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The Optimal Ball and Horoball Packings to the Coxeter Honeycombs in the Hyperbolic *d*-space

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Abstract. In a former paper [18] a method is described that determines the data and the density of the optimal ball or horoball packing to each Coxeter tiling in the hyperbolic 3-space. In this work we extend this procedure – based on the projective interpretation of the hyperbolic geometry – to higher dimensional Coxeter honeycombs in $\mathbb{H}^d$, $(d = 4, 5)$, and determine the metric data of their optimal ball and horoball packings, respectively.

1. Introduction

In [3], Böröczky and Florian determined the densest horosphere packing of $\mathbb{H}^3$ without any symmetry assumption. They proved that this provides the general density upper bound for all sphere packings (more precisely ball packings) of $\mathbb{H}^3$, where the density is related to the Dirichlet-Voronoi cell of every ball, as follows:

$$s_0 = (1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \; - + + \cdots)^{-1} \approx 0.85327609.$$

This limit is achieved by the 4 horoballs touching each other in the ideal regular simplex whose honeycomb has the Schläfli symbol $(3, 3, 6)$, the horoball centres are just in the 4 vertices of the simplex. Beyond the universal upper bound there are a few results in this topic ([4], [15], [16], [17]), therefore our method seems

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to be timely for determining local optimal ball and horoball packings for given hyperbolic tilings.

In [18] we investigated the regular Coxeter honeycombs and their optimal ball and horoball packings in the hyperbolic space $\mathbb{H}^3$. These Coxeter tilings are the following:

$$(p,\ q,\ r) = (3,5,3),\ (4,3,5),\ (5,3,4),\ (5,3,5),$$
$$(3,3,6),\ (3,4,4),\ (4,3,6),\ (5,3,6),$$
$$(3,6,3),\ (4,4,4),\ (6,3,6),$$
$$(4,4,3),\ (6,3,3),\ (6,3,4),\ (6,3,5).$$

In each case we have determined the metric data of the cell, moreover, we have computed the density of the optimal ball or horoball packing.

A d-dimensional honeycomb $\mathcal{P}$ (or solid tessellation, or tiling) is an infinite set of congruent polyhedra (polytopes) fitting together to fill all space ($\mathbb{H}^d$ $(d \geq 2)$) just once, so that every face of each polyhedron (polytope) belongs to another polyhedron as well. At present the cells are congruent regular polyhedra. A honeycomb with cells congruent to a given regular polyhedron P exists if and only if the dihedral angle of P is a submultiple of 2π (in the hyperbolic plane zero angle is also possible). All honeycombs with bounded cells were first found by Schlegel in 1883, those with unbounded cells by H. S. M. Coxeter in his famous article [5]. Such honeycombs exist only for $d \leq 5$.

Another approach to describing honeycombs involves the analysis of their symmetry groups. If $\mathcal{P}$ is such a honeycomb, then any motion taking one cell into another maps the whole honeycomb onto itself. The symmetry group of a honeycomb is denoted by $Sym\mathcal{P}$. Therefore the characteristic simplex $\mathcal{F}$ of any cell $P \in \mathcal{P}$ is a fundamental domain of the group $Sym\mathcal{P}$ generated by reflections in its facets ($(d-1)$-dimensional hyperfaces).

The scheme of a regular polytope P is a weighted graph (characterizing $P \subset \mathbb{H}^d$ up to congruence) in which the nodes, numbered by $0, 1, \dots, d$ correspond to the bounding hyperplanes of $\mathcal{F}$. Two nodes are joined by an edge if the corresponding hyperplanes are not orthogonal. Let the set of weights $(n_1, n_2, n_3, \dots, n_{d-1})$ be the Schläfli symbol of P, and n_d the weight describing the dihedral angle of P that equals $\frac{2\pi}{n_d}$. Then $\mathcal{F}$ is the Coxeter simplex with the scheme

.

The ordered set $(n_1, n_2, n_3, \dots, n_{d-1}, n_d)$ is said to be the Schläfli symbol of the honeycomb $\mathcal{P}$. To every scheme there is a corresponding symmetric matrix (b^{ij}) of size $(d+1) \times (d+1)$ where $b^{ii} = 1$ and, for $i \neq j \in \{0, 1, 2, \dots, d\}$, b^{ij} equals $-\cos \frac{\pi}{n_{ij}}$ with all angles between the facets i,j of $\mathcal{F}$; then $n_k =: n_{k-1,k}$, too. Reversing the numbers of the nodes in the scheme of $\mathcal{P}$ (but keeping the weights), leads to the so called dual honeycomb $\mathcal{P}^*$ whose symmetry group coincides with $Sym\mathcal{P}$.

In this paper we investigate regular Coxeter honeycombs and their optimal ball and horoball packings in the hyperbolic space $\mathbb{H}^d, (d = 4, 5)$. By $Sym\mathcal{P}$ we denote the symmetry group of the honeycomb $\mathcal{P}_{n_1 n_2 \dots n_d}$, thus

$$P_{n_1 n_2 \dots n_d} = \{ \bigcup_{\gamma \in Sym\mathcal{P}_{n_1 n_2 \dots n_{d-1}}} \gamma(\mathcal{F}_{n_1 n_2 \dots n_d}) \}.$$

For the density, we relate each ball or horoball, respectively, to its regular polytope $P_{n_1 n_2 \dots n_d}$ that contains it (not necessarily assumed to be a Dirichlet-Voronoi cell). The 4-dimensional Coxeter tilings are the following:

$$(n_1, n_2, n_3, n_4) = (5, 3, 3, 3),\ (3, 3, 3, 5),\ (5, 3, 3, 4), \tag{1.1}$$

$$(4, 3, 3, 5),\ (5, 3, 3, 5),\ (3, 4, 3, 4);$$

$$(n_1, n_2, n_3, n_4) = (4, 3, 4, 3); \tag{1.2}$$

The 5-dimensional Coxeter tilings are the following:

$$(n_1, n_2, n_3, n_4, n_5) = (3, 3, 3, 4, 3), \tag{1.3}$$

$$(n_1, n_2, n_3, n_4, n_5) = (3, 4, 3, 3, 3),\ (3, 4, 3, 3, 4), \tag{1.4}$$

$$(4, 3, 3, 4, 3),\ (3, 3, 4, 3, 3).$$

From these, in Section 3 of this paper, we shall consider every tiling, where a horosphere is inscribed in each regular polyhedron which is infinite centred and its vertices are proper points or lie at infinity. Thus we obtain of the parameters (1.2), (1.4) satisfying the above mentioned properties.

In Section 4 we consider the Coxeter honeycombs with parameters (1.1) and (1.3). In these cases the cells have proper centres and its vertices are proper points or lie at infinity, thus we investigate the ball packings where each ball lies in its regular polyhedron $P_{n_1 n_2 \dots n_d}$.

With our method, based on the projective interpretation of hyperbolic geometry [12], [14], in each case we have determined the metric data of the cell, moreover, we have computed the density of the optimal ball and horoball packing, respectively.

The computations were carried out by *Maple V Release 5* up to 30 decimals.

2. The projective model

Let X denote either the d-dimensional sphere $\mathbb{S}^d$, the d-dimensional Euclidean space $\mathbb{E}^d$ or the hyperbolic space $\mathbb{H}^d$, $d \geq 2$. We use for $\mathbb{H}^d$ the projective model in the Lorentz space $\mathbb{E}^{1,d}$ of signature $(1, d)$, i.e. $\mathbb{E}^{1,d}$ denotes the real vector space $\mathbf{V}^{d+1}$ equipped with the bilinear form of signature $(1, d)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0 y^0 + x^1 y^1 + \cdots + x^d y^d \tag{2.1}$$

where the non-zero vectors

$$\mathbf{x} = (x^0, x^1, \dots, x^d) \in \mathbf{V}^{d+1} \text{ and } \mathbf{y} = (y^0, y^1, \dots, y^d) \in \mathbf{V}^{d+1},$$

are determined up to real factors, for representing points of $\mathcal{P}^d(\mathbb{R})$. Then $\mathbb{H}^d$ can be interpreted as the interior of the quadric

$$Q = \{[\mathbf{x}] \in \mathcal{P}^d | \langle \mathbf{x}, \mathbf{x} \rangle = 0\} =: \partial \mathbb{H}^d \tag{2.2}$$

in the real projective space $\mathcal{P}^d(\mathbf{V}^{d+1}, \boldsymbol{V}_{d+1})$. Any proper interior point $\mathbf{x} \in \mathbb{H}^d$ is characterized by $\langle \mathbf{x}, \mathbf{x} \rangle < 0$.

The points of the boundary $\partial \mathbb{H}^d$ in $\mathcal{P}^d$ are called points at infinity of $\mathbb{H}^d$, the points $\mathbf{y}$ with $\langle \mathbf{y}, \mathbf{y} \rangle > 0$ lying outside $\partial \mathbb{H}^d$ are said to be outer points of $\mathbb{H}^d$. Let $P([\mathbf{x}]) \in \mathcal{P}^d$, a point $[\mathbf{y}] \in \mathcal{P}^d$ is said to be conjugate to $[\mathbf{x}]$ relative to Q if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ holds. The set of all points which are conjugate to $P([\mathbf{x}])$ form a projective (polar) hyperplane

$$pol(P) := \{[\mathbf{y}] \in \mathcal{P}^d | \langle \mathbf{x}, \mathbf{y} \rangle = 0.\} \tag{2.3}$$

Thus the quadric Q (by the symmetric bilinear form or scalar product in (2.1)) induces a bijection (linear polarity $\mathbf{V}^{d+1} \to \boldsymbol{V}_{d+1}$) from the points of $\mathcal{P}^d$ onto its hyperplanes.

The point $X[\mathbf{x}]$ and the hyperplane $\alpha[\boldsymbol{a}]$ are called incident if $\mathbf{x}\boldsymbol{a} = 0$ i.e. the value of the linear form $\boldsymbol{a}$ on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \mathbf{V}^{d+1} \setminus \{\mathbf{0}\}$, $\boldsymbol{a} \in \boldsymbol{V}_{d+1} \setminus \{\mathbf{0}\}$). The straight lines of $\mathcal{P}^d$ are characterized by 2-subspaces of $\mathbf{V}^{d+1}$ or by $d-1$-spaces of $\boldsymbol{V}_{d+1}$, i.e. by 2 points or dually by $d-1$ hyperplane, respectively [12].

Let $P \subset \mathbb{H}^d$ denote a polyhedron bounded by hyperplanes H^i, which are characterized by unit normal vectors $\boldsymbol{b}^i \in \boldsymbol{V}_{d+1}$ directed inwards with respect to P:

$$H^i := \{\mathbf{x} \in \mathbb{H}^d | \langle \mathbf{x},\ \boldsymbol{b}^i \rangle = 0\} \text{ with } \langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1. \tag{2.4}$$

We always assume that P is acute-angled polyhedron and the vertices are proper points or lie at infinity.

The Gram matrix $G(P) := (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle)\ i, j \in \{0, 1, 2, \dots, d\}$ of the normal vectors $\boldsymbol{b}^i$ associated to P is an indecomposable symmetric matrix of signature $(1, d)$ with entries $\langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1$ and $\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle \leq 0$ for $i \neq j$, having the following geometrical meaning

$$\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{cases} 0 & \text{if } H^i \perp H^j, \\ -\cos \alpha^{ij} & \text{if } H^i, H^j \text{ intersect on } P \text{ at angle } \alpha^{ij}, \\ -1 & \text{if } H^i, H^j \text{ are parallel in hyperbolic sense}, \\ -\cosh l^{ij} & \text{if } H^i, H^j \text{ admit a common perpendicular of length } l^{ij}. \end{cases}$$

Definition 2.1. *An orthoscheme $\mathcal{O}$ in X is a simplex bounded by $d+1$ hyperplanes $H^0, \dots, H^d$ such that* ([8], [1])

$$H^i \perp H^j, \text{ for } j \neq i-1, i, i+1.$$

A plane orthoscheme is a right-angled triangle, whose area can be expressed by the well known defect formula. For an orthoscheme we denote the $(d-1)$-hyperface opposite to the vertex A_i by H^i $(0 \le i \le d)$. An orthoscheme $\mathcal{O}$ has d dihedral angles which are not right angles. Let α^{ij} denote the dihedral angle of $\mathcal{O}$ between the faces H^i and H^j. Then we have

$$\alpha^{ij} = \frac{\pi}{2}, \quad \text{if} \ \ 0 \le i < j-1 \le d.$$

The remaining d dihedral angles $\alpha^{i,i+1}$, $(0 \le i \le d-1)$ are called the essential angles of $\mathcal{O}$. The initial and final vertices, A_0 and A_d of the orthogonal edge-path

$$\bigcup_{i=0}^{d-1} A_i A_{i+1}$$

are called principal vertices of the orthoscheme.

In our cases the characteristic simplex $\mathcal{F}$ of any honeycomb $\mathcal{P}$ with Schläfli symbol $(n_1, n_2, n_3, \dots, n_d)$ is an orthoscheme.

The matrix $(b^{ij}) = G(P)$ is the so called Coxeter-Schläfli matrix of such an orthoscheme $\mathcal{F}$ with parameters $n_1, n_2, n_3, \dots, n_d$:

$$(b^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{n_1} & 0 & \dots & 0 \\ -\cos\frac{\pi}{n_1} & 1 & -\cos\frac{\pi}{n_2} & \dots & 0 \\ 0 & -\cos\frac{\pi}{n_2} & 1 & \dots & 0 \\ 0 & 0 & -\cos\frac{\pi}{n_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -\cos\frac{\pi}{n_d} & 1 \end{pmatrix}. \tag{2.5}$$

Inverting the Coxeter-Schläfli matrix (b^{ij}) (see (2.5) and Section 1) of an orthoscheme we get the matrix (a_{ij}) and we can express any distance between two vertices by the following formula [10]:

$$\cosh\frac{d_{ij}}{k} = \frac{-a_{ij}}{\sqrt{a_{ii}a_{jj}}}, \tag{2.6}$$

at present paper we choose $k=1$, $K=-k^2$ is the sectional curvature of $\mathbb{H}^d$. The distance s of two proper points $(\mathbf{x})$ and $(\mathbf{y})$ can be calculated by the following formula:

$$\cosh\frac{s}{k} = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \tag{2.7}$$

2.1. Description of a horosphere in the hyperbolic space $\mathbb{H}^d$

We shall use the Cayley-Klein ball model with centre $A_{d-1}(1, 0, \dots, 0)$ of the hyperbolic space $\mathbb{H}^d$ in a Cartesian homogeneous rectangular coordinate system $\{\mathbf{e}_i\}$ $i = 0, \dots, d$ to (2.1). We have illustrated in Figure 2.a the site of the horosphere in the 3-dimensional Cayley-Klein ball model. The equation of the horosphere with centre $A_d(1, 0, \dots, 1)$ through the point $S(1, 0, \dots, s)$ in the projective coordinates $(x^0, x^1, x^2, \dots, x^d)$ is the following [18]:

$$0 = -2s(x^0)^2 - 2(x^d)^2 + 2(s+1)(x^0 x^d) + (s-1)((x^1)^2 + \cdots + (x^{d-1})^2). \quad (2.7)$$

In the Cartesian rectangular coordinate system this equation is the following:

$$\frac{2(\sum_{i=1}^{d-1} h_i^2)}{1-s} + \frac{4(h_d - \frac{s+1}{2})^2}{(1-s)^2} = 1, \text{ where } h_i := \frac{x^i}{x^0}, i = 1, 2, \dots, d. \quad (2.8)$$

The site of this horosphere in the part of the infinite regular polyhedron is illustrated in Figure 1 (d=3).

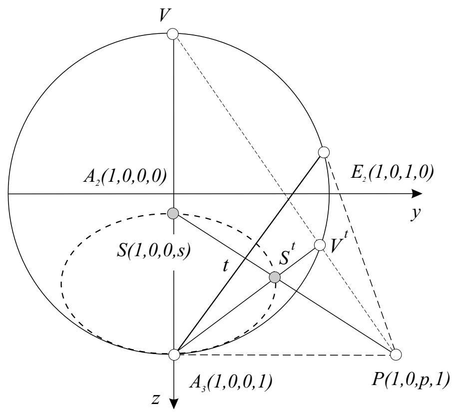


Figure 1.

3. The *d*-dimensional optimal horoball packings

In this section we consider those Coxeter tilings in the 4- and the 5-dimensional hyperbolic space, where an infinite regular polyhedron (polytope) is circumscribed about a horosphere and the polyhedron has proper vertices or the vertices lie at infinity. These honeycombs are given by their Schläfli symbols with parameters (1.2) and (1.4) where the facets are regular 3- and 4-dimensional polyhedra, respectively. In Figure 2.a we illustrate a part of a 3-dimensional Coxeter honeycomb, where A_3 is the centre of a horosphere, the centre of a regular polygon is denoted by A_2 (A_2 is also the common point of this face and the optimal horosphere), A_0 is one of its vertices, and we denote by A_1 the footpoint of A_2 on an edge of this face (see [18]). Analogously in Figure 2.b, we display a part of the infinite regular polyhedron of a Coxeter tiling in 4-dimensional hyperbolic space, where A_4 is the centre of a horosphere, A_3 is the centre of the facet-polyhedron (A_3 is also the common point of this facet-polyhedron and the optimal horosphere), the centre of its regular polygon is denoted by A_2, A_0 is one of its vertices, and A_1 is the centre of an edge of this face where A_0 is one of its endpoints. It is sufficient to consider the optimal horoball packing in the orthoscheme $A_0 A_1 A_2 \dots A_d$ because the tiling can be constructed from such orthoschemes as fundamental domain of $Sym\mathcal{P}_{n_1 n_2 \dots n_d}$. We introduce a Cartesian rectangular projective coordinate system, by a vector

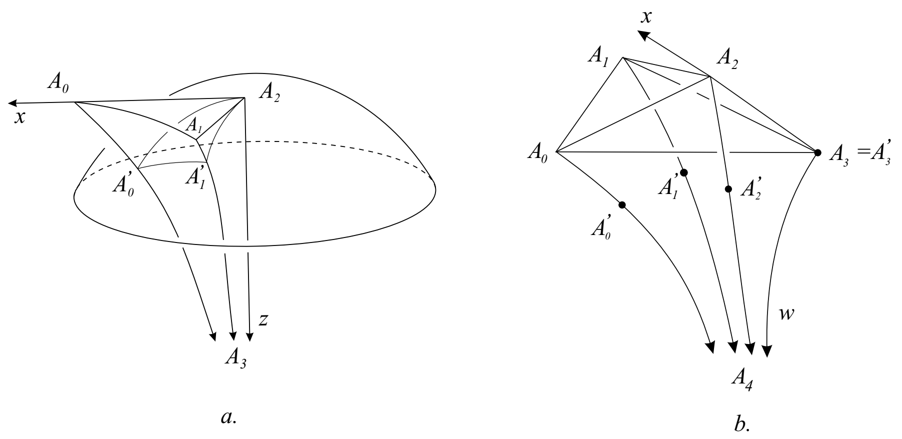


Figure 2.

basis $A_i(\mathbf{v}_i)$ $(i = 0, 1, 2, \dots, d)$ for $\mathbb{P}^d$, with the following coordinates of the points of the infinite regular polyhedron (in the 4-dimensional case see Figure 2.b),

$$A_0(\mathbf{v}_0)(1, v_0^1, \dots, v_0^{d-1}, 0),\ A_1(\mathbf{v}_1)(1, v_1^1, \dots, v_1^{d-2}, 0, 0),$$
$$A_2(\mathbf{v}_2)(1, v_2^1, \dots, v_2^{d-3}, 0, 0, 0),\ A_3(\mathbf{v}_3)(1, v_3^1, \dots, v_3^{d-4}, 0, 0, 0, 0), \dots$$
$$A_{d-1}(\mathbf{v}_{d-1})(1, 0, \dots, 0, 0),\ A_d(\mathbf{v}_d)(1, 0, \dots, 0, 1).$$

3.1. The data of a cell of a regular honeycomb

By the formulas (2.5), (2.6) and (2.7) and by the above introduced coordinate system we get a system of equations for $i, j = 0, 1, 2, \dots, d-1,\ i \neq j$, for the coordinates:

$$\frac{-\langle \mathbf{v_i}, \mathbf{v_j} \rangle}{\sqrt{\langle \mathbf{v_i}, \mathbf{v_i} \rangle \langle \mathbf{v_j}, \mathbf{v_j} \rangle}} = \frac{-a_{ij}}{\sqrt{a_{ii} a_{jj}}}. \tag{3.1}$$

Solving this system of equations we get the coordinates in our basis $\{\mathbf{e}_i\}$, $i = 0, \dots, d$, as follows in Table 1:

Table 1

$(n_1, n_2, \dots, n_d)$	$v_0^1 = v_1^1 = v_2^1 = v_3^1$	$v_0^2 = v_1^2 = v_2^2$	$v_0^3 = v_1^3$	v_0^4
$(4, 3, 4, 3)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	–
$(3, 4, 3, 3, 3)$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{2}}$
$(3, 4, 3, 3, 4)$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2}$
$(4, 3, 3, 4, 3)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(3, 3, 4, 3, 3)$	$\frac{1}{2}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{2}$

3.2. On the optimal horoballs

It is clear that the optimal horosphere has to touch the faces of its containing infinite regular polyhedron. Thus the optimal horoball passes through the point $A_{d-1}(1,0,\dots,0,0)$ and the parameter s in the equation of the optimal horosphere is 0 (see Section 2.1). The orthoscheme $A_0A_1\dots A_d$ and its images under $Sym\mathcal{P}_{n_0n_1\dots n_d}$ divide the optimal horosphere into congruent horospherical simplices (see Figure 2). The vertices $A'_0, A'_1, A'_2, \dots, A'_{d-1} = A_{d-1}(1,0,\dots,0,0)$ of such a simplex are in the edges A_0A_d, $A_2A_d, \dots,$ $A_{d-1}A_d$, and on the optimal horosphere, respectively. Therefore, their coordinates can be determined in the Cayley-Klein model. We have summarized the coordinates of the points A'_i $(i=0,1,\dots,d-1)$ for the investigated honeycombs in the following:

$$(4,3,4,3):\ A'_0(1,\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{3}{11}),\ A'_1(1,\frac{2}{5},\frac{2}{5},0,\frac{1}{5}),\ A'_2(1,\frac{4}{9},0,0,\frac{1}{9}).$$

$$(3,4,3,3,3):A'_0(1,\frac{2}{5},\frac{2\sqrt{3}}{15},\frac{\sqrt{6}}{15},\frac{\sqrt{2}}{5},\frac{1}{5}),\ A'_1(1,\frac{8}{19},\frac{8\sqrt{3}}{57},\frac{4\sqrt{6}}{57},0,\frac{3}{19}),$$
$$A'_2(1,\frac{3}{7},\frac{\sqrt{3}}{7},0,0,\frac{1}{7}),\ A'_3(1,\frac{4}{9},0,0,0,\frac{1}{9}),$$

$$(3,4,3,3,4):A'_0(1,\frac{\sqrt{2}}{3},\frac{\sqrt{6}}{9},\frac{\sqrt{3}}{9},\frac{1}{3},\frac{1}{3}),\ A'_1(1,\frac{4\sqrt{2}}{11},\frac{4\sqrt{6}}{33},\frac{4\sqrt{3}}{33},0,\frac{3}{11}),$$
$$A'_2(1,\frac{3\sqrt{2}}{8},\frac{\sqrt{6}}{8},0,0,\frac{1}{4}),\ A'_3(1,\frac{2\sqrt{2}}{5},0,0,0,\frac{1}{5}),$$

$$(4,3,3,4,3):A'_0(1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}),\ A'_1(1,\frac{4}{11},\frac{4}{11},\frac{4}{11},0,\frac{3}{11}),$$
$$A'_2(1,\frac{2}{5},\frac{\sqrt{2}}{5},0,0,\frac{1}{5}),\ A'_3(1,\frac{4}{9},0,0,0,\frac{1}{9}),$$

$$(3,3,4,3,3):A'_0(1,\frac{1}{3},\frac{1}{3\sqrt{3}},\frac{\sqrt{6}}{9},\frac{2}{3},\frac{1}{3}),\ A'_1(1,\frac{2}{5},\frac{2\sqrt{3}}{15},\frac{2\sqrt{6}}{15},0,\frac{1}{5}),$$
$$A'_2(1,\frac{3}{7},\frac{\sqrt{3}}{7},0,0,\frac{1}{7}),\ A'_3(1,\frac{4}{9},0,0,0,\frac{1}{9}).$$

The lengths of the edges of such a horospherical polyhedron (the edges are horocycle segments) are determined by the classical formula of J. Bolyai (see Figure 3):

$$l(x) = k\sinh\frac{x}{k}\ \text{(at present } k=1). \tag{3.2}$$

The volume of the horoball pieces in the d-dimensional hyperbolic space can be calculated by the formula (3.3) which is the generalization of the classical formula of J. Bolyai to higher dimensions (see [19]). If the volume of the polyhedron A

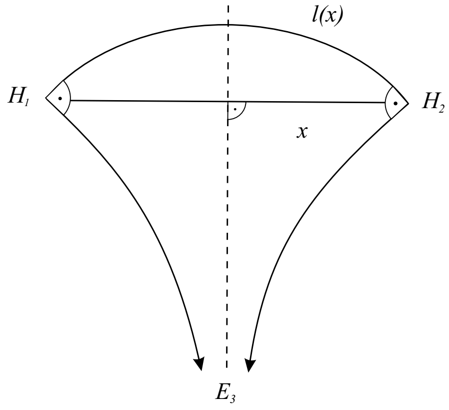


Figure 3.

on the horosphere is $\mathcal{A}$, the volume determined by A and the aggregate of axes drawn from A is equal to

$$V = \frac{1}{d-1} k\mathcal{A} \ \text{ (we assume that } k = 1 \text{ here).} \tag{3.3}$$

It is well known that the intrinsic geometry of the horosphere is Euclidean, therefore, the volume $\mathcal{A}_{n_0 n_1 \dots n_d}$ of the horospherical $d-1$-dimensional simplex $A'_0 A'_1 \dots A'_{d-1}$ can be calculated from the lengths of edges implied by (2.7) and (3.2).

For the density of the packing it is sufficient to relate the volume of the optimal horoball piece to that of its containing orthoscheme $A_0 A_1 \dots A_d$ (see Figure 3) because the tiling can be constructed of such simplex.

The volume of a Coxeter orthoscheme with Schläfli symbol $(n_0, \dots, n_d)$ is denoted by $W_{n_0 n_1 \dots n_d}$. The volumes of all hyperbolic Coxeter simplex (where the vertices are proper points or lie at infinity) were determined by N. W. Johnson, R. Kellerhals, J. G. Ratcliffe and S. T. Tschantz in their nice work [7]. The volumes are summarized in Table 2.

Definition 3.1. *The density of the horoball packing for the regular honeycombs* (1.2), (1.4) *is defined by the following formula:*

$$\delta_{n_0 n_1 \dots n_d} := \frac{\frac{1}{d-1} k \mathcal{A}_{n_0 n_1 \dots n_d}}{W_{n_0 n_1 \dots n_d}}. \tag{3.4}$$

In Table 2 we have collected the results of the optimal horoball packings for the Coxeter honeycombs of Schläfli symbols (1.2) and (1.4):

Table 2

$(n_0, n_1, \dots, n_d)$	$\mathcal{A}_{n_0 n_1 \dots n_d}$	$W_{n_0 n_1 \dots n_d}$	$\delta_{n_0 n_1 \dots n_d}$
$(4,3,4,3)$	$\frac{\sqrt{3}}{108} \sinh \frac{\operatorname{arcosh} \frac{11}{8}}{2}$	$\frac{\pi^2}{864}$	≈ 0.60792710
$(3,4,3,3,3)$	$\frac{\sqrt{3}}{2304} \sinh \frac{\operatorname{arcosh} \frac{17}{16}}{2}$	$\frac{7\zeta(3)}{46080}$	≈ 0.59421955
$(3,4,3,3,4)$	$\frac{1}{144} \sinh \frac{\operatorname{arcosh} \frac{9}{8}}{2}$	$\frac{7\zeta(3)}{4608}$	≈ 0.23768782
$(4,3,3,4,3)$	$\frac{1}{384} \sinh \frac{\operatorname{arcosh} \frac{9}{8}}{2}$	$\frac{7\zeta(3)}{4608}$	≈ 0.35653173
$(3,3,4,3,3)$	$\frac{\sqrt{2}}{1152} \sinh \frac{\operatorname{arcosh} \frac{5}{4}}{2}$	$\frac{7\zeta(3)}{9216}$	≈ 0.47537564

Remark 3.2. In the 5-dimensional cases ζ is Riemann's zeta function:

$$\zeta(n) := \sum_{r=1}^{\infty} \frac{1}{r^n}.$$

4. The d-dimensional optimal ball packings

In this section we investigate the Coxeter honeycombs with Schläfli symbols in (1.1) and (1.3).

In Figure 4 we have illustrated a part of the 3- and 4-dimensional regular polyhedron (polytope) of a Coxeter tiling. In the 3-dimensional case (Figure 4.a) A_3 is the centre of a cell, the centre of a regular polygon is denoted by A_2, A_0 is one of its vertices and we denote by A_1 the midpoint of an edge of this face. Analogously in Figure 4.b, we display a part of the regular polyhedron of a Coxeter tiling in 4-dimensional hyperbolic space, where A_4 is the centre of a regular polyhedron (polytope), A_3 is the centre of the facet-polyhedron (A_3 is also the common point of this facet-polyhedron and the optimal ball), the centre of its regular polygon is denoted by A_2, A_1 is the centre of an edge of this face where A_0 is one of its. In general, it is sufficient to consider the optimal ball packing in the orthoscheme $A_0A_1A_2 \dots A_d$ because the tiling can be constructed from such orthoschemes as fundamental domain of $Sym\mathcal{P}_{n_1 n_2 \dots n_d}$.

The cells for these parameters have proper centres and the vertices are proper points or lie at infinity. The volume of every regular polyhedron of $\mathcal{P}_{n_1 n_2 \dots n_d}$ is denoted by $V(P_{n_1 n_2 \dots n_d})$. In this section we are interested in ball packings, where the congruent balls with radius $R = R_{n_1 n_2 \dots n_d}$ lie in cells of the above mentioned tilings.

Definition 4.1. *The density of the ball packing to any Coxeter honeycomb* (1.1) *and* (1.3) *can be defined by the following formula:*

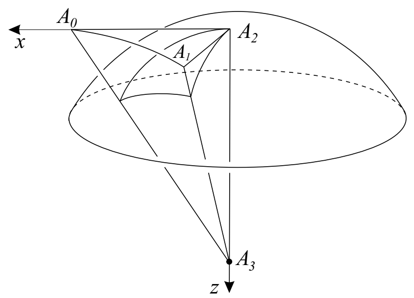
$$\delta_{n_1 n_2 \dots n_d} := \frac{2\pi^{d/2} \int_0^R \sinh^{d-1}(x) dx}{\Gamma(\frac{d}{2}) V(P_{n_1 n_2 \dots n_d})}. \tag{4.1}$$

Remark 4.2. The Gamma function is defined for $Re(z) > 0$ by:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \mathrm{d}t$$

and is extended to the rest of the complex plane by analytic continuation.

It is clear that the optimal ball with centre A_d has to touch the facets of its regular polyhedron (see Figure 4). Thus the optimal ball passes through the point A_{d-1}, and the optimal radius $A_{d-1}A_d$ of these tilings can be calculated by the projective method [10], [14], where $(a_{ij}) = (b^{ij})^{-1}$ and $b^{ij} = -\cos\frac{\pi}{n_{ij}}$ (see Section 1 and (2.5), (2.6)).



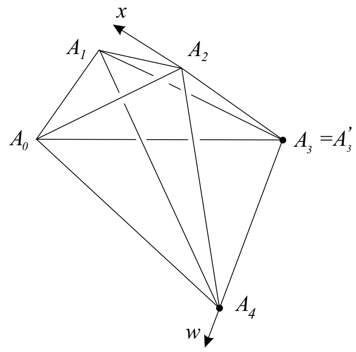


Figure 4

$$R^{opt}_{n_1 n_2 \dots n_d} := A_{d-1}A_d = \operatorname{arcosh}\frac{-a_{(d-1)d}}{\sqrt{a_{(d-1)(d-1)}a_{dd}}}. \tag{4.2}$$

Again, we have calculated the volume $W_{n_1 n_2 \dots n_d}$ of the orthoschemes $A_0A_1 \dots A_d$ (see [7]) for the parameters (1.1) and (1.3).

The volumes $W_{n_1 n_2 \dots n_d}$ and the volumes $V(P_{n_1 n_2 \dots n_d})$ of the regular polyhedra $P_{n_1 n_2 \dots n_d} \in \mathcal{P}_{n_1 n_2 \dots n_d}$ are summarized in Table 3.

Table 3

$(n_1, n_2, \dots, n_d)$	$W_{n_1 n_2 \dots n_d}$	$V(\mathcal{P}_{n_1 n_2 \dots n_d})$
$(5,3,3,3)$	$\frac{\pi^2}{10800}$	$14400 \cdot W_{5333} = \frac{4}{3}\pi^2 \approx 13.15947253$
$(3,3,3,5)$	$\frac{\pi^2}{10800}$	$120 \cdot W_{3335} = \frac{1}{90}\pi^2 \approx 13.15947253$
$(5,3,3,4)$	$\frac{17\pi^2}{21600}$	$14400 \cdot W_{5334} = \frac{34}{3}\pi^2 \approx 111.85551655$
$(4,3,3,5)$	$\frac{17\pi^2}{21600}$	$384 \cdot W_{4335} = \frac{68}{225}\pi^2 \approx 2.98281378$
$(5,3,3,5)$	$\frac{13\pi^2}{5400}$	$14400 \cdot W_{5335} = \frac{104}{3}\pi^2 \approx 342.14628590$
$(3,4,3,4)$	$\frac{\pi^2}{864}$	$1152 \cdot W_{3434} = \frac{4}{3}\pi^2 \approx 13.15947253$
$(3,3,3,4,3)$	$\frac{7\zeta(3)}{46080}$	$3840 \cdot W_{33343} = \frac{35}{6}\zeta(3) \approx 7.01199860$

The optimal radius and optimal density is summarized by the formulas (4.1), (4.2) in the following table:

Table 4

$(n_1, n_2, \dots, n_d)$	$R^{opt}_{n_1 n_2 \dots n_d}$	$\delta^{opt}_{n_1 n_2 \dots n_d}$
$(5,3,3,3)$	$\operatorname{arcosh}\frac{3-\sqrt{5}}{\sqrt{(3-\sqrt{5})(7-3\sqrt{5})}}$	≈ 0.69098301
$(3,3,3,5)$	$\operatorname{arcosh}\frac{1+\sqrt{5}}{\sqrt{10}}$	≈ 0.09877254
$(5,3,3,4)$	$\operatorname{arcosh}\frac{\sqrt{2}(3-\sqrt{5})}{\sqrt{(3-\sqrt{5})(7-3\sqrt{5})}}$	≈ 0.41862781
$(4,3,3,5)$	$\operatorname{arcosh}\frac{\sqrt{2}(\sqrt{5}+1)}{4}$	≈ 0.14406128
$(5,3,3,5)$	$\operatorname{arcosh}\frac{-1+\sqrt{5}}{\sqrt{(3-\sqrt{5})(7-3\sqrt{5})}}$	≈ 0.23250327
$(3,4,3,4)$	$\operatorname{arcosh}(\sqrt{2})$	≈ 0.29289322
$(3,3,3,4,3)$	$\operatorname{arcosh}\frac{\sqrt{5}}{2}$	≈ 0.02162577

Analogous questions for determining the optimal ball and horoball packings of tilings in hyperbolic d-space ($d > 2$) seem to be interesting and timely. Our projective method suites to studying these problems.

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