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# The Intersection Conics of Six Straight Lines

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**Abstract.** We investigate and visualize the manifold $M$ of planes that intersect six straight lines of real projective three space in points of a conic section. It is dual to the apex-locus of the cones of second order that have six given tangents. In general $M$ is algebraic of dimension two and class eight. It has 30 single and six double lines. We consider special cases, derive an algebraic equation of the manifold and give an efficient algorithm for the computation of solution planes.

## 1. Introduction

Line geometry of projective three space is a well-established but still active field of geometric research. Right now the time seems to be right for tackling previously impossible computational problems of line space by merging profound theoretical knowledge with the computational power of modern computer algebra systems. An introduction and detailed overview of recent developments can be found in [5]. The present paper is a contribution to this area. It deals with conic sections that intersect six fixed straight lines of real projective three space $P^3$.

The history of this problem dates back to the $19^{th}$ century when A. Cayley and L. Cremona tried to determine ruled surfaces of degree four to six straight lines of a linear complex (compare the references in [4]). These surfaces carry a one parameter set of conic sections that are solutions to our problem. Cayley and Cremona could prove the existence of a finite number of solution surfaces but were unable to provide further details concerning, e.g., the number of solutions or algorithms for their computation.

In [9] the author deals with surfaces of conic sections that carry planar families of curves that induce projective relations between any two surface conics (families of cross ratios). He gives an example of a special class of surfaces with five straight lines that lie in a degenerate line complex whose axis is contained in the surface. Every surface of that kind yields a one parameter manifold of solution conics for this special line configuration.

As our problem is of projective nature, it can be dualized. The dual task consists of finding the quadratic cones to six given tangents. Recently, related questions have been considered in Euclidean settings: Cylinders of revolution to three, four or five tangents rose some interest among Canadian and Austrian geometers but did not result in publications. Cylinders and cones on a certain number of points are the topic of several papers ([3, 6, 7, 10, 11, 13, 14]). The projective task of finding the cones of second order to six points comes much closer to our topic. It has already been solved in the $19^{th}$ century ([2, 12]).

In case of cones of revolution or quadratic cones in general, the apex locus got special attention. With respect to our problem this means that we will have to investigate the dual apex locus, i.e., the set $M$ of all planes that carry a solution conic. It is a manifold in dual space $P^{3\star}$ and deserves interest not only from the theoretical point of view: The direct computation of solution conics (e.g., with the help of conic coordinates as presented in [1]) is quite hard while it is elementary to find the solution conic in a given plane of $M$.

We will start our investigation with the characterization of those line configurations that yield a three parameter variety of solution planes (Section 2). They turn out to be trivial and will be excluded from further considerations. Then we recall the well-known theorem of Pascal that, together with a result of [8], will be the main tool for all further considerations. In Section 3 we present an algorithm for deriving an algebraic equation of $M$. It is a little bit lengthy, but it poses no problems to current computer algebra systems. Subsequently, we investigate the Pascal curves of pencils of planes, we prove that $M$ is of class eight and characterize all straight lines on $M$ for the general case. This will result in an algorithm for the efficient computation of all solution planes in Section 7. Its most costly step is the solution of an algebraic equation of degree four.

## 2. Prerequisites

Let $S_0, \ldots, S_5$ be six straight lines in real projective three space $P^3$. They will be referred to as *base lines*. Our aim is to determine those regular or singular conic sections that intersect all base lines. Any such conic will be called a *solution conic*. In general it is uniquely determined by its carrier plane. Therefore any such plane will be called a *solution plane*.

Whenever we perform algebraic calculations, we will embed $P^3$ in complex projective three space $P^3(\mathbb{C})$ without explicitly mentioning this. Any results are to be understood "in the sense of algebraic geometry", i.e., admitting complex solutions and counting the respective multiplicities.

### 2.1. Klein map, Plücker quadric and reguli

When dealing with straight lines of projective three space $P^3$ it is often useful to transfer them to the points of the Plücker quadric $M_2^4 \subset P^5$ via the Klein map $\gamma$ (see [5], p. 133 ff). If a straight line $L$ is spanned by two points with homogeneous coordinates $(a_0, a_1, a_2, a_3)$ and $(b_0, b_1, b_2, b_3)$, its Klein image has coordinates $(l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$ where $l_{ij} = a_i b_j - b_i a_j$.

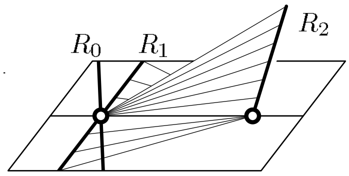


Figure 1: A degenerate regulus.

The line coordinates satisfy the defining relation $x_0x_3+x_1x_4+x_2x_5 = 0$ of the Plücker quadric $M_2^4$.

The interpretation of straight lines as points of $M_2^4$ allows the investigation of line space by methods of projective point geometry and, since its introduction in the $19^th$ century, has been exploited in numerous ways. We will use it for defining the notion of a regulus.

Commonly, a *regulus* $\mathcal{R}$ is defined as the set of lines that intersect three pairwise skew lines $R_0$, $R_1$, $R_2$. It corresponds to a non-tangential planar section of $M_2^4$, i.e., a regular conic on the Plücker quadric. Its adjoint regulus $\tilde{\mathcal{R}}$ is the set of lines that intersect the elements of $\mathcal{R}$. It corresponds to the conjugate intersection of $M_2^4$.

For our purposes it will be useful to extend this notion of a regulus to degenerate cases as well. If one or two pairs of the straight lines $R_0$, $R_1$ and $R_2$ intersect, there still exists a one parameter set of common intersection lines. It consists of two pencils of lines so that the vertex of one lies in the supporting plane of the other (Figure 1). We will call this set of lines a *degenerate regulus.* It corresponds to a tangential planar intersection of $M_2^4$ (a degenerate conic on the Plücker quadric) and equals its adjoint regulus. When we talk of a regulus, we will usually refer to this extended concept.

A plane containing an element of a regulus will be called a *tangent plane.* The union of all points on elements of a regulus is called its *carrier quadric.* It is singular in case of a degenerate regulus. Note that any three lines $R_0$, $R_1$ and $R_2$ determine a *unique* regulus as long as they are not concurrent or coplanar.

### 2.2. Trivial configurations

A generic plane $\varepsilon$ intersects the base lines $S_i$ in six points $s_i$ that in general do not lie on a conic section. There are several ways of seeing that this "conic restriction" defines an algebraic manifold $M$ of solution planes that will usually be of dimension two (compare, e.g., Section 3). However, there are base line configurations where all planes in $P^3$ are solution planes:

**Theorem 1.** *The manifold $M$ of solution planes is of dimension three iff the base lines have a common carrier quadric or if at least four of them are coplanar.*

*Proof.* Due to the algebraic character of the problem, a three parameter variety of solution planes implies that all planes of $P^3$ are solution planes. Clearly, this is the case for the two configurations mentioned in the theorem. We have to show that there are no further possibilities.

We assume that no four base lines are coplanar. It is an elementary task to see that there exist two skew base lines $S_i$ and $S_j$. They and a further base line $S_k$ span a regulus $\mathcal{R}$. Now we have to distinguish two cases:

If $S_k$ does not intersect both base lines $S_i$ and $S_j$, there exist infinitely many elements of $\mathcal{R}$ that intersect $S_i$, $S_j$ and $S_k$ in three pairwise different points. Therefore, the intersection points of infinitely many tangent planes of $\mathcal{R}$ with the remaining base lines must be collinear as well. This is not possible, if they are concurrent or coplanar. Consequently, they have a well-defined carrier quadric that is identical to the carrier quadric of $S_i$, $S_j$ and $S_k$.

If $S_k$ intersects both $S_i$ and $S_j$ and if we cannot find a base line $S_{k'}$ not exhibiting this behavior, the base lines are necessarily the edges of a tetrahedron. However, since there exist coplanar base line triples, a generic plane contains exactly three collinear intersection points with base lines and is no solution plane. □

Configurations with a three parameter manifold of solution planes are of little geometric interest. We will therefore exclude them from our further considerations. I.e., we will assume that the base lines have no common carrier quadric and that no four of them are coplanar.

### 2.3. Pascal's theorem

A conic section $C \subset P^3$ is uniquely determined by five pairwise different coplanar points $a_0, \ldots, a_4$. In order to test whether a sixth point $a_5$ is contained in $C$ one can use Pascal's theorem (B. Pascal, 1639). It states that the six points $a_0, \ldots, a_5$ lie on a conic section iff the three *Pascal points*

$$p_1 = (a_0 \wedge a_1) \cap (a_3 \wedge a_4), \quad p_2 = (a_1 \wedge a_2) \cap (a_4 \wedge a_5), \quad p_3 = (a_2 \wedge a_3) \cap (a_5 \wedge a_0)$$

are collinear.[1] If this is the case, their connecting line is called the *Pascal axis* of $a_0, \ldots, a_5$ (Figure 2). Note that Pascal points and Pascal axis depend on the point sequence rather then the point set. An index permutation of $a_0, \ldots, a_5$ leads to different Pascal points and a different Pascal axis. For later reference, we state a simple lemma:

**Lemma 1.** *Two Pascal points coincide iff two base line triples* $(a_i, a_{i+1}, a_{i+2})$ *and* $(a_{i+3}, a_{i+4}, a_{i+5})$*, respectively, are collinear.*[2]

Pascal's theorem is a good tool for characterizing solution planes. If the Pascal points of the intersection points $s_i$ of a plane $\varepsilon$ with the base lines $S_i$ are well-defined we call them the *Pascal points* of $\varepsilon$. Otherwise (if $\varepsilon$ contains a base line, an intersection line of four base lines $S_i$, $S_{i+1}$, $S_{i+3}$ and $S_{i+4}$ or a possible intersection point of two subsequent base lines) its Pascal points are undefined. If this is the case, $\varepsilon$ contains a solution conic anyway and we may state:

*A plane is solution plane iff its Pascal points are either collinear or undefined.*

[1] The wedge symbol '$\wedge$' denotes the span of two projective subspaces.

[2] Here and in the following indices that are out of range have to be read modulo six.

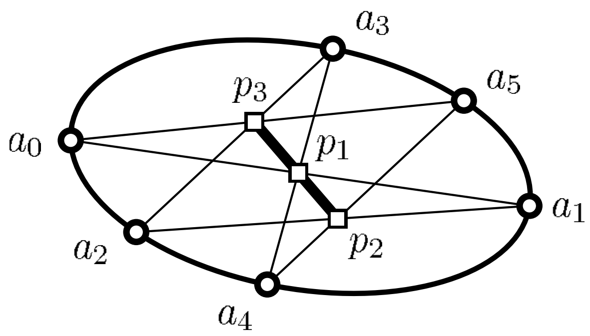


Figure 2: The theorem of Pascal.

## 3. The algebraic equation of *M*

Now we want to derive an algebraic equation of $M$. It will characterize the collinearity of a plane's Pascal points in terms of homogeneous plane coordinates $\mathbb{R}\mathsf{u} = \mathbb{R}(u_0, u_1, u_2, u_3)$.[3] Later in Section 7 we will propose a method for computing solution planes that only requires solving an algebraic equation of degree four. It will be based on considerations of this section and Section 5.

For the computation with mixed point, line and plane coordinates, an affine interpretation of $P^3$ is useful. We define a plane at infinity $x_0 = 0$ and use the notations

$$\mathsf{x}\mathbb{R} = (x_0, \mathbf{x})\mathbb{R}, \quad \mathbb{R}\mathsf{u} = \mathbb{R}(u_0, \mathbf{u}) \quad \text{and} \quad \mathsf{g}\mathbb{R} = (\mathbf{g}, \overline{\mathbf{g}})\mathbb{R}$$

for homogeneous point, plane and line coordinates. In this equation, $x_0$ and $u_0$ are scalars while $\mathbf{x}$, $\mathbf{u}$, $\mathbf{g}$ and $\overline{\mathbf{g}}$ are vectors of dimension three.

The connecting line of two points $(p_0, \mathbf{p})\mathbb{R}$ and $(q_0, \mathbf{q})\mathbb{R}$, the intersection point of a line $(\mathbf{l}, \bar{\mathbf{l}})\mathbb{R}$ and a plane $\mathbb{R}(u_0, \mathbf{u})$ and the intersection point of two concurrent lines $(\mathbf{l}, \bar{\mathbf{l}})\mathbb{R}$ and $(\mathbf{k}, \overline{\mathbf{k}})\mathbb{R}$ are obtained as

$$(p_0, \mathbf{p})\mathbb{R} \wedge (q_0, \mathbf{q})\mathbb{R} = (p_0\mathbf{q} - q_0\mathbf{p}, \mathbf{p} \times \mathbf{q})\mathbb{R}, \tag{1}$$

$$(\mathbf{l}, \bar{\mathbf{l}})\mathbb{R} \cap \mathbb{R}(u_0, \mathbf{u}) = (\mathbf{ul}, -u_0\mathbf{l} + \mathbf{u} \times \bar{\mathbf{l}})\mathbb{R}, \tag{2}$$

$$(\mathbf{l}, \bar{\mathbf{l}})\mathbb{R} \cap (\mathbf{k}, \overline{\mathbf{k}})\mathbb{R} = (\bar{\mathbf{l}}\mathbf{k}, \bar{\mathbf{l}} \times \overline{\mathbf{k}})\mathbb{R}. \tag{3}$$

These formulae fail, if the span or intersection is not properly defined. Additionally, formula (3) cannot be used if $\bar{\mathbf{l}}$ and $\overline{\mathbf{k}}$ are linearly dependent (compare [5], p. 137 ff).

With the help of (1), (2) and (3) it is not difficult to compute the Pascal points $p_0$, $p_1$ and $p_2$ of a plane in terms of the plane coordinates $\mathbb{R}\mathsf{u}$. In order to test the Pascal points for collinearity, we can compute the coordinate determinant $D = D(\mathsf{u})$ of $p_0$, $p_1$, $p_2$ and an arbitrary fourth point $p_3$. The roots of $D(\mathsf{u}) = 0$ indicate either collinearity of the Pascal points $p_0$, $p_1$ and $p_2$ of a plane or incidence with $p_3$. Thus, we can derive an algebraic equation that describes all points of $M$ in the following way:

[3] Our notation follows the conventions of [5] where a plane with coordinate vector $\mathsf{u}$ is denoted by $\mathbb{R}\mathsf{u}$. The '$\mathbb{R}$'-symbol reminds us that $\mathsf{u}$ is determined up to non-zero scalar factors. Points will be denoted by symbols of the shape $\mathsf{p}\mathbb{R}$ and can be distinguished from plane symbols by the position of the $\mathbb{R}$.

**Step 1:** Compute the Pascal points $p_0$, $p_1$, $p_2$ of an indeterminate plane $\varepsilon = \mathbb{R}\mathsf{u}$ by means of formulae (1), (2) and (3). Their coordinates are homogeneous polynomials of degree four in $\mathsf{u}$.

**Step 2:** Choose an arbitrary point $p_3 = \mathsf{p}_3\mathbb{R}$ and compute the coordinate determinant $D(\mathsf{u})$ of $p_0$, $p_1$, $p_2$ and $p_3$. It is a homogeneous polynomial of degree twelve in $\mathsf{u}$.

**Step 3:** The equation $D(\mathsf{u}) = 0$ describes not only the solution planes but also the additional bundle of planes through $p_3$. Thus, $M$ is also described by the algebraic equation $E(\mathsf{u}) := D(\mathsf{u})/(\mathsf{p}_3 \cdot \mathsf{u}^{\mathrm{T}}) = 0$ of degree eleven.

The last step eliminates a bundle of "virtual" solution planes that comes from our collinearity test for the Pascal points. But there exist further unwanted roots of $D(\mathsf{u})$: The computation of the Pascal points fails for all planes of the bundle $u_0 = 0$ since, in this case, the connecting line $T_{ij}$ of the intersection points $S_i \cap \varepsilon$ and $S_j \cap \varepsilon$ has line coordinates $(\mathbf{t}_{ij}, \overline{\mathbf{t}}_{ij})$ where $S_i = (\mathbf{s}_i, \overline{\mathbf{s}}_i)\mathbb{R}$ and $\overline{\mathbf{t}}_{ij} = \det(\mathbf{u}, \overline{\mathbf{s}}_i, \overline{\mathbf{s}}_j)\mathbf{u}$. Thus, all vectors $\overline{\mathbf{t}}_{ij}$ are proportional and the intersection formula (3) produces zero vectors. As a result, $u_0^3$ is a factor of $E(\mathsf{u})$ and we can further simplify the equation of $M$:

**Step 4:** Set $F(\mathsf{u}) := u_0^{-3}E(\mathsf{u})$. This eliminates a bundle of virtual solution planes of multiplicity three. The resulting equation $F(\mathsf{u}) = 0$ of $M$ is of degree eight.

In Section 5 we will see that $M$ is of class eight. Thus, the degree of $F(\mathsf{u})$ cannot be further reduced.[4] The algorithm's computational details are not difficult and may be left to a computer algebra system. We have implemented it on average PC hardware and obtain both, numeric and symbolic results within a few seconds.

A dual image of $M$ is displayed on the left-hand side of Figure 3. It has been produced by identifying plane coordinates $\mathbb{R}\mathsf{u}$ with point coordinates $\mathsf{u}\mathbb{R}$. This surface is an example of the generic case. The special case depicted on the right-hand side is explained on page 443.

## 4. The Pascal curves of a pencil of planes

The results of this section will be used in Section 5 for the investigation of straight lines on $M$. However, they are interesting in their own right as well. In contrast to the preceding section we will henceforth (until Section 7) use synthetic reasoning. Thereby, we will make the additional assumption that *no three base lines are coplanar or concurrent.* We will refer to this as *regularity condition.* It ensures that any three base lines define a (possibly degenerate) regulus.[5]

Consider a pencil of planes $\mathbb{E}$. Due to our regularity condition, the three Pascal points $p_i = p_i(\varepsilon)$ of almost all planes $\varepsilon \in \mathbb{E}$ are well-defined. The closure of the union of all Pascal points $p_i$ is a curve $C_i \subset P^3$ that will be called the *i-th Pascal curve of* $\mathbb{E}$.

**Theorem 2.** *In general the Pascal curves of a pencil of planes are twisted cubics.*

[4]This means that possible factors of $F(\mathsf{u})$ are not the result of a specific computation technique but have a *geometric* meaning.

[5]In most cases it is sufficient to require the existence of a certain number of non-coplanar and non-concurrent base line triples. However, in order to simplify things and to avoid too many cases we will not stay as general as possible. A few results for the excluded cases will be presented in Section 8.1.

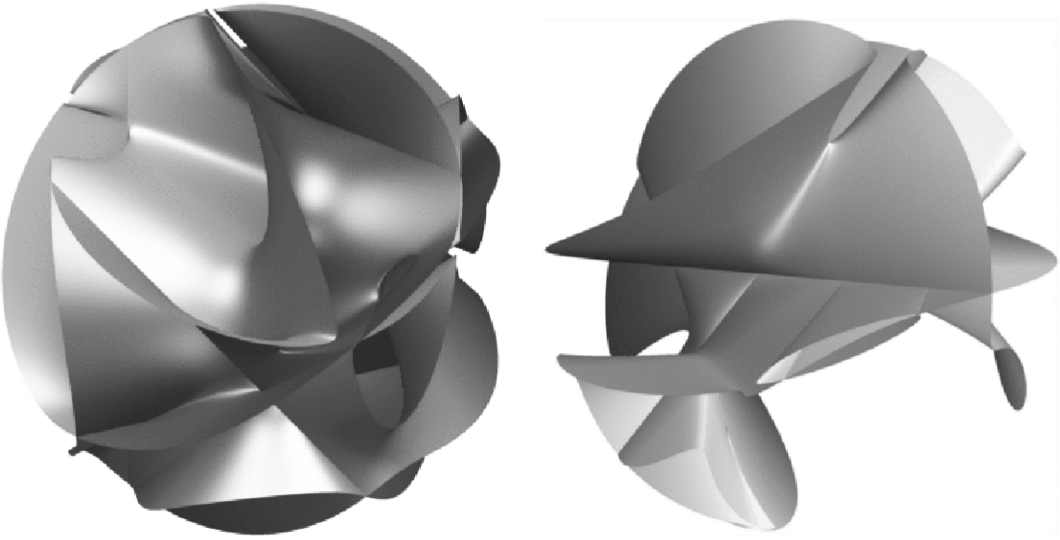

Figure 3: Two dual images of $M$: The left image shows a generic case, while the surface on the right has a special property: The horizontal line in the center of the image is a triple line (compare Section 6).

*Proof.* Consider the pencil of planes $\mathbb{E}$ with axis $E$. The planes of $\mathbb{E}$ induce a projective relation between any two base lines $S_i$ and $S_j$. In general (i.e., if $S_i$, $S_j$ and $E$ are pairwise skew) this projectivity generates a quadric $Q_{i,j}$ through $E$. The Pascal curve $C_i$ is now contained in the intersection $Q_{i,i+1} \cap Q_{i+2,i+3}$ which consist of $E$ and a cubic remainder. □

From the proof of Theorem 2 we may draw conclusions for the non-generic case as well. The Pascal curve $C_0$ is not cubic if $Q_{0,1}$ or $Q_{3,4}$ are not regular quadrics or if their intersection contains a straight line besides $E$. A discussion of both possibilities yields the following result:

**Theorem 3.** *The Pascal curve $C_i$ of a pencil of planes with axis $E$ is a conic iff either the base lines $S_i$ and $S_{i+1}$ or $S_{i+3}$ and $S_{i+4}$ are concurrent, if $E$ intersects one of these lines or one of the straight lines that intersect $S_i$, $S_{i+1}$, $S_{i+3}$ and $S_{i+4}$. The Pascal curve $C_i$ is a straight line, iff two of these incidences come together.*

To complete the picture, we mention that the Pascal curve $C_i$ consists of a single point only iff $E$ intersects three base lines $S_i$, $S_{i+1}$, $S_{i+3}$ (or $S_i$, $S_{i+1}$, $S_{i+4}$). In this case we will speak of a *degenerate* Pascal curve.

The points of the Pascal curve $C_i$ of a pencil of planes $\mathbb{E}$ are related to the planes of $\mathbb{E}$ in a natural way. A rational parameter representation $\mathbb{R}\mathsf{e}(s) = \mathbb{R}\mathsf{e}_0 + s\mathbb{R}\mathsf{e}_1$ of $\mathbb{E}$ induces rational parameterizations $C_i \dots \mathsf{c}_i\mathbb{R}(s)$ of the Pascal curves so that $\mathsf{c}_i\mathbb{R}(s) \in \mathbb{R}\mathsf{e}(s)$. Since each plane of $\mathbb{E}$ corresponds to exactly one point of $C_i$, the Pascal curve must intersect the axis $E$ of $\mathbb{E}$ in exactly two (possible coinciding or complex) points. The situation for conic sections and straight lines is similar (compare Figure 4) and we get:

**Theorem 4.** *If the Pascal curve $C_i$ of a pencil of planes $\mathbb{E}$ is not degenerate, it intersects the axis of $\mathbb{E}$ in $\delta - 1$ points where $\delta \in \{1, 2, 3\}$ is the degree of $C_i$.*

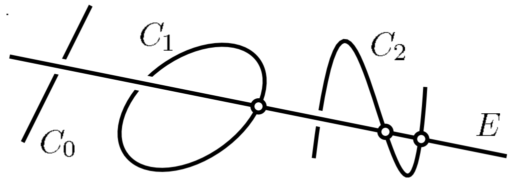


Figure 4: A Pascal curve of degree $\delta$ intersects the pencil axis in $\delta - 1$ points.

## 5. The class of *M*

In this section we will show that $M$ is of class eight. This has already been indicated (but not proved!) by the computations in Section 3. Here, we will use a more geometric approach that will turn out to be very useful. We begin with the following crucial lemma. Its proof (together with further clarifications concerning certain notions) is given in [8].

**Lemma 2.** *Let $D_0, \ldots, D_g$ be rational curves in real projective n-space $P^n$. The degree of $D_i$ be $\delta_i$. The curves $D_0, \ldots, D_g$ be in projective relation such that they generate a one parameter set $\mathbb{G}$ of subspaces $U(s) \subset P^n$ of generic dimension $g$. Then the class $\gamma$ of $\mathbb{G}$ and the (finite) numbers $\nu_i$ of subspaces $U(s)$ of dimension $g - i$ are linked via the equation $\gamma + \sum i\nu_i = \sum \delta_i$.*

**Theorem 5.** *The manifold $M$ of solution planes is of class eight.*

*Proof.* We have to show that a generic test line $E \in P^3$ is incident with eight solution planes. In order to do this, we consider the pencil of planes $\mathbb{E}$ with axes $E$. The class of $M$ equals the number of collinear Pascal points in the planes of $\mathbb{E}$.

At first, we assume that no two base lines $S_i$ and $S_{i+1}$ intersect. Theorem 3 and the generic position of $E$ guarantee, that the Pascal curves of $E$ are twisted cubics. At the end of Section 4 we saw that the Pascal curves are projectively related by the planes of $\mathbb{E}$. Thus, Lemma 2 may be applied to them with $g = 2$, $D_i = C_i$, $\mathbb{G} = \mathbb{E}$ and, consequently, $\gamma = 1$ and $\sum \delta_i = 9$. Since coinciding triples are not possible because of Lemma 1 and the assumed general position of $E$, we have $\nu_i = 0$ for $i > 1$. This results in $\nu_1 = 8$ collinear triples of Pascal points. Consequently, the class of $M$ is eight.

Now we assume that exactly one pair of base lines $S_i$ and $S_{i+1}$ is concurrent. In this case $C_i$ is of degree two. Using the same arguments as above, we obtain $\nu_1 = 7$ collinear triples of Pascal points. Since the bundle of planes through $S_i \cap S_{i+1}$ is an irreducible part of $M$, the total class is eight as well. Further intersection points of consecutive base lines lead to further bundles of planes as irreducible parts of $M$ but do not change the total class. □

## 6. Straight lines in *M*

In this section we will investigate straight lines in $M$. A line $L$ is said to be contained in $M$ if all planes of the pencil with axis $L$ are contained in $M$. This is just the dual of a straight line being contained in a two-dimensional manifold of $P^3$. It is not difficult to find straight lines in $M$. The base lines are obvious candidates. In fact, we even have

**Theorem 6.** *The base lines are double lines of $M$.*

*Proof.* Let $E$ be a straight line concurrent with the base line $S_i$. The two Pascal curves $C_i$ and $C_{i+2}$ of the pencil of planes $\mathbb{E}$ through $E$ are conic sections. Following the ideas of the proof of Theorem 5 we see that there exist six solution planes besides $E \wedge S_i$ in $\mathbb{E}$. Hence, $E \wedge S_i$ counts twice and $S_i$ is of multiplicity two. □

An intersection line $L$ of at least four base lines is contained in $M$ as well. The planes of the pencil $\mathbb{L}$ through $L$ contain singular solution conics. In general, there exist 30 lines of that type (two for each of the 15 base line quartupels). If four base lines lie in a regulus, there exist infinitely many. From Theorem 3 it follows that $L$ is a single line in general and a triple line iff it intersects all six base lines. Figure 3 displays an example of the latter case.

In general there exist no further straight lines on $M$. Before proving this we introduce a few useful notions: The regularity condition on page 440 guarantees that any three base lines $S_i$, $S_j$, $S_k$ lie on a unique regulus $\mathcal{R}_{i,j,k}$. We will call it a *base line regulus*. Its adjoint regulus $\tilde{\mathcal{R}}_{i,j,k}$ will be called *adjoint base line regulus*. If $\{i, j, k\}$ and $\{\bar{\imath}, \bar{\jmath}, \bar{k}\}$ are disjoint subsets of $\{0, \ldots, 5\}$, the base line reguli $\mathcal{R}_{i,j,k}$ and $\mathcal{R}_{\bar{\imath},\bar{\jmath},\bar{k}}$ are called *complementary*, $\mathcal{R}_{i,j,k}$ and $\tilde{\mathcal{R}}_{\bar{\imath},\bar{\jmath},\bar{k}}$ are called *adjoint complementary*.

**Theorem 7.** *In general $M$ contains six double lines (the base lines) and* 30 *single lines (the intersection lines of four base lines).*

*Proof.* Consider a straight line $L$ that is contained in $M$. If it intersects exactly three base lines $S_i$, $S_j$ and $S_k$ or is element of $\mathcal{R}_{i,j,k}$, it must be contained in the complementary or adjoint complementary base line regulus of $\mathcal{R}_{i,j,k}$. In general this is not possible since the intersection of two complementary or adjoint complementary base line reguli does not contain straight lines. Therefore, there exist exactly two tangent planes of $\mathcal{R}_{i,j,k}$ through $L$. They must be tangent to the complementary base line regulus as well which, again, is impossible in the general case. □

This proof shows that straight lines on $M$ different from those mentioned in Theorem 7 might be possible for special base line configurations. In particular, we can say that the straight line $L$ lies in $M$ if it is contained in

- the bundle of lines through an intersection point of two base lines,
- two complementary base line reguli or
- a base line regulus and its adjoint complementary regulus.

Our usual argument (using Lemma 2) shows that $L$ is of generic multiplicity one in any of these cases. The question whether there exist base line configurations with further straight lines on $M$ remains open. At any rate, the tangent planes to any two complementary base line reguli must be identical. We conjecture that this is not possible.

## 7. Computation of solution planes

Having learned more about the structure of the manifold $M$ we are ready for the effective computation of solution planes. The algorithm of Section 3 and Theorem 5 provide the theoretical background for our strategy. The latter guarantees that every straight line $E$ that is concurrent with two base lines $S_i$ and $S_j$ contains exactly four solution planes differentfrom $E \wedge S_i$ and $E \wedge S_j$. For their computation it is sufficient to solve an algebraic equation of degree four. As $M$ is of dimension two, the union of these planes contains all planes of $M$ (or at least a non-trivial component if $M$ is reducible). We propose the following steps:

**Step 1:** Choose a straight line $E$ that intersects two skew base lines $S_i$ and $S_j$.

**Step 2:** For $k = 0, \ldots, 5$ define $\mathbb{R}\mathsf{e}_k = E \wedge S_k$ and parameterize the pencil of planes $e$ through $E$ according to $\varepsilon(s) = \mathbb{R}\mathsf{e}(s) = \mathbb{R}\mathsf{e}_i + s\mathbb{R}\mathsf{e}_j$.

**Step 3:** Insert the plane coordinates of $\varepsilon(s)$ in the algebraic equation of $F(\mathsf{u}) = 0$ of $M$. The result will be a polynomial $F(s)$ of degree six with 0 as zero of multiplicity two (i.e., the two tailing coefficients vanish).

**Step 4:** Divide $F(s)$ by $s^2$ and solve the resulting algebraic equation of degree four. Its four roots lead to the solution planes in $e$.

This algorithm can still be optimized: Firstly, we can replace any polynomial division by index shifts if we choose $p_3 = \mathbb{R}(1, 0, 0, 0)$ in Step 2 of the algorithm in Section 3. The polynomial $D(\mathsf{u})$ will then have the factor $u_0^4$. Secondly, it might not be necessary to compute the algebraic equation of $M$. In this case, we can perform all steps of the algorithm in Section 3 directly with the plane coordinates of $\varepsilon(s)$.

The computation of solution planes simplifies if two base lines intersect or four base lines have a common carrier quadric. In this case the bundle of planes through the intersection point or the set of tangent planes of the quadric are components of $M$. The class of the remaining part reduces by one or two, respectively. An example is depicted on the right hand-side of Figure 5. It shows the dual image of the manifold of solution planes to the six straight lines

$$S_0 = (0, 1, 1, 0, -5, 5)\mathbb{R}, \quad S_1 = (0, -1, 1, 0, 5, 5)\mathbb{R}, \quad S_2 = (-1, 0, 1, -5, 0, 5)\mathbb{R},$$
$$S_3 = (1, 0, 1, -5, 0, 5)\mathbb{R}, \quad S_4 = (1, 0, 0, 0, 8, 0)\mathbb{R}, \quad S_5 = (0, 1, 0, 0, 8, 0)\mathbb{R}.$$

It is easy to verify that the base line quadruples $(S_0, S_1, S_2, S_3)$, $(S_0, S_3, S_4, S_5)$ and $(S_1, S_2, S_4, S_5)$, respectively, lie on reguli. Therefore, $M$ is the union of four dual quadrics.

## 8. Final remarks

### 8.1. Special cases

From Section 4 onwards we have assumed that no three base lines are coplanar or concurrent. At the same time, we have mentioned that we could prove most results with weaker assumptions. Actually, the generic case is even more complicated and, without going into detail, we can summarize a few results for the neglected special cases:

1. If three base lines (say $S_0$, $S_1$ and $S_2$) are coplanar, the manifold $M$ of solution planes consists of six bundles of planes and a dual quadric (possible degenerate). The bundles have vertices $S_0 \cap S_1$, $S_1 \cap S_2$, $S_2 \cap S_0$ and $S_i \cap \sigma$ where $\sigma$ is the carrier plane of $S_0$, $S_1$ and $S_2$ and $i \in \{3, 4, 5\}$. The quadric is defined by the base lines $S_3$, $S_4$ and $S_5$. If more than three base lines are coplanar or if $S_3$, $S_4$ and $S_5$ have a common supporting plane, all planes of $P^3$ are solution planes.
2. If two base lines (say $S_0$ and $S_1$) intersect in a common point $x$, the bundle of planes $p$ through $x$ is an irreducible part of $M$. If a third base line (say $S_2$) is incident with

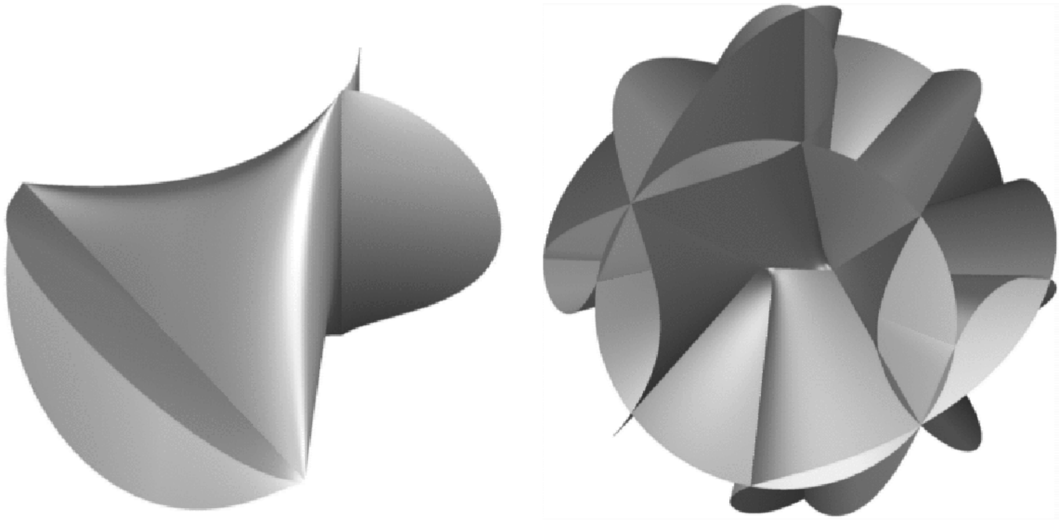

Figure 5: If four base lines are concurrent, $M$ splits in a bundle of planes of multiplicity four and a remaining part of class four that is visualized in the left image. The right hand side displays an example with three quadruples of base lines that lie on quadrics. In this case $M$ consists of four dual quadrics.

$x$, one would expect that the multiplicity of $p$ is higher than one. In fact, it is not difficult to see (compare Section 6) that $p$ is of multiplicity two in general. Similarly, four base lines through $x$ raise the multiplicity of $p$ to four, five concurrent base lines to six and six concurrent base lines to eight (i.e., $M$ consists of a bundle of planes with multiplicity eight).

In Figure 5 we depict an example of with four concurrent base lines. The manifold of solution planes consists of a bundle of planes of multiplicity four and a quartic remainder.

### 8.2. Future research

We have investigated the manifold of planes that intersect six given lines in points of a conic section. The presented algorithms can be used for their efficient computation. Open questions of interest concern solution conics with additional constraints. We mention a few examples:

1. Any two solution conics that are projectively related via the base lines lead to solutions of Cayley's and Cremona's problem (Section 1). According to [4], p. 246, this is only possible, if the base lines belong to a linear line complex. So this case deserves special attention. As a first step towards the solution, one may investigate the two parameter manifold of conics that are projectively related by five given lines.
2. The algebraic equation $F(\mathsf{u})$ of $M$ can be used to determine the solution planes in a bundle or, dually, to find the cones of second order to six tangents with vertex in a given plane $\omega$. In an appropriate affine interpretation, these solution cones are cylinders. Thus, we can compute those cylinders of second order that are tangent to six given lines. In general there exists a one parameter variety of solution cylinders.

3. Additionally one may want to impose Euclidean constraints on the solution conics or cones. There should be a finite number of circles that intersect six given lines or a finite number of cylinders of revolution to six given tangents. Removing one base line will lead to one parameter sets of circles and cylinders of revolution.
4. Finally, one can increase the number of base lines. Eight base lines will, in general, result in a finite number $n$ of solution conics. From Theorem 5 we obtain the upper boundary $8^3 = 512$ for $n$ but, actually, it might be smaller.

So far, we do not know whether our results will help answering at least some of these questions (especially the Cayley-Cremona problem). The computational effort seems to be rather high but perhaps a closer investigation of $M$ or similar manifolds will yield further results.

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