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On Converses of Napoleon's Theorem and a Modified Shape Function

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Abstract. The (negative) Torricelli triangle $\mathcal{T}_1(ABC)$ of a non-degenerate (positively oriented) triangle ABC is defined to be the triangle $A_1B_1C_1$, where ABC_1, BCA_1, and CAB_1 are the equilateral triangles drawn outwardly on the sides of ABC. It is known that not every triangle is the Torricelli triangle of some initial triangle, and triangles that are not Torricelli triangles are characterized in [28]. In the present article it is shown that, by extending the definition of $\mathcal{T}_1$ such that degenerate triangles are included, the mapping $\mathcal{T}_1$ becomes bijective and every triangle is then the Torricelli triangle of a unique triangle. It is also shown that $\mathcal{T}_1$ has the *smoothing property*, i.e., that the process of iterating the operations $\mathcal{T}_1$ converges, in shape, to an equilateral triangle for any initial triangle. Analogous statements are obtained for internally erected equilateral triangles, and the proofs give rise to a slightly modified form of June Lester's shape function which is expected to be useful also in other contexts. Several further results pertaining to the various triangles that arise from the configuration created by ABC_1, BCA_1, and CAB_1 are derived. These refer to Brocard angles, perspectivity properties, and (oriented) areas.

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Keywords: anti-medial triangle, anti-similar triangle, area, Brocard angle, complementary triangle, complex number, Gaussian plane, medial triangle, Möbius transformation, Napoleon configuration, Napoleon triangle, oriented area, perspectivity, shape function, similar triangles, smoothing iteration, Torricelli configuration, Torricelli triangle

1. Introduction

For an arbitrary triangle ABC in the Euclidean plane let ABC_1, BCA_1 and CAB_1 denote the equilateral triangles erected externally on the sides AB, BC, and CA of ABC, and α_1, β_1, and γ_1 be their circumcenters, respectively. This yields the so-called "Napoleon figure" of ABC, and the famous Napoleon theorem says that the triangle $\alpha_1\beta_1\gamma_1$ is equilateral and that the analogous statement holds for internally erected equilateral triangles. For the history and many generalizations of this theorem we refer to the survey [20] and to [26]. For example, instead of equilateral erected triangles one might consider similar ones (see [15], [16], [26], and of [20], § 4), or one can start with affine regular n-gons and erect regular n-gons (cf. [1], [10], and § 6 of [20]). And even figures given like in the original Napoleon theorem, but with only two erected triangles have interesting geometric properties; see [8] and § 8 in [20].

However, closer to the converses of Napoleon's theorem considered in the present paper are the so-called Petr-Douglas-Neumann theorems (cf. [29], Chapters 6, 7, 8, and 9, [9], and [20], § 7), since they refer to the free vertices of triangles erected on the sides of given n-gons yielding, by iterations, vertex sets of regular n-gons. Converses to this were explicitly studied in [5].

In the spirit of such converses one can also ask the following: If only the triangle $A_1B_1C_1$ of free vertices of the "Napoleon figure" described above is given, to what extent is the original triangle determined? Moreover, can the vertices of any triangle be free vertices of such a figure? Results in this direction were obtained in the papers [17], [31], [34], and [35]; see also [28] and [20], § 2.

We want to complete related contributions, particularly given [34], [35], and [28], to the following new and in a sense final results: Any triangle $A_1B_1C_1$ can be interpreted as the triangle of free vertices in a unique Napoleon figure (i.e., the initial triangle ABC of $A_1B_1C_1$ is unique), if the construction of $A_1B_1C_1$ is extended such that also degenerate triangles are taken into consideration. Analogous statements are presented for internally erected triangles, and also a related perspectivity result is derived. Furthermore we will show that iterations of such extended constructions have the so-called smoothing property, i.e., by iterating the described construction (with $A_1B_1C_1$ as starting point, etc.) we get a convergence to the shape of an equilateral triangle. For getting this and further results in our paper, we present a new modification of June Lester's shape function (see [14], [15], and [16]). We continue by using this new shape function for deriving a sequence of theorems on Brocard angles and (oriented) areas of different triangles occurring in Napoleon figures created from ex- and internally erected equilateral triangles. This seems to manifest that this new shape function can be successfully used also in many other related contexts. So it is our hope that the methods developed here can also be generalized or applied to "more general Napoleon figures", e.g., with respect to higher dimensions or non-Euclidean geometries; see [23], [21], and [22]. And one might also look for possible extensions of other results, related to polygons with erected triangles or erected n-gons; see the surveys and papers [18], [19], [20], [14], [8], and [30] for many results in this direction.

Since our results should be expressed in terms of oriented triangles, we have to continue with some more precise notation regarding triangles.

2. Terminology regarding triangles

A *triangle* ABC is defined to be any ordered triple (A, B, C) of points in the Euclidean plane. Thus, in general there are six different triangles having the same set of vertices.

A triangle is called *degenerate* if its vertices are collinear, and *non-degenerate* otherwise. It is called *trivial* if the three vertices coincide.

A non-degenerate triangle ABC is said to be *positively oriented* if the motion $A \to B \to C$ is counterclockwise, and *negatively oriented* if this motion is clockwise. Since a degenerate triangle has no apriori orientation, and since we need all our triangles to be oriented, we stipulate that there are two copies of every degenerate triangle ABC; one of them is positively oriented, and the other one is negatively oriented. Thus, if we refer to a degenerate triangle ABC, we assume that its orientation is also specified. From now on, all our triangles are oriented and non-trivial, but not necessarily non-degenerate.

Two triangles ABC and $A'B'C'$ are said to be *similar* if they have the same orientation and

$$\|A - B\| : \|A' - B'\| = \|B - C\| : \|B' - C'\| = \|C - A\| : \|C' - A'\| \qquad (1)$$

holds. They are said to be *anti-similar* if (1) holds and they have different orientations.

3. Napoleon-Torricelli configurations

In a refined manner we will now describe and analyse the geometric configuration which is usually called "Napoleon configuration" (or can be extended to the so-called "Torricelli configuration"; see [20], [26], and [23]).

Let ABC be a given triangle, and let ABC_1, BCA_1, and CAB_1 be the negatively oriented equilateral triangles drawn on the sides of ABC. These are the triangles erected outwardly or inwardly on the sides of ABC according to the case whether ABC is positively or negatively oriented, respectively. They will be referred to as the *negative ear triangles* of ABC. The triangle $A_1B_1C_1$, formed by the new or free vertices, will be called the *negative Torricelli triangle* of ABC and denoted by $\mathcal{T}_1(ABC)$. Figure 1 shows the negative ear triangles ABC_1, BCA_1,

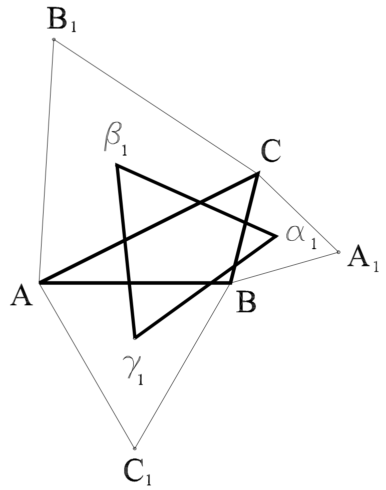

Figure 1

and CAB_1 for a positively oriented triangle ABC. Note that the negative ear triangles of ACB are the inwardly erected triangles AC_2B, BA_2C, and CB_2A.

If α_1, β_1, and γ_1 are the circumcenters of the triangles ABC_1, BCA_1, and CAB_1, respectively, then a well-known theorem, customarily attributed to Napoleon Bonaparte, states that $\alpha_1\beta_1\gamma_1$ is equilateral; see again Figure 1. We shall call $\alpha_1\beta_1\gamma_1$ the *negative Napoleon triangle* of ABC and denote it by $\mathcal{N}_1(ABC)$; see [35]. Note once more that the negative ear, Torricelli, and Napoleon triangles are defined for all triangles.

As already mentioned, the negative Napoleon configuration described above corresponds to what is known as the *outward Napoleon configuration* if ABC is positively oriented, and to the *inward Napoleon configuration* if ABC is negatively oriented. Our Figure 2 consists of two pictures. In each of them $A_1B_1C_1$ is the negative Torricelli triangle $\mathcal{T}_1(ABC)$, and $A_2B_2C_2$ is the positive Torricelli triangle $\mathcal{T}_2(ABC)$ of ABC.

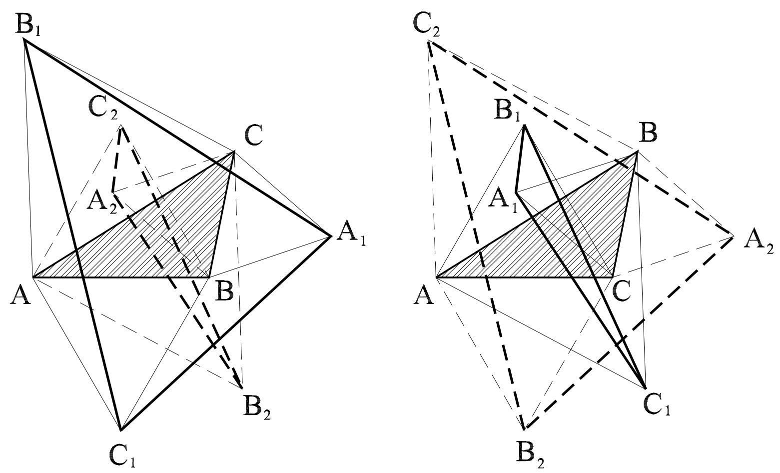

Figure 2

In studying the process of iterating the Napoleon operations, it is relevant to note that the negative Torricelli and Napoleon triangles $\mathcal{T}_1(ABC)$ and $\mathcal{N}_1(ABC)$ of ABC do not necessarily have the same orientation as ABC. However, it is proved in Corollary 5.1 that $\mathcal{T}_1$ eventually preserves orientation in the sense that for a given triangle ABC, the derived triangles $\mathcal{T}_1^n(ABC)$ will all have the same orientation for all n that are sufficiently large.

The *positive ear, Torricelli*, and *Napoleon triangles* are defined analogously, and the corresponding statements can be easily formulated and checked. The positive Torricelli and Napoleon triangles of ABC will be denoted by $\mathcal{T}_2(ABC)$ and $\mathcal{N}_2(ABC)$, respectively.

It should be remarked here that when ABC is negatively oriented, then the negative and positive Torricelli triangles $\mathcal{T}_1(ABC)$ and $\mathcal{T}_2(ABC)$ correspond to what the author of [4] calls the *first Fermat* and *second Fermat* triangles of ABC, respectively. Things are reversed when ABC is positively oriented.

4. The arbitrariness of the Torricelli triangles

It is very useful to identify the Euclidean plane with the Gaussian plane $\mathbb{C}$ of complex numbers; see [7], [11], and [12] for many related and elegant approaches to interesting theorems. A triangle is then an ordered triple (A, B, C) of complex numbers, and still we will denote it by ABC, except when there is a possibility of misinterpreting this (to mean the product ABC).

The next theorem expresses the vertices of the negative and positive Torricelli and Napoleon triangles of ABC in terms of A, B, and C, and conversely. Here, and throughout this paper, ζ will denote the primitive third root $e^{2\pi i/3}$ of 1. Thus

$$\zeta = \frac{-1 + i\sqrt{3}}{2}, \quad \zeta^2 = \frac{-1 - i\sqrt{3}}{2}.$$

Theorem 4.1. *Let $A_1B_1C_1$ and $A_2B_2C_2$ be the negative and positive Torricelli triangles of triangle ABC, and let $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$ be the negative and positive Napoleon triangles. Then*

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & -\zeta^2 & -\zeta \\ -\zeta & 0 & -\zeta^2 \\ -\zeta^2 & -\zeta & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \quad \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & -\zeta & -\zeta^2 \\ -\zeta^2 & 0 & -\zeta \\ -\zeta & -\zeta^2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

$$\begin{bmatrix} 2A \\ 2B \\ 2C \end{bmatrix} = \begin{bmatrix} 1 & -\zeta^2 & -\zeta \\ -\zeta & 1 & -\zeta^2 \\ -\zeta^2 & -\zeta & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & -\zeta & -\zeta^2 \\ -\zeta^2 & 1 & -\zeta \\ -\zeta & -\zeta^2 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix},$$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} = \frac{1-\zeta}{3} \begin{bmatrix} 0 & -\zeta^2 & 1 \\ 1 & 0 & -\zeta^2 \\ -\zeta^2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

$$\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \frac{1-\zeta}{3} \begin{bmatrix} 0 & 1 & -\zeta^2 \\ -\zeta^2 & 0 & 1 \\ 1 & -\zeta^2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}.$$

Proof. Since AC_1 is obtained by rotating AB counterclockwise by 60°, it follows that $C_1 - A = -\zeta(B - A)$, and therefore $C_1 = -\zeta^2 A - \zeta B$. The remaining relations are analogous. □

Corollary 4.1. *Every triangle is the negative Torricelli triangle of a unique triangle. An analogous statement holds for positive triangles.*

Proof. This follows from Theorem 4.1, where one can also read off a method for recovering a triangle from each of its Torricelli triangles. □

Note 4.1. The corollary above would take a much less pleasant form if we would exclude degenerate triangles. Thus the question which non-degenerate triangle is the negative Torricelli triangle of some non-degenerate triangle has a two-fold drawback, and it constitutes the Monthly's Problem 3257. According to the solution in [28], such a triangle ABC is characterized by either of the equivalent conditions

$$10 \sin\alpha \sin\beta \sin\gamma > \sqrt{3}\,(\sin^2\alpha + \sin^2\beta + \sin^2\gamma), \tag{2}$$

$$11(a^2 + b^2 + c^2)^2 > 25(a^4 + b^4 + c^4) \tag{3}$$

with a, b, and c as side lengths and α, β, and γ as angles of ABC.

Corollary 4.2. *If $A_1B_1C_1$ and $A_2B_2C_2$ are the negative and positive Torricelli triangles of ABC, and if $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$ are the corresponding negative and positive Napoleon triangles, then the centroids of ABC, $A_1B_1C_1$, $A_2B_2C_2$, $\alpha_1\beta_1\gamma_1$, and $\alpha_2\beta_2\gamma_2$ coincide.*

Proof. It follows from Theorem 4.1 that

$$A_1 = -\zeta^2 B - \zeta C, \quad B_1 = -\zeta A - \zeta^2 C, \quad C_1 = -\zeta^2 A - \zeta B.$$

Therefore

$$A_1 + B_1 + C_1 = (-\zeta^2 - \zeta)(A + B + C) = (A + B + C). \tag{4}$$

Analogously,

$$A_2 + B_2 + C_2 = (-\zeta^2 - \zeta)(A + B + C) = (A + B + C). \tag{5}$$

The relations (4) and (5) mean that the centroids of ABC, $A_1B_1C_1$, and $A_2B_2C_2$ coincide. The other triangles are treated similarly. □

We end this section by remarking that the importance of the Napoleon-Torricelli configuration has much to do with its role in locating the *Fermat-Torricelli point* F of a given triangle ABC, i.e., the unique point having minimal sum of distances of A, B, and C (see Chapter II of [3]). Referring to Figure 1, it turns out that F is nothing but the intersection point of the lines AA_1, BB_1, and CC_1. In terms of perspectivities, this says that the triangles ABC and $A_1B_1C_1$ are perspective and that their point of perspectivity is F. In addition it is known that for all triangles ABC the negative and positive Napoleon triangles are perspective with respect to the circumcenter of ABC, see [33], and that the negative Torricelli triangle and ABC are perspective with respect to F, cf. [13]. The following simple proposition exhibits one more perspectivity in the Napoleon-Torricelli configuration.

Proposition 4.1. *The negative and positive Torricelli triangles of any triangle ABC are perspective and their point of perspectivity is the circumcenter of ABC. In fact, this is still true in the more general configuration where the six relevant ear triangles are isosceles of arbitrary shapes.*

Proof. Let $A_1B_1C_1$ and $A_2B_2C_2$ be the negative and positive Torricelli triangles of a triangle ABC. Then A_1BCA_2 is a rhombus and therefore A_1A_2 is the perpendicular bisector of BC. Similar statements hold for B_1B_2 and C_1C_2, and the rest follows from the fact that the perpendicular bisectors of the sides of ABC concur at the circumcenter.

For the general statement, the quadrilateral A_1BCA_2 is not necessarily a rhombus but it has the properties that $A_1B = A_1C$ and $A_2B = A_2C$. Letting M be the point of intersection of A_1A_2 and BC, one uses the obvious congruence of the triangles A_1BA_2 and A_1CA_2 to show that the triangles A_1BM and A_1CM are also congruent, and to conclude again that A_1A_2 is the perpendicular bisector of BC. □

5. The smoothing property of the Torricelli iterations and a new shape function ϕ

Using the SAS similarity theorem and the geometric interpretation of the quotient of two complex numbers, it is easy to see that the non-degenerate triangles ABC and $A'B'C'$ are similar (respectively, anti-similar) if and only if the fractions $(A-C)/(A-B)$ and $(A'-C')/(A'-B')$ are equal (respectively, reciprocal). In fact, this still holds for all non-zero triangles as long as the fraction $(A-C)/(A-B)$ is assigned the value ∞ when $A = B \neq C$. The quantity $(A-C)/(A-B)$ is called the *shape* of the triangle ABC and is studied in great detail in [14], [15], and [16]. Denoting the shape of ABC by $S(ABC)$, it is easy to see that S can assume all values in the extended complex plane $\mathbb{C}_\infty$. In fact, for fixed B and C the function $(A-B)/(A-C)$ is a *Möbius transformation* (see [25], pp. 206–223) in A and therefore surjective. (The shape of the zero triangle $A = B = C$ is not defined.)

We summarize the properties of the shape function S in the first two columns of Table 1 below, where statements in the same row are equivalent, and where $\zeta = e^{2\pi i/3}$, as before. The third column refers to the shape function ϕ defined below. Note that the positively and negatively oriented copies of a degenerate triangle are both similar and anti-similar.

When studying Torricelli triangles (and expectedly in other contexts), we find it much more convenient to work with the modified shape ϕ defined by

$$\phi(ABC) = \frac{A + \zeta B + \zeta^2 C}{A + \zeta^2 B + \zeta C}; \tag{6}$$

see also the beginning of Section 6 below. It is easy to check that $(A + \zeta B + \zeta^2 C)$ and $(A + \zeta^2 B + \zeta C)$ are both zero if and only if $A = B = C$, in which case $\phi(ABC)$ is not defined. It is also easy to see that S and ϕ are related by the Möbius transformations (cf. again [25], pp. 206–223)

$$\phi = \frac{1 + \zeta S}{\zeta + S}, \qquad S = \frac{\zeta\phi - 1}{\zeta - \phi}.$$

It follows that ϕ and S define each other uniquely, and also that ϕ assumes all the values in the extended complex plane $\mathbb{C}_\infty$. The properties of ϕ are exhibited in Table 1, where again the three entries in each row are equivalent. They are immediate, except, possibly, for the properties in rows 3–5 which are proved in Theorem 6.1.

It is obvious that if two triangles are similar, then so are their negative Torricelli triangles. Similar statements hold for anti-similar triangles and for positive Torricelli triangles. Also, we point out that the negative (analogously, positive) Torricelli triangle of a non-degenerate triangle can be degenerate, and vice versa.

The next theorem shows that $\phi \circ \mathcal{T}_1$ and $\phi \circ \mathcal{T}_2$ are, as functions on the similarity classes of triangles, one-to-one onto. In other words, every triangle is similar to the negative (similarly, positive) Torricelli triangle of some triangle that is unique up to shape. It also shows that iterating $\mathcal{T}_1$ (similarly $\mathcal{T}_2$) is a smoothing process.

Theorem 5.1. *Let $\mathcal{T}_1(ABC)$ and $\mathcal{T}_2(ABC)$ be the negative and positive Torricelli triangles of a triangle ABC, respectively. Then*

$$\phi(\mathcal{T}_1(ABC)) = \frac{-1}{2}\,\phi(ABC), \qquad \phi(\mathcal{T}_2(ABC)) = -2\phi(ABC).$$

Consequently, the process of constructing either of the Torricelli triangles is a smoothing iteration in the sense that it always results in an equilateral triangle. In other words, the limit of the shapes of each of $\mathcal{T}_1^n(ABC)$ and $\mathcal{T}_2^n(ABC)$ is the shape of the equilateral triangle.

Proof. Using the first equations in Theorem 4.1 and the definition of ϕ, we see that

$$\phi(A_1B_1C_1) = \frac{A_1 + \zeta B_1 + \zeta^2 C_1}{A_1 + \zeta^2 B_1 + \zeta C_1} = \frac{A + \zeta B + \zeta^2 C}{-2(A + \zeta^2 B + \zeta C)} = \frac{-\phi(ABC)}{2}.$$

Analogously, we can go on with $\phi(A_2B_2C_2)$.

The last statement follows from the fact that the limits of $(-2s)^n$ and of $(-s/2)^n$, as n tends to infinity, are either 0 or infinity for all s in the extended complex plane $\mathbb{C}_\infty$. □

1	ABC and UVW are similar.	$S(ABC) = S(UVW)$.	$\phi(ABC) = \phi(UVW)$.
2	ABC and UVW are anti-similar.	$S(ABC) = \overline{S(UVW)}$.	$\phi(ABC)\ \phi(UVW) = 1$.
3	ABC is degenerate.	$S(ABC)$ is real.	$\lVert\phi(ABC)\rVert = 1$.
4	ABC is non-degenerate and positively oriented.	$\mathrm{Im}(S) > 0$.	$\lVert\phi(ABC)\rVert < 1$.
5	ABC is non-degenerate and negatively oriented.	$\mathrm{Im}(S) < 0$.	$\lVert\phi(ABC)\rVert > 1$.
6	The vertices C and A coincide.	$S(ABC) = 0$.	$\phi(ABC) = \zeta^2$.
7	The vertices B and A coincide.	$S(ABC) = \infty$.	$\phi(ABC) = \zeta$.
8	The vertices B and C coincide.	$S(ABC) = 1$.	$\phi(ABC) = 1$.
9	ABC is degenerate and isosceles with apex at A.	$S(ABC) = -1$.	$\phi(ABC) = \pm 1$.
10	ABC is equilateral.	$S(ABC) = -\zeta$ or $-\zeta^2$.	$\phi(ABC) = 0$ or ∞.
11	ABC is isosceles with vertex angle $120°$ at A.	$S(ABC) = \zeta$ or ζ^2.	$\phi(ABC) = -2$ or $-1/2$.

Table 1

Corollary 5.1. *Let ABC be a given non-equilateral triangle. Then there exists an n_0 such that $\mathcal{T}_1^n(ABC)$ is positively oriented and $\mathcal{T}_2^n(ABC)$ is negatively oriented for all $n \geq n_0$.*

Proof. Since $\phi(ABC)$ is not 0 and since $\|\phi(\mathcal{T}_2^n(ABC))\| = 2^n\|\phi(ABC)\|$, it follows that for sufficiently large n we have $\|\phi(\mathcal{T}_2^n(ABC))\| > 1$, and therefore $\mathcal{T}_2^n(ABC)$ is negatively oriented. Here we have used Theorem 5.1 and row 5 of Table 1.

For the statement about $\mathcal{T}_1$ we use row 4 of Table 1 and the fact that $\|\phi(ABC)\| \neq \infty$. We remark that the statements in rows 3–5 of Table 1 are proved in Theorem 6.1. □

Another immediate consequence of Theorem 5.1 is that both $\mathcal{T}_1(\mathcal{T}_2(ABC))$ and $\mathcal{T}_2(\mathcal{T}_1(ABC))$ have the same shape as ABC (since (-2)(-1/2) = 1). However, much more can be said about both the sizes and locations of these triangles relative to ABC. Recalling that the *medial triangle* $\mathcal{M}(XYZ)$ of a triangle XYZ is the triangle whose vertices are the mid-points of the sides YZ, ZX, and XY, respectively, the next corollary says that $\mathcal{T}_1(\mathcal{T}_2(ABC))$ and $\mathcal{T}_2(\mathcal{T}_1(ABC))$ coincide, and that they coincide with what one may call the *anti-medial* triangle $\mathcal{M}^{-1}(ABC)$ of ABC. This is the triangle $A_0B_0C_0$ whose medial triangle is ABC, as shown in Figure 3 below. In particular, the composition $\mathcal{T}_2 \circ \mathcal{T}_1$, identical with $\mathcal{T}_1 \circ \mathcal{T}_2$, has a linear magnification factor 2. It may be added that medial and anti-medial triangles are sometimes referred to as *complementary* and *anti-complementary* triangles; see [2], p. 122.

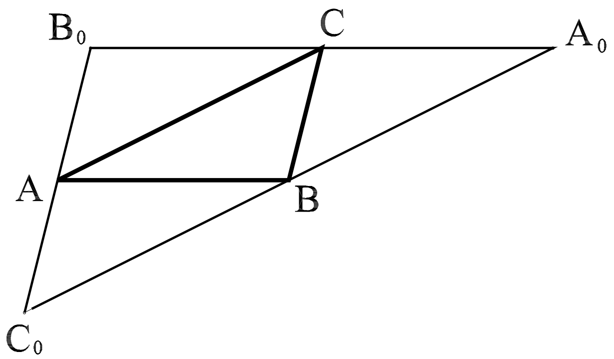

Figure 3

Corollary 5.2. *The negative Torricelli triangle of the positive Torricelli triangle of ABC and the positive Torricelli triangle of the negative Torricelli triangle of ABC coincide, and ABC is their medial triangle. In other words,*

$$\mathcal{T}_1(\mathcal{T}_2(ABC)) = \mathcal{T}_2(\mathcal{T}_1(ABC)) = \mathcal{M}^{-1}(ABC),$$

where $\mathcal{M}^{-1}(ABC)$ is the pre-medial triangle $A_0B_0C_0$ of ABC shown in Figure 3.

Proof. Let $\mathcal{T}_1(\mathcal{T}_2(ABC)) = A_{21}B_{21}C_{21}$ be the negative Torricelli triangle of the positive Torricelli triangle of ABC. From Theorem 4.1 we get

$$\begin{bmatrix} A_{21} \\ B_{21} \\ C_{21} \end{bmatrix} = \begin{bmatrix} 0 & -\zeta^2 & -\zeta \\ -\zeta & 0 & -\zeta^2 \\ -\zeta^2 & -\zeta & 0 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\zeta^2 & -\zeta \\ -\zeta & 0 & -\zeta^2 \\ -\zeta^2 & -\zeta & 0 \end{bmatrix} \begin{bmatrix} 0 & -\zeta & -\zeta^2 \\ -\zeta^2 & 0 & -\zeta \\ -\zeta & -\zeta^2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

or

$$A_{21} = -A + B + C, \quad B_{21} = A - B + C, \quad C_{21} = A + B - C.$$

Therefore

$$A = \frac{B_{21} + C_{21}}{2}, \ B = \frac{C_{21} + A_{21}}{2}, \ C = \frac{A_{21} + B_{21}}{2},$$

and ABC is the medial triangle of $A_{21}B_{21}C_{21}$, i.e., of $\mathcal{T}_1(\mathcal{T}_2(ABC))$. The same holds for $\mathcal{T}_2(\mathcal{T}_1(ABC))$. □

It is easy to see that

$$\phi(BCA) = \zeta\phi(ABC), \quad \phi(ACB) = \frac{1}{\phi(ABC)}.$$

Using Theorem 5.1, it follows that if ABC is not equilateral, then neither of the triangles $\mathcal{T}_1(ABC)$ and $\mathcal{T}_2(ABC)$ can be similar to any cyclic permutation of ABC. However, the next theorem says that the same is not true for a general permutation.

Corollary 5.3. *There is a positively oriented triangle ABC for which the triangle $\mathcal{T}_1(ABC)$ is similar to the triangle ACB. This triangle is unique up to shape and its angles α, β, and γ (in the standard order) are such that $\alpha = \frac{\pi}{6}$ and*

$$\cos\beta = \frac{3 - 2\sqrt{6}}{2\sqrt{9 - 3\sqrt{6}}}, \quad \cos\gamma = \frac{3 + 2\sqrt{6}}{2\sqrt{9 + 3\sqrt{6}}}.$$

Similar statements hold for negatively oriented triangles and for $\mathcal{T}_2$.

Proof. In view of Theorem 5.1 and the fact that $\phi(ACB) = 1/\phi(ABC)$, it follows that $\mathcal{T}_1(ABC)$ and ACB are similar if and only if $\phi^2(ABC) = \frac{-1}{2}$. For positively oriented ABC, this is equivalent, by row 4 of Table 1, to $\phi(ABC) = i/\sqrt{2}$. Without loss of generality, take $A = 0$. Then

$$\begin{aligned} \phi(ABC) = \frac{i}{\sqrt{2}} &\iff \sqrt{2}(\zeta B + \zeta^2 C) = i(\zeta^2 B + \zeta C) \\ &\iff (\sqrt{2} - i\zeta)B = (i - \zeta\sqrt{2})C \\ &\iff B = i - \zeta\sqrt{2} \text{ and } C = \sqrt{2} - i\zeta, \text{ up to similarity.} \end{aligned}$$

For these A, B, and C we have

$$\begin{aligned} \|B\|^2 &= (i - \zeta\sqrt{2})(-i - \zeta^2\sqrt{2}) \;=\; 3 - \sqrt{6}, \\ \|C\|^2 &= (\sqrt{2} - i\zeta)(\sqrt{2} + i\zeta^2) \;=\; 3 + \sqrt{6}, \\ \|B - C\|^2 &= \|-\zeta^2(i + \sqrt{2})\|^2 \;=\; 3. \end{aligned}$$

The rest follows by using the law of cosines. □

6. Relations of $\|\phi\|$ to the Brocard angle and the areas of the Napoleon and Torricelli triangles

The definition of ϕ given earlier in (6) was, in a sense, forced by Theorem 5.1. Specifically, using the ordinary shape function S defined by $S = (A-C)/(A-B)$, we find that the shape S' of the negative Torricelli triangle of a triangle with shape S is given by

$$S' = \frac{S - (1-\zeta)}{(1-\zeta)S + \zeta}. \tag{7}$$

Using the standard method for diagonalizing the associated system

$$f' = f - (1-\zeta)g, \quad g' = (1-\zeta)f + \zeta g$$

of difference equations, one finds that the eigenfunction of (7) is

$$\Phi = \frac{S + \zeta^2}{S + \zeta},$$

with $\Phi' = \frac{-1}{2}\,\Phi$. Writing Φ in terms of A, B, and C, we obtain

$$\Phi = \frac{A + \zeta B + \zeta^2 C}{A + \zeta^2 B + \zeta C},$$

i.e., $\Phi = \phi$. Thus the shape function ϕ is the right function for dealing with the Torricelli triangle iterations. However, due to its symmetry and simplicity, we expect it to be useful in other contexts, too.

In general, one may define a *shape function* (or simply a *shape*) to be a function σ that assigns to every triangle ABC an extended complex number $\sigma(ABC)$ in such a way that ABC and $A'B'C'$ are similar if and only if $\sigma(ABC)$ and $\sigma(A'B'C')$ are equal. Then it is trivial that every triangle is completely determined by its shape (for any shape function) and its area, and that any two shape functions determine each other uniquely. Thus, if ABC is a triangle, say of area 1 for simplicity, then all the elements of ABC and of any configurations arising from ABC are functions of $\sigma(ABC)$ for every shape function σ. However, it would be an advantage for a shape function σ if the various elements of ABC have simple expressions in terms of $\sigma(ABC)$. It would also be an advantage for σ if certain natural elements of ABC can be expressed in terms of $\|\sigma(ABC)\|$, or equivalently if $\|\sigma(ABC)\|$ has an interesting geometric significance. With this in mind, Theorems 6.2–6.5 below must come as pleasant surprises and also as further testimony to the advantage that the shape ϕ has over the usual shape S. It should be emphasized here that if $\sigma_1(ABC)$ and $\sigma_2(ABC)$ are two shape functions, then $\|\sigma_1(ABC)\|$ and $\|\sigma_2(ABC)\|$ do not necessarily determine each other. This is true in particular for the aforementioned shapes ϕ and S, and it explains why Theorems 6.2–6.5 have no analogues in terms of S.

Before proving the main theorems, we introduce the notion of oriented area and prove a simple theorem.

Definition 6.1. *Let (a_1, a_2), (b_1, b_2), and (c_1, c_2) be the cartesian coordinates of the vertices A, B, and C, respectively, of a triangle ABC. The oriented area of ABC, denoted by $[ABC]$, is defined by*

$$[ABC] = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

It is clear that the oriented area of ABC is numerically equal to the usual area of ABC and that the oriented area is positive or negative depending on whether ABC is positively or negatively oriented, respectively; see [27, (9.2.8), p. 198 and 9.7.6, p. 220]. For ease of reference, we include this in the next theorem.

Theorem 6.1. *Let ABC be a triangle and let*

$$x = A\overline{B} + B\overline{C} + C\overline{A}, \tag{8}$$
$$y = \overline{A}B + \overline{B}C + \overline{C}A \ (= \overline{x}), \tag{9}$$
$$K = x - y, \tag{10}$$

where $\overline{z}$ denotes the complex conjugate of z. Then

$$[ABC] = \frac{iK}{4}. \tag{11}$$

If ABC is non-degenerate, then

$$ABC \text{ is positively oriented} \iff \|\phi(ABC)\| < 1 \iff [ABC] > 0 \iff iK > 0,$$
$$ABC \text{ is negatively oriented} \iff \|\phi(ABC)\| > 1 \iff [ABC] < 0 \iff iK < 0.$$

If ABC is degenerate, then $\|\phi(ABC)\| = 1$ and $\operatorname{area}(ABC) = [ABC] = K = 0$.

Proof. Using the relations $2a_1 = A + \overline{A}$, $2ia_2 = A - \overline{A}$, etc., we see that

$$\begin{aligned}
[ABC] &= \frac{1}{8i}\begin{vmatrix} 1 & 1 & 1 \\ A+\overline{A} & B+\overline{B} & C+\overline{C} \\ A-\overline{A} & B-\overline{B} & C-\overline{C} \end{vmatrix} \\
&= \frac{-i}{8}(-2)\left(\left(\overline{B}A + \overline{C}B + \overline{A}C\right) - \left(\overline{A}B + \overline{B}C + \overline{C}A\right)\right) \\
&= \frac{iK}{4}.
\end{aligned}$$

Next, let

$$v = \|A\|^2 + \|B\|^2 + \|C\|^2. \tag{12}$$

Then

$$\begin{aligned}\|\phi(ABC)\|^2 &= \frac{(A+\zeta B+\zeta^2 C)(\overline{A}+\zeta^2\overline{B}+\zeta\overline{C})}{(A+\zeta^2 B+\zeta C)(\overline{A}+\zeta\overline{B}+\zeta^2\overline{C})}\\ &= \frac{v+\zeta^2 x+\zeta y}{v+\zeta x+\zeta^2 y}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\phi(ABC)\|^2 < 1 &\iff v+\zeta^2 x+\zeta y < v+\zeta x+\zeta^2 y \iff (\zeta-\zeta^2)(x-y) > 0\\ &\iff (\zeta-\zeta^2)K > 0 \iff i\sqrt{3}K > 0 \iff iK > 0.\end{aligned}$$

The remaining implications follow, as mentioned earlier, from [27, (9.2.8), p. 198 and 9.7.6, p. 220]. □

The next theorem shows the close relation between $\|\phi(ABC)\|$ and the Brocard angle ω of ABC. Here, the *Brocard angle* ω of a triangle ABC is defined to be the angle $\angle BAP$ where P is the (unique) point inside ABC for which

$$\angle BAP = \angle CBP = \angle ACP;$$

see [32].

Theorem 6.2. *Let ω be the Brocard angle of a triangle ABC and let $\rho = \|\phi(ABC)\|^2$.*

1. *If ABC is positively oriented, then*

$$\rho = \frac{\cos(60^\circ+\omega)}{\cos(60^\circ-\omega)}, \quad \cot\omega = \frac{(1+\rho)\sqrt{3}}{1-\rho}.$$

2. *If ABC is negatively oriented, then*

$$\rho = \frac{\cos(60^\circ-\omega)}{\cos(60^\circ+\omega)}, \quad \cot\omega = \frac{(\rho+1)\sqrt{3}}{\rho-1}.$$

Proof. Let x, y, K, and v be as defined in (8), (9), (10), and (12), and let

$$V = \|A-B\|^2 + \|B-C\|^2 + \|C-A\|^2. \tag{13}$$

Then

$$\begin{aligned}V &= (A-B)(\overline{A}-\overline{B}) + (B-C)(\overline{B}-\overline{C}) + (C-A)(\overline{C}-\overline{A})\\ &= 2v - x - y\\ &= 2(v-y) - K,\end{aligned}$$

and therefore

$$2(v-y) \;=\; V+K. \tag{14}$$

Using (13) and (14), we see that

$$\rho = \frac{v-y+\zeta^2K}{v-y+\zeta K} = \frac{V+K+2\zeta^2K}{V+K+2\zeta K} = \frac{V-iK\sqrt{3}}{V+iK\sqrt{3}}.$$

Therefore

$$\frac{1-\rho}{1+\rho} \;=\; \frac{iK\sqrt{3}}{V}. \tag{15}$$

(a) Suppose now that ABC is positively oriented. Thus $\rho < 1$ and $[ABC] = \text{area}(ABC)$. By [32, Proposition 3], we have $4[ABC]\cot\omega = V$. From this and (11), it follows that

$$V = iK\cot\omega. \tag{16}$$

Plugging this in (15), we see that

$$\begin{aligned} \rho &= \frac{iK\cot\omega - iK\sqrt{3}}{iK\cot\omega + iK\sqrt{3}} \\ &= \frac{\cot\omega-\sqrt{3}}{\cot\omega+\sqrt{3}} \\ &= \frac{\cos\omega-\sqrt{3}\sin\omega}{\cos\omega+\sqrt{3}\sin\omega} = \frac{\cos\omega\cos 60^\circ - \sin\omega\sin 60^\circ}{\cos\omega\cos 60^\circ + \sin\omega\sin 60^\circ} = \frac{\cos(60^\circ+\omega)}{\cos(60^\circ-\omega)}. \end{aligned} \tag{17}$$

This proves the first statement. The second one follows from (17).
(b) If ABC is negatively oriented, then ACB is positively oriented and has the same Brocard angle. Applying (a) to ACB and using the fact that $\phi(ACB)$ $\phi(ABC) = 1$, we get the desired result. □

It follows from Corollary 5.2 that

$$\left|\frac{[\mathcal{T}_1(\mathcal{T}_2(ABC))]}{[ABC]}\right| = \left|\frac{[\mathcal{T}_2(\mathcal{T}_1(ABC))]}{[ABC]}\right| = 4.$$

The following theorem answers natural questions that this relation raises.

Theorem 6.3. *Let $\mathcal{T}_1(ABC)$ and $\mathcal{T}_2(ABC)$ be the negative and positive Torricelli triangles of a triangle ABC, and let $\mathcal{N}_1(ABC)$ and $\mathcal{N}_2(ABC)$ be the negative and positive Napoleon triangles of ABC. Let $\rho = \|\phi(ABC)\|^2$. Then*

$$\frac{[\mathcal{T}_1(ABC)]}{[ABC]} = \frac{4-\rho}{1-\rho},\ \frac{[\mathcal{T}_2(ABC)]}{[ABC]} = \frac{1-4\rho}{1-\rho}, \tag{18}$$

$$\frac{[\mathcal{N}_1(ABC)]}{[ABC]} = \frac{1}{1-\rho},\ \frac{[\mathcal{N}_2(ABC)]}{[ABC]} = \frac{\rho}{1-\rho}. \tag{19}$$

Proof. Let x, y, K, and v be defined as in (8), (9), (10), and (12), and let $A_1B_1C_1 = \mathcal{T}_1(ABC)$. Letting

$$\begin{aligned} x_1 &= A_1\overline{B_1} + B_1\overline{C_1} + C_1\overline{A_1} \\ y_1 &= \overline{A_1}B_1 + \overline{B_1}C_1 + \overline{C_1}A_1 \ (= \overline{x_1}) \\ K_1 &= x_1 - y_1 \end{aligned}$$

and using Theorem 4.1, we see that

$$\begin{aligned} x_1 &= (\zeta^2 B + \zeta C)(\zeta^2\overline{A} + \zeta\overline{C}) + (\zeta^2 C + \zeta A)(\zeta^2\overline{B} + \zeta\overline{A}) + (\zeta^2 A + \zeta B)(\zeta^2\overline{C} + \zeta\overline{B}) \\ &= \zeta^2 v + 2x + \zeta y. \end{aligned}$$

From this and the definition $K = x - y$ we see that

$$\begin{aligned} K_1 &= x_1 - y_1 \ = \ x_1 - \overline{x_1} \\ &= (\zeta^2 - \zeta)v + 2(x - y) + \zeta y - \zeta^2 x \\ &= (\zeta^2 - \zeta)v + 2K + \zeta y - \zeta^2(y + K) \\ &= (\zeta^2 - \zeta)(v - y) + (2 - \zeta^2)K. \end{aligned} \tag{20}$$

It also follows from (13) that

$$\rho = \frac{v - y + \zeta^2 K}{v - y + \zeta K},$$

and therefore

$$(v - y)(1 - \rho) = (\zeta\rho - \zeta^2)K. \tag{21}$$

Multiplying (20) by $(1 - \rho)$ and using (21), we see that

$$(1 - \rho)K_1 = (\zeta^2 - \zeta)(\zeta\rho - \zeta^2)K + (2 - \zeta^2)(1 - \rho)K \ = \ (4 - \rho)K,$$

and therefore

$$\frac{K_1}{K} = \frac{4 - \rho}{1 - \rho}.$$

Using (11), we conclude that

$$\frac{[\mathcal{T}_1(ABC)]}{[ABC]} = \frac{[A_1B_1C_1]}{[ABC]} = \frac{K_1}{K} = \frac{4 - \rho}{1 - \rho},$$

as desired.

To prove the statement pertaining to $\mathcal{T}_2$, let $\mathcal{T}_2(ABC) = A_2B_2C_2$. Then it is easy to see that $\mathcal{T}_1(ACB) = A_2C_2B_2$ and that $\phi(ACB) = 1/\phi(ABC)$. Using this and the part that we have just proved, we see that

$$\frac{[\mathcal{T}_2(ABC)]}{[ABC]} = \frac{[A_2B_2C_2]}{[ABC]} = \frac{-[A_2C_2B_2]}{-[ACB]} = \frac{[\mathcal{T}_1(ACB)]}{[ACB]} = \frac{4 - 1/\rho}{1 - 1/\rho} = \frac{1 - 4\rho}{1 - \rho},$$

as desired. This proves (18).

To prove (19), let $\alpha_1\beta_1\gamma_1 = \mathcal{N}_1(ABC)$ and let

$$\begin{aligned}\xi_1 &= \alpha_1\overline{\beta_1} + \beta_1\overline{\gamma_1} + \gamma_1\overline{\alpha_1}\\ \eta_1 &= \overline{\alpha_1}\beta_1 + \overline{\beta_1}\gamma_1 + \overline{\gamma_1}\alpha_1\ (= \overline{\xi_1})\\ \kappa_1 &= \xi_1 - \eta_1.\end{aligned}$$

Using Theorem 4.1, we see that

$$\xi_1 = \frac{1-\zeta}{3}\,\frac{1-\zeta^2}{3}\,Q = \frac{1}{3}\,Q,$$

where

$$\begin{aligned}Q &= (-\zeta^2B + C)(\overline{A} - \zeta\overline{C}) + (-\zeta^2C + A)(\overline{B} - \zeta\overline{A}) + (-\zeta^2A + B)(\overline{C} - \zeta\overline{B})\\ &= -\zeta v + 2x - \zeta^2 y.\end{aligned}$$

Therefore

$$\begin{aligned}3\kappa_1 &= 3(\xi_1 - \eta_1) \ = \ 3(\xi_1 - \overline{\xi_1}) \ = \ Q - \overline{Q}\\ &= v(\zeta^2 - \zeta) + 2(x - y) - \zeta^2 y + \zeta x\\ &= v(\zeta^2 - \zeta) + 2K - \zeta^2 y + \zeta(y + K)\\ &= (\zeta^2 - \zeta)(v - y) + (2 + \zeta)K. \qquad (22)\end{aligned}$$

Multiplying (22) by $(1-\rho)$ and using (21), we obtain $(1-\rho)\kappa_1 = K$, and therefore

$$\frac{\kappa_1}{K} = \frac{1}{1-\rho}.$$

Using (11), we see that

$$\frac{[\mathcal{N}_1(ABC)]}{[ABC]} = \frac{[\alpha_1\beta_1\gamma_1]}{[ABC]} = \frac{\kappa_1}{K} = \frac{1}{1-\rho},$$

as desired.

To prove the statement pertaining to $\mathcal{N}_2$, let $\mathcal{N}_2(ABC) = \alpha_2\beta_2\gamma_2$. Then it is easy to see that $\mathcal{N}_1(ACB) = \alpha_2\gamma_2\beta_2$ and that $\phi(ACB) = 1/\phi(ABC)$. Using this and the part that we have just proved, we see that

$$\frac{[\mathcal{N}_2(ABC)]}{[ABC]} = \frac{[\alpha_2\beta_2\gamma_2]}{[ABC]} = -\frac{[\alpha_2\gamma_2\beta_2]}{[ABC]} = -\frac{[\mathcal{N}_1(ACB)]}{[ABC]} = -\frac{1}{1-1/\rho} = \frac{\rho}{1-\rho},$$

as desired. This completes the proof. □

It is well-known that the difference between the areas of the negative and positive Napoleon triangles of ABC is equal to that of ABC. This, as well as the seemingly unknown analogue for the Torricelli triangles, follows immediately from the previous theorem. We record these in Theorem 6.4.

Theorem 6.4. *Let U and W be the areas of the negative and positive Torricelli triangles of a triangle ABC, and let u and w be the areas of the negative and positive Napoleon triangles. Let Δ be the area of ABC and let $\rho = \|\phi(ABC)\|^2$. Then*

(a) $|u - w| = \Delta$.

(b) *If* $\frac{1}{4} < \rho < 4$, *then* $|U - W| = 5\Delta$. *Otherwise,* $U + W = 5\Delta$.

Part (b) of Theorem 6.4 raises a natural question regarding a geometric equivalent of the condition $\frac{1}{4} < \rho < 4$. The following theorem provides an answer.

Theorem 6.5. *Let ω be the Brocard angle of a triangle ABC and let $\rho = \|\phi(ABC)\|^2$. Then*

$$\begin{aligned} \rho = 4 \text{ or } \rho = \frac{1}{4} &\iff \cot\omega = \frac{5\sqrt{3}}{3}, \\ \rho > 4 \text{ or } \rho < \frac{1}{4} &\iff \cot\omega < \frac{5\sqrt{3}}{3}, \\ \frac{1}{4} < \rho < 4 &\iff \cot\omega > \frac{5\sqrt{3}}{3}. \end{aligned}$$

Proof. Since $\|\phi(ABC)\|\ \|\phi(ACB)\| = 1$ and since ABC and ACB have the same Brocard angles, we may restrict our attention to positively oriented triangles. So let ABC be positively oriented. Letting V and K be as defined in (13) and (10), we see that

$$\begin{aligned} \rho < \frac{1}{4} &\iff \frac{V - iK\sqrt{3}}{V + iK\sqrt{3}} < \frac{1}{4}, \text{ by (15)}, \\ &\iff 4V - 4iK\sqrt{3} < V + iK\sqrt{3} \\ &\iff 3V < 5iK\sqrt{3} \\ &\iff 3iK\cot\omega < 5iK\sqrt{3}, \text{ by (16)}, \\ &\iff \cot\omega < \frac{5\sqrt{3}}{3}, \text{ because } iK > 0 \text{ by Theorem 6.1.} \end{aligned}$$

This proves the theorem. □

It is not apparent whether the condition $\cot\omega < \frac{5\sqrt{3}}{3}$ has an interpretation that is more geometric. However, letting Δ be the area of ABC and V be the sum $a^2 + b^2 + c^2$ of the squares of its side-lengths, it follows from (16) that

$$4\Delta\cot\omega = V, \tag{23}$$

and it follows from Heron's formula [27, (9.2.9), p. 198] that

$$16\Delta^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4). \tag{24}$$

Squaring (23) and using (24), we obtain

$$\big(2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\big)\cot^2\omega = V^2,$$

or equivalently

$$\cot^2\omega = \frac{V^2}{V^2 - 2P},$$

where $P = a^4 + b^4 + c^4$. It follows that

$$\cot\omega < \frac{5\sqrt{3}}{3} \iff \cot^2\omega < \frac{25}{3} \iff \frac{V^2}{V^2 - 2P} < \frac{25}{3} \iff 25P < 11V^2.$$

The surprise that the last inequality $25P < 11V^2$ is nothing but inequality (3) of Note 4.1 has an explanation. In fact, a continuity argument applied to Theorem 6.5 shows that the condition $\cot\omega = \frac{5\sqrt{3}}{3}$ takes place when U or W changes sign, i.e., when U or W is 0. This happens when one of the Torricelli triangles is degenerate – an unacceptable triangle according to [28]. For ease of reference, we include this in the next theorem.

Theorem 6.6. *Let ABC be a triangle with side-lengths a, b, and c, and with Brocard angle ω. Then the following statements are equivalent.*

(a) $\mathcal{T}_1(ABC)$ *is degenerate.*
(b) $\mathcal{T}_2(ABC)$ *is degenerate.*
(c) $\mathcal{T}_1^{-1}(ABC)$ *is degenerate.*
(d) $\mathcal{T}_2^{-1}(ABC)$ *is degenerate.*
(e) $\cot\omega = \frac{5\sqrt{3}}{3}$.
(f) $\|\phi(ABC)\| = 2$ *or* $\frac{1}{2}$.
(g) $25(a^4 + b^4 + c^4) = 11(a^2 + b^2 + c^2)^2$.

Proof. The equivalence of (a), (e), (f), and (g) is established above. The equivalence of (a) and (b) follows from the fact that (e) holds for ABC if and only if it holds for ACB, the fact that if $\mathcal{T}_1(ABC) = A_1B_1C_1$, then $\mathcal{T}_2(ACB) = A_1C_1B_1$, and the fact that $A_1B_1C_1$ is degenerate if and only if $A_1C_1B_1$ is degenerate. As for (c), one uses that

$$\mathcal{T}_1^{-1}(ABC) \sim \mathcal{M}^{-1}(\mathcal{T}_1^{-1}(ABC)) = (\mathcal{T}_2\mathcal{T}_1)(\mathcal{T}_1^{-1}(ABC)) = \mathcal{T}_2(ABC).$$

Similarly for (d). □

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