $(\alpha)$ has length $d(B) - d(B/P'B) + 1$, by [12]. Let us prove that $d(B/P) = d(B/P'B) - 1$. As $B'/P'$ is a subalgebra of $B/P$, we have $d(B'/P') \le d(B/P)$; so $d(B/P'B) - 1 \le d(B/P)$. On the other hand, $P/P'B$ is a nonzero $\Delta_{m+1,n}$-prime ideal of the $\Delta_{m+1,n}$-prime $\Delta_{m+1,n}$-hypernormal ring $B/P'B$. By Corollary 1.3, $P/P'B$ contains a regular element. By [3, Proposition 3.15], $d(B/P) \le d(B/P'B) - 1$. This proves the proposition. □

The main result of the paper can be formulated as the following

**Theorem 2.5.** *Let $R$ be noetherian, affine and $\Delta_{1,n}$-hypernormal with finite Gelfand-Kirillov dimension and let $P$ be a $\Delta_{m+1,n}$-prime ideal of $B = R_m; 0 \le m \le n$. Assume that Tauvel's height formula is valid for all $\Delta_{1,n}$-prime ideals of $R$. Then*

$$d(B) = d(B/P) + ht(P).$$

*Proof.* By Lemma 1.9 and Proposition 2.4, we have $\Delta_{m+1,n}$-$ht(P) \le d(B) - d(B/P) \le \Delta_{m+1,n}$-$ht(P)$. By Proposition 1.8, we have $\Delta_{m+1,n}$-$ht(P) = ht(P)$. □

**Remark 2.6.** Because of Remark 1.4, we are unable to establish Proposition 2.4 and Theorem 2.5 in a more general setting as in the characteristic 0 case.

We shall deduce from Theorem 2.5 some corollaries.

**Corollary 2.7.** *Let $R$ be a noetherian, affine, $\Delta_{1,n}$-prime, $\Delta_{1,n}$-hypernormal P.I. algebra and let $P$ be a $\Delta_{m+1,n}$-prime ideal of $B = R_m$; $0 \leq m \leq n$. Then*

$$d(B) = d(B/P) + ht(P).$$

**Corollary 2.8.** *Let $R$ be noetherian, affine and $\Delta_{1,n}$-simple with finite Gelfand-Kirillov dimension and let $P$ be a $\Delta_{m+1,n}$-prime ideal of $B = R_m; 0 \leq m \leq n$. Then*

$$d(B) = d(B/P) + ht(P).$$

## 3. Application of the main result

Let $g$ be a $k$-Lie algebra of finite dimension $n$ and $U(g)$ the enveloping algebra of $g$. We suppose that $g$ acts by derivations on $R$ and we denote by $A = R \star g$ the crossed product of $R$ by $U(g)$ (see [4, 13]).

For each $X \in g$, we denote by $\bar{X}$ the canonical image of $X$ in $R \star g$ and we set $\delta(X) = \delta_X$. We recall that there exists a linear map $\delta$ from $g$ to the $k$-Lie algebra of $k$-derivations of $R$ and a bilinear map $t : g \times g \to R$ such that $[\bar{X}, \bar{Y}] - \overline{[X, Y]} = t(X, Y)$. Let $h$ be an ideal of $g$. We extend the action of $g$ on $R \star h$ by setting $\delta_X(Y) = [\bar{X}, \bar{Y}]$ for all $X \in g$ and $Y \in h$. It is well known that $R \star g = (R \star h) \star g/h$.

The notions of $g$-invariant ideal, $g$-prime ideal, $g$-normal element and $g$-hypernormal ring are well known in the literature [1], [13], [14] and [15].

If $g$ is completely solvable, we fix a composition series of $g$; i.e. a chain

$$0 = g_0 \subset g_1 \subset \cdots \subset g_n = g$$

of ideals of $g$ such that $g_{i+1}/g_i$ has dimension one. We shall set $R_i = R \star g_i$; $0 \leq i \leq n$; so $R_0 = R$ and $R \star g_n = R \star g$. Choose $X_i$ in $g_i - g_{i-1}$ such that $X_i + g_{i-1}$ is a basis of $g_i/g_{i-1}$. So $R_i \simeq R_{i-1}[\theta_i, \delta_i]$ the Ore extension of $R_{i-1}$ by $\delta_i$; where $\bar{X}_i$ is sent to $\theta_i$ and $\delta_i(r) = \delta_{X_i}(r)$ for any $r \in R_{i-1}$. Note that $\Delta_{1,n} = \{\delta_1, \delta_2, \ldots, \delta_n\}$ is a set of derivations of $R_i$; $0 \leq i \leq n$. Each $X_i + g_{i-1}$ is a $g$-eigenvector of $g_i/g_{i-1}$; so $\overline{[X, X_i]} - \lambda_i(X)\bar{X}_i \in R_{i-1}$ for any $X \in g$; where $\lambda_i(X) \in k$ is the $g$-eigenvalue of $X_i + g_{i-1}$. Hence $\delta_X(\bar{X}_i) - \lambda_i(X)\bar{X}_i \in R_{i-1}$.

It thereby follows that a crossed product of a $k$-algebra $R$ by the enveloping algebra of a completely solvable finite-dimensional $k$-Lie algebra is an iterated differential operator $k$-algebra over $R$ satisfying the conditions (i) and (ii).

Now we assume that $g$ is completely solvable and we keep the above notations. Then our main result may be applied to the ring $R_m = R \star g_m$; $0 \leq m \leq n$ and the following remark enables us to improve the result.

**Remark 3.1.** For each $0 \leq i \leq n$,

(1) an ideal of $R_i$ is

- $g/g_i$-invariant if and only if it is $g$-invariant.
- $\Delta_{i+1,n}$-invariant if and only if it is $g$-invariant.

- $\Delta_{i+1,n}$-prime if and only if it is $g$-prime.

(2) $R_i$ is $\Delta_{i+1,n}$-hypernormal if and only if $R_i$ is $g$-hypernormal.

(3) $R_i$ is $\Delta_{i+1,n}$-simple if and only if $R_i$ is $g$-simple.

The main result of this section is

**Theorem 3.2.** *Let $k$ be a field of positive characteristic, $g$ completely solvable and $R$ noetherian, affine and $g$-hypernormal with finite Gelfand-Kirillov dimension. Let $P$ be a $g$-prime ideal of $B = R_m = R \star g_m$; $0 \leq m \leq n$. Assume that Tauvel's height formula holds for the $g$-prime ideals of $R$. Then*

$$d(B) = d(B/P) + ht(P).$$

**Corollary 3.3.** *Let $k$ be a field of positive characteristic, $g$ completely solvable and $h$ an ideal of $g$. Let $P$ be a prime ideal of $A = U(h) \star g$. Then*

$$d(A) = d(A/P) + ht(P).$$

**Corollary 3.4.** *Let $k$ be a field of positive characteristic, $g$ completely solvable and $R$ noetherian, affine and $g$-simple with finite Gelfand-Kirillov dimension. Let $P$ be a $g$-prime ideal of $B = R_m = R \star g_m$; $0 \leq m \leq n$. Then*

$$d(B) = d(B/P) + ht(P).$$

**Remark 3.4.** By the proof of Theorem 2.5, we have assumed that $R$ is affine to ensure that $d(R_{i+1})/IR_{i+1} = d(R_i/I) + 1$ for all $g$-invariant ideals $I$ of $R_i = R \star g_i$. But if $R$ is $g$-locally finite, $R_i$ and $R_i/I$ are $g$-locally finite [15, Corollary 1.4]; so it is $g_{i+1}/g_i$-locally finite. By [15, Corollary 1.5], $d(R_{i+1})/IR_{i+1} = d(R_i/I) + 1$. We deduce from this remark that all the results of this section are also true if we replace the assumption that $R$ is affine by $R$ is $g$-locally finite.

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